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# A solvable version of the inverse problem of dynamics

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**Abstract**

The particular version of the inverse problem of dynamics considered here is: given the 'slope function' $\gamma = f_y/f_x$, representing uniquely a family of planar curves $f(x, y) = c$, find, if possible, potentials of the form $V(x, y) = v(\gamma(x, y))$ which give rise to this family. Such potentials $V$ will then have as equipotential curves the isoclinic curves $\gamma =$ const of the family $f(x, y) = c$. We show that, for the problem of admitting a solution, a necessary and sufficient condition must be satisfied by the given $\gamma(x, y)$. Inferring by reasoning from particular to more general forms, we find analytically a very rich set of slope functions $\gamma(x, y)$ satisfying this condition. In contrast to the (not always solvable) general case $V = V(x, y)$, in all these cases we can find the potential $v = v(\gamma)$ analytically by quadratures. Several examples of pairs $(\gamma, v(\gamma))$ are presented.

## 1. Introduction

The inverse problem of dynamics in a broad sense consists of the determination of forces, parameters and constraints which are required for the realization of a motion of a mechanical system with some properties given in advance (Galiullin 1984). As Santilli (1978) pointed out, for a Newtonian system such an inverse problem looks for necessary and sufficient conditions for the existence of a Lagrangian for which the given system represents the Euler–Lagrange equations.

The inverse problem considered in the present study seeks potentials $V = V(x, y)$ which, for adequate initial conditions, give rise to a preassigned family of curves, traced in the Cartesian plane by a material point of unit mass. This old problem has its origin in the determination by Newton in 1687 of the force law compatible with Kepler's laws of planetary motion. It was afterwards treated by Joukovsky, as reported by Whittaker (1961, section 56), and then brought to the scene again by Szebehely (1974). An account of the history of the various versions of the problem and of the progress made during the two decades after

Szebehely can be found in Bozis' review paper (1995). Many papers from the last decade may be found in Anisiu's report (2003). In addition, we mention here the study of certain isoenergetic families of orbits (Puel 1999, Borghero and Bozis 2002), the estimate of the role of the inverse problem in the framework of celestial mechanics (Szebehely 1997, Bozis 2003) and Agekyan's (2003) work in galactic dynamics. A purely mathematical account of the various versions of the inverse problem was given recently by Ramirez and Sadovskaia (2004).

The version of the planar inverse problem we are dealing with concerns the motion of one material point of unit mass, moving in the $xy$ inertial Cartesian plane. For a given family of curves

$$f(x, y) = c \tag{1}$$

we denote by

$$\gamma = \frac{f_y}{f_x} \qquad \text{and} \qquad \Gamma = \gamma\gamma_x - \gamma_y. \tag{2}$$

The 'slope function' $\gamma$ represents the family (1), in the sense that if the family (1) is given, $\gamma$ is obtained uniquely; conversely, from a given $\gamma$ we can obtain a unique family (1). The inverse problem consists of finding potentials $V$ which can produce the planar family of orbits (1) or, equivalently, of finding potentials $V$ compatible with a given $\gamma$. This means that a material point of unit mass, whose motion is governed by the Newtonian conservative system

$$\ddot{x} = -V_x, \qquad \ddot{y} = -V_y, \tag{3}$$

will describe, with appropriate initial conditions, the curves of the family (1), i.e., $f(x(t), y(t)) = c$ for $t$ in a real interval.

Szebehely's equation (1974) relating the total energy function $E(f)$, the potential $V$ and the 'slope function' $\gamma$ reads (Bozis 1995)

$$E = V - \frac{(1+\gamma^2)(V_x + \gamma V_y)}{2\Gamma}. \tag{4}$$

In what follows we shall assume that $\Gamma \neq 0$. From the viewpoint of this paper, the case $\Gamma = 0$ is commented upon in remark 5 of section 4.

We shall adopt for the partial derivatives the notation $V_{ij} = \frac{\partial^{i+j} V}{\partial x^i \partial y^j}$, which will be used for $\gamma$ too.

The potential $V$ also satisfies the second-order linear partial differential equation (free of the energy $E(f)$) (Bozis 1995, Anisiu 2004)

$$V_{02} - V_{20} + \kappa V_{11} = \lambda V_{10} + \mu V_{01} \tag{5}$$

where

$$\kappa = \frac{1}{\gamma} - \gamma, \qquad \lambda = \frac{\Gamma_y - \gamma\Gamma_x}{\gamma\Gamma}, \qquad \mu = \lambda\gamma + \frac{3\Gamma}{\gamma}. \tag{6}$$

If $V$ is a solution of Szebehely's equation (4) or of Bozis' equation (5), the solution of the system (3) with initial conditions $t_0, x_0, y_0, \dot{x}_0 = \tilde{g}(x_0, y_0) f_y(x_0, y_0)$ and $\dot{y}_0 = -\tilde{g}(x_0, y_0) f_x(x_0, y_0)$ will have the property that

$$f(x(t), y(t)) = f(x_0, y_0) = c_0.$$

The two-variable function $\tilde{g}$ can be determined by solving a linear second-order partial differential equation in $\tilde{g}^2$ (see Gonzales-Gascon *et al* (1984)). The function $\tilde{g}$ is related to the potential $V$ by

$$\tilde{g}^2 = -\frac{V_x + \gamma V_y}{f_x^2 \Gamma}.$$

Real motion of the particle will trace the curves of the family (1) in the region given by $(V_x + \gamma V_y)/\Gamma \leqslant 0$ (Bozis and Ichtiaroglou 1994).

In spite of its linearity, in general, for a given family $\gamma(x, y)$, equation (5) cannot be solved for $V = V(x, y)$. For this reason, we consider in this paper the following special version of the inverse problem described above: supposing that a family $\gamma = \gamma(x, y)$ is given, let us examine if there exist solutions of (5) of the form

$$V(x, y) = v(\gamma(x, y)). \tag{7}$$

This is our basic assumption. Its implications are mostly of a mathematical nature. Indeed, it will appear that only for adequate families $\gamma(x, y)$ (those satisfying condition (15) of section 2) equation (5) has solutions of the form (7). The partial differential equation (5) now becomes ordinary but, as it is still of the second order, it is not generally expected to be solvable. Yet, in our case, it turns out that its general solution is always found by quadratures.

It is easily seen from (5) that if $V$ is a solution, $c_1 V + c_2$ is also a solution ($c_1$, $c_2$ constants). For the reasons of simplicity, in what follows the constants $c_1$ and $c_2$ will be omitted.

From the physical point of view, the meaning of the assumption (7) is that for $\gamma(x, y) =$ constant it is also $V(x, y) =$ constant, i.e., the isoclinic curves of the orbits in the $xy$ plane coincide with the equipotential curves.

The pertinent direct problem is the following: given a potential $V = V(x, y)$, to find monoparametric families of the form $\gamma = h(V(x, y))$, traced in its presence by a unit mass point. For this problem, of course,

(i) not any given potential would be ‘adequate’;
(ii) the calculations involved would be much more complicated. In fact, due to the nonlinearity of the pertinent differential equation in $\gamma = h(V)$, one would not generally be able to find solutions by quadratures.

## 2. A differential condition for the given families

Let us suppose that for a given $\gamma$ there exist potentials of the form (7). We can then write the partial derivatives of $V$ as follows (with $v' = \frac{\mathrm{d}v}{\mathrm{d}\gamma}$ and with $\gamma_{ij} = \frac{\partial^{i+j}\gamma}{\partial x^i \partial y^j}$)

$$V_{10} = v'\gamma_{10}, \qquad V_{01} = v'\gamma_{01}$$
$$V_{20} = v''\gamma_{10}^2 + v'\gamma_{20}, \qquad V_{11} = v''\gamma_{10}\gamma_{01} + v'\gamma_{11}, \qquad V_{02} = v''\gamma_{01}^2 + v'\gamma_{02}.$$

Equation (5) becomes

$$\frac{v''}{v'} = R, \tag{8}$$

where

$$R = \frac{R_1}{R_2}, \tag{9}$$

with

$$R_1 = (1+\gamma^2)(\gamma\gamma_{01}\gamma_{20} - (\gamma\gamma_{10} + \gamma_{01})\gamma_{11} + \gamma_{10}\gamma_{02}) + (\gamma\gamma_{10} - \gamma_{01})\big(\gamma_{10}^2 - 2\gamma\gamma_{10}\gamma_{01} + 3\gamma_{01}^2\big),$$
$$R_2 = (\gamma\gamma_{10} - \gamma_{01})^2(\gamma\gamma_{01} + \gamma_{10}). \tag{10}$$

**Remark 1.** We shall solve the differential equation (8), when possible, in domains where $R_2$ has no zeros. On the other hand, we note that the expression $R_2$ is *identically zero* if and only if

$$\text{(i)}\quad \Gamma = \gamma\gamma_{10} - \gamma_{01} = 0 \qquad \text{or} \qquad \text{(ii)}\quad \gamma\gamma_{01} + \gamma_{10} = 0. \tag{11}$$

For (11(i))—a case already excluded in section 1 from our study—$R_1$ also vanishes, so, the ratio $v''/v'$ becomes indeterminate. If in (11(ii)) we have $\gamma_{01} = 0$, we must also have $\gamma_{10} = 0$, hence $\gamma$ = constant, excluded by the condition $\Gamma \neq 0$. It follows that we can express $\gamma = -\gamma_{10}/\gamma_{01}$ from (11(ii)). We differentiate (11(ii)) with respect to $x$, then with respect to $y$, and substitute $\gamma_{20} = -(\gamma_{10}\gamma_{01} + \gamma\gamma_{11})$ and $\gamma_{02} = -\left(\gamma_{11} + \gamma_{01}^2\right)/\gamma$ in $R_1$ from (10); then we insert $\gamma = -\gamma_{10}/\gamma_{01}$ in the result and obtain after some calculations $R_1 = -2\left(\gamma_{10}^2 + \gamma_{01}^2\right)^2/\gamma_{01}$. Therefore, for the functions $\gamma$ which satisfy (11(ii)) we have $R_1 \neq 0$. From (8) we get in this case for $v$, hence also for $V$, only the trivial solution of a constant potential.

**Remark 2.** The condition (11(ii)) is associated with families (1) for which

$$f_x^2 + f_y^2 = A(f), \qquad A = \text{arbitrary functional} \tag{12}$$

known as families of *parallel curves* (Goursat 1945, p 42). Indeed, each of (11(ii)) and (12) amounts to the same condition: $f_x f_y (f_{xx} - f_{yy}) = \left(f_x^2 - f_y^2\right) f_{xy}$.

**Remark 3.** From equation (8), it is clear that if a function $\gamma$ is given such that $R_1 = 0$ (therefore, according to remark 1, $R_2 \neq 0$), $v = \gamma$ will be its solution. It is not, of course, an easy or even possible task to find all solutions $\gamma = \gamma(x, y)$ of the equation $R_1 = 0$. Trying to find, e.g., a function $\gamma(x, y)$ of the form $\gamma = \alpha + \beta y/x$ for which $R_1 = 0$, we obtain the complex families (see Contopoulos and Bozis (2000)) $\gamma = \pm 2\mathrm{i} - y/x$, compatible with $V = \pm 2\mathrm{i} - y/x$, for which the energy is $E = \mp\mathrm{i}$.

Generalizing slightly and trying to make $R_1 = 0$ with $\gamma = \gamma(y/x)$, we come to the first-order differential equation

$$(w^2 + 2w\gamma + 3)\frac{\mathrm{d}\gamma}{\mathrm{d}w} = 1 + \gamma^2, \tag{13}$$

where $w = y/x$.

The function $v$, as it can be seen from (7), depends on $\gamma$ only, hence by necessity from (8) it follows that $R\,(x, y) = r\,(\gamma\,(x, y))$, i.e.,

$$R_y/R_x = \gamma_{01}/\gamma_{10}. \tag{14}$$

Working out this condition, we obtain

$$\begin{aligned}
(1 + \gamma^2)\big(&a_{30}\gamma_{30} + a_{21}\gamma_{21} + a_{12}\gamma_{12} + a_{03}\gamma_{03} + a_{2020}\gamma_{20}^2 + a_{1111}\gamma_{11}^2 + a_{0202}\gamma_{02}^2 \\
&+ a_{2011}\gamma_{20}\gamma_{11} + a_{2002}\gamma_{20}\gamma_{02} + a_{1102}\gamma_{11}\gamma_{02}\big) \\
&= (\gamma\gamma_{10} - \gamma_{01})\big((3\gamma^2 - 1)\gamma_{10}^2 - 8\gamma\gamma_{10}\gamma_{01} - (\gamma^2 - 3)\gamma_{01}^2\big) \\
&\quad \times \left(\gamma_{01}^2\gamma_{20} - 2\gamma_{10}\gamma_{01}\gamma_{11} + \gamma_{10}^2\gamma_{02}\right),
\end{aligned} \tag{15}$$

with

$$\begin{aligned}
a_{30} &= \gamma\gamma_{01}^2(\gamma\gamma_{01} + \gamma_{10})(\gamma\gamma_{10} - \gamma_{01}) \\
a_{21} &= -\gamma_{01}(\gamma\gamma_{01} + \gamma_{10})(\gamma\gamma_{10} - \gamma_{01})(2\gamma\gamma_{10} + \gamma_{01}) \\
a_{12} &= \gamma_{10}(\gamma\gamma_{01} + \gamma_{10})(\gamma\gamma_{10} - \gamma_{01})(\gamma\gamma_{10} + 2\gamma_{01}) \\
a_{03} &= -\gamma_{10}^2(\gamma\gamma_{01} + \gamma_{10})(\gamma\gamma_{10} - \gamma_{01}) \\
a_{2020} &= -\gamma\gamma_{01}^2(2\gamma^2\gamma_{01} + 3\gamma\gamma_{10} - \gamma_{01}) \\
a_{1111} &= -2\gamma\left(\gamma\gamma_{10}^3 + 3\gamma\gamma_{01}^2\gamma_{10} + 3\gamma_{01}\gamma_{10}^2 + \gamma_{01}^3\right) \\
a_{0202} &= \gamma_{10}^2(\gamma^2\gamma_{10} - 3\gamma\gamma_{01} - 2\gamma_{10}) \\
a_{2011} &= \gamma_{01}\left(3\gamma^3\gamma_{01}\gamma_{10} + 5\gamma^2\gamma_{01}^2 + 6\gamma^2\gamma_{10}^2 + 3\gamma\gamma_{01}\gamma_{10} - \gamma_{01}^2\right) \\
a_{2002} &= -\gamma\left(\gamma\gamma_{10}^3 + 3\gamma\gamma_{01}^2\gamma_{10} + 3\gamma_{01}\gamma_{10}^2 + \gamma_{01}^3\right) \\
a_{1102} &= -\gamma_{10}\left(\gamma^3\gamma_{10}^2 - 3\gamma^2\gamma_{01}\gamma_{10} - 5\gamma\gamma_{10}^2 - 6\gamma\gamma_{01}^2 - 3\gamma_{01}\gamma_{10}\right)
\end{aligned} \tag{16}$$

expressed in terms of first-order partial derivatives of $\gamma$.

So, condition (15) is necessary for $\gamma$ in order that (8) possesses a solution $V = v(\gamma(x, y))$. In fact, if $\gamma$ is a solution of (15) for which $R_2 \neq 0$, we obtain from (8)

$$v(\gamma) = c_1 \int \exp\left(\int r(\gamma)\,\mathrm{d}\gamma\right) \mathrm{d}\gamma + c_2, \tag{17}$$

with $c_1, c_2$ constants, and with $V$ given by (7). In what follows we shall omit $c_1$ and $c_2$, as stated at the end of section 1. The meaning of (17) is that the inverse problem (for orbits satisfying (15) and $R_2 \neq 0$) is solved by quadratures, in the sense that solutions of the form (7) can be found.

It is then natural to focus our attention to condition (15) as the *tank* from which we can and we must select adequate families $\gamma(x, y)$ for which (17) would be a compatible potential.

## 3. Certain adequate classes of families

By 'adequate' we mean families $\gamma(x, y)$ satisfying condition (15), i.e., families for which the version of the inverse problem considered here does indeed give a solution of the form (7).

It appeared to us impossible to obtain the totality of solutions $\gamma = \gamma(x, y)$ of (15), a nonlinear partial differential equation of the third order in the unknown function $\gamma(x, y)$. We did, however, manage to find solutions of (15) of certain forms, as expounded in the present section and then, in turn, of more general forms as reported in section 4.

(a) We start by observing that all terms in (15) include as a factor second- or third-order derivatives of $\gamma$. Therefore, all slope functions

$$\gamma = \gamma_0 + \gamma_1 x + \gamma_2 y \tag{18}$$

($\gamma_0, \gamma_1, \gamma_2$ constants) satisfy (15).

With (18), we obtain from (10)

$$\begin{aligned} R_1 &= (\gamma\gamma_1 - \gamma_2)\left(-2\gamma_1\gamma_2\gamma + \gamma_1^2 + 3\gamma_2^2\right) \\ R_2 &= (\gamma\gamma_1 - \gamma_2)\left(\gamma_1\gamma_2\gamma^2 + \left(\gamma_1^2 - \gamma_2^2\right)\gamma - \gamma_1\gamma_2\right), \end{aligned} \tag{19}$$

and

$$r(\gamma) = \frac{-2k_0\gamma + 1 + 3k_0^2}{(\gamma - k_0)(k_0\gamma + 1)}, \qquad k_0 = \frac{\gamma_2}{\gamma_1}. \tag{20}$$

As stated in section 2, from the functions of the form (18), we choose only those with $R_2 \neq 0$, hence the ones for which at least one of $\gamma_1$ and $\gamma_2$ is different from zero. Let us suppose that both $\gamma_1$ and $\gamma_2$ are different from zero; the case when $\gamma$ is a function of one variable ($x$ or $y$) will be treated later.

From (17), we obtain

$$v(\gamma) = \frac{2k_0\gamma - k_0^2 + 1}{2k_0^2(k_0\gamma + 1)^2}. \tag{21}$$

The equipotential lines are in this case parallel straight lines.

We remark that for $\gamma$ given by (18), the expression

$$\mathcal{A} = \gamma\gamma_{01}\gamma_{20} - (\gamma\gamma_{10} + \gamma_{01})\gamma_{11} + \gamma_{10}\gamma_{02}$$

in (10) is equal to zero. We have in general that

$$\mathcal{A} = \gamma_{01}^2 \left(\gamma \left(\frac{\gamma_{10}}{\gamma_{01}}\right)_x - \left(\frac{\gamma_{10}}{\gamma_{01}}\right)_y\right),$$

hence it will also be zero for $\gamma = g(\gamma_0 + \gamma_1 x + \gamma_2 y)$, with $g$ an arbitrary function. It can be easily checked that for

$$\gamma = g(\gamma_0 + \gamma_1 x + \gamma_2 y), \tag{22}$$

the values of $R_1$ and $R_2$ are those from (19) multiplied by $\dot{g}$, where $\dot{g}$ denotes the derivative of $g$ with respect to its unique argument. It follows that $r(\gamma)$ will be given by the same formula (20), independently of the arbitrariness of the function $g$. Therefore, the potential corresponding to (22) is again given by (21).

All the functions $\gamma$ of the form (22) give rise to potentials (21), for which all isoclinic curves of the family (1) are also equipotential. From (4) we find that all families (22), independently of the selection of the arbitrary function $g(\gamma_0 + \gamma_1 x + \gamma_2 y)$, are isoenergetic, i.e., all their members are traced with the constant value for the energy $E = k_0/2$.

**Example 1.** For $\gamma = (x + y)^2$ (corresponding to the family of orbits (1) with $f(x, y) = \frac{x+y-1}{x+y+1}\exp(2y)$) we obtain from (21) $v(\gamma) = \frac{\gamma}{(\gamma+1)^2}$, i.e., $V(x, y) = \frac{(x+y)^2}{((x+y)^2+1)^2}$ and $E = 1/2$. The very same potential $v(\gamma) = \frac{\gamma}{(\gamma+1)^2}$ is also compatible with $\gamma = \frac{1}{(x+y)^2}$, corresponding to the family $f^*(x, y) = \frac{x+y-1}{x+y+1}\exp(2x)$, traced also with $E = 1/2$.

(b) Another general result regarding condition (15) is the following: equation (15) is satisfied for any arbitrary function of the form

$$\gamma = g(w), \qquad w = \frac{y}{x} \quad (g(w) \neq w). \tag{23}$$

It turns out that the corresponding function $r$ is

$$r = \frac{g^2 + 1 - \dot{g}(w^2 + 2gw + 3)}{\dot{g}(g - w)(1 + gw)}, \tag{24}$$

where $\dot{g}$ is the derivative of $g$ with respect to its unique argument $w$.

Keeping in mind that $R_1$ was the numerator of $r$, equating it to zero will give equation (13) obtained directly in section 2. We remark that the condition $\gamma\gamma_{01} + \gamma_{10} \neq 0$, discussed in remark 1 of section 2, becomes $g(w) \neq w$, which means that no circles $x^2 + y^2 = c$ are allowed.

Since we know $\gamma = g(w)$ (hence $\mathrm{d}\gamma = \dot{g}\,\mathrm{d}w$), we calculate the integral $\int R\,\mathrm{d}\gamma = \int R\dot{g}\,\mathrm{d}w$, i.e.,

$$i_1 = \int R\,\mathrm{d}\gamma = \int \frac{1 + g^2 - \dot{g}(w^2 + 2gw + 3)}{(g - w)(1 + gw)}\,\mathrm{d}w. \tag{25}$$

Then, we proceed to the calculation of the potential

$$V(w) = \int \exp(i_1)\dot{g}\,\mathrm{d}w. \tag{26}$$

The equipotential lines are straight lines through the origin.

**Application 1.** Let us apply the last two formulae for

$$\gamma = k_0 w^m \tag{27}$$

with $k_0$, $m$ constants (for $m = 1$ we take $k_0 \neq 1$, to exclude the circles $x^2 + y^2 = c$).

For $m \neq 1$, we find from (25)

$$i_1 = \frac{1}{m-1} \ln \frac{w^{2m}(1 + k_0 w^{m+1})^{m-1}}{(w - k_0 w^m)^{3m-1}}$$

and from (26) we obtain

$$V(w) = \int \frac{(1 + k_0 w^{m+1}) w^{m-2}}{(1 - k_0 w^{m-1})^{\frac{3m-1}{m-1}}} \, \mathrm{d}w \tag{28}$$

which, apart from a multiplicative and an additive constant (depending on $m$, $k_0$), leads to

$$V(w) = \left(1 + k_0^2 w^{2m}\right)(1 - k_0 w^{m-1})^{\frac{2m}{1-m}} \tag{29}$$

valid (for any $m \neq 1$ and any $k_0$) for families of the form (27). With the aid of (4), it can be shown that, independently of the value of $m$, all families (27) are isoenergetic, traced with the total energy $E = 0$.

**Example 2.** Let us consider in (27) $m = -1$. It is found that each family $\gamma = k_0/w$ (corresponding to $f(x, y) = x^{\frac{1}{k_0}} y$) is compatible with the potential

$$V(x, y) = \frac{k_0^2 x^2 + y^2}{k_0 x^2 - y^2}. \tag{30}$$

The total energy of all members of the family is, as expected, $E = 0$.

For $k_0 = -1/4$ one obtains the family $f(x, y) = y/x^4$, which was proved (Bozis *et al* 1997) to be traced under the Hénon–Heiles (1964) potential

$$V_1(x, y) = \frac{x^2}{2} + 8y^2 + x^2 y + \frac{16}{3} y^3 \tag{31}$$

with the energy $E_1 = -x^4/(24y)$. It follows that this family is common to the Hénon–Heiles potential $V_1$ and to the potential (which can be expressed in terms of $\gamma$) obtained from (30), namely

$$V_2(x, y) = -(x^2 + 16y^2)/(x^2 + 4y^2). \tag{32}$$

**Example 3.** For $m = 1$ and $k_0 \neq 1$, we have the family of curves (27) $\gamma = k_0 y/x$ (corresponding to $f(x, y) = x^2 + k_0 y^2$). From (25), we obtain

$$i_1 = \frac{1}{k_0 - 1} \ln(w^{1-3k_0}(k_0 w^2 + 1)^{k_0 - 1})$$

and

$$V(w) = -w^{\frac{2k_0}{1-k_0}} \left(1 + k_0^2 w^2\right). \tag{33}$$

**Remark 1.** Slope functions of the form $\gamma = \gamma(x)$ or $\gamma = \gamma(y)$ satisfy (15). In fact, all functions $\gamma$ of the form (22) with $\gamma_1 = 0$ or $\gamma_2 = 0$ belong to this class. We obtain $r = 1/\gamma$ and the potential $V = V(x) = -\gamma^2(x)$, compatible with $\gamma$ for the first case. In the second case, $r = -3/\gamma$ and the potential $V = V(y) = -\gamma^{-2}(y)$ is compatible with $\gamma = \gamma(y)$. In both cases, the energy is equal to 1.

(c) The following result generalizes and covers the previous two cases of this section.

If $\gamma_0, \gamma_1, \gamma_2, \delta_0, \delta_1, \delta_2$ are constants, any function $\gamma(x, y)$ of the form

$$\gamma = g\left(\frac{\gamma_0 + \gamma_1 x + \gamma_2 y}{\delta_0 + \delta_1 x + \delta_2 y}\right), \qquad g = \text{arbitrary} \tag{34}$$

satisfies condition (15). This can be shown by direct computations.

For $\delta_0 = 1, \delta_1 = \delta_2 = 0$ (34) reduces to (22), whereas for $\gamma_0 = \gamma_1 = \delta_0 = \delta_2 = 0$, $\delta_1 = \gamma_2$ (34) reduces to (23). The ratio $r = R_1/R_2$ (not given here) for this case can be calculated and can be shown to constitute a generalization of formulae (20) and (24).

The examples in this section are new; the homogeneous potential (32) has the interesting property that it produces the family $\gamma = -x/(4y)$ which is also compatible with the quasihomogeneous Hénon–Heiles potential (31).

## 4. A richer solution of equation (15)

We observed that all slope functions (34) separately make each side of equation (15) zero. In fact any function $\gamma$ given by (34) satisfies the second-order partial differential equation

$$\gamma_{01}^2\gamma_{20} - 2\gamma_{10}\gamma_{01}\gamma_{11} + \gamma_{10}^2\gamma_{02} = 0 \tag{35}$$

whose left-hand side appears as a factor in the right-hand side of equation (15). This led us to try to find the general solution of this equation. To this end, we write (35) as

$$\gamma_{10}\left(\frac{\gamma_{01}}{\gamma_{10}}\right)_y - \gamma_{01}\left(\frac{\gamma_{01}}{\gamma_{10}}\right)_x = 0 \tag{36}$$

whose general solution is

$$\frac{\gamma_{01}}{\gamma_{10}} = A(\gamma) \tag{37}$$

with $A$ arbitrary function of $\gamma = \gamma(x, y)$.

Then from (37), we readily obtain

$$x + yA(\gamma) = B(\gamma) \tag{38}$$

where $B(\gamma)$ is also arbitrary. So, all functions $\gamma$ defined by (38) satisfy (35), as expected. But, to our surprise, it turns out that all $\gamma$ defined by (38) satisfy the third-order equation (15) as well.

On the other hand, in view of (38), we can calculate $\gamma_{10}, \gamma_{01}, \gamma_{20}, \gamma_{11}, \gamma_{02}$ in terms of $\gamma$, $A, A', A'', B', B''$ (where primes denote differentiation with respect to $\gamma$) and insert them into (10) in order to calculate $R = r(\gamma)$ from (9). In so doing we find

$$r = \frac{(1+\gamma^2)A' - 3A^2 + 2\gamma A - 1}{(A-\gamma)(\gamma A + 1)}. \tag{39}$$

In conclusion, we see that, for all families $\gamma$ given by (38), there exist potentials of the form (7) which generate them and which can be found by quadratures. From (37) we obtain the ratio $\gamma_{01}/\gamma_{10} = A(\gamma)$ and insert it in Szebehely's equation (4), also taking into account that $V(x, y) = v(\gamma(x, y))$. So, we write (4) as

$$E = v + \frac{(1+\gamma^2)(1+\gamma A)v'}{2(A-\gamma)}. \tag{40}$$

From (40), and in view of (8), we compute $E_x$, $E_y$ and we find them identically equal to zero. This means that $E$ = constant for all members of each family $\gamma$. It is called that the isoenergeticity above was established for families $\gamma(x, y)$ which make each member of equation (15) zero. It is plausible that (not known to us) the solutions of (15) which are not included in the set (38) correspond to nonisoenergetic families.

**Remark 1.** We assume that $A \neq \gamma$ and $A \neq -1/\gamma$. For $A = \gamma$, the above ratio $r$ becomes indeterminate. The case $A = -1/\gamma$ corresponds to families of straight lines and has been excluded up to this point. (See remark 5.)

**Remark 2.** It is striking that the arbitrary function $B(\gamma)$ of (38) does not appear explicitly in (39). This means that, for two different $\gamma$ defined by (38) for the same $A(\gamma)$ but different $B(\gamma)$, the functions $V = v(\gamma)$ which satisfy equation (8) are the same. However, just because the two $\gamma$ are different, the corresponding potentials $V = V(x, y)$ defined by (7) will be, as expected, different.

**Remark 3.** Equation (38) introduces two arbitrary functions $A(\gamma)$, $B(\gamma)$ and, as such, is of course less from what one should expect as general solution of the third-order equation (15). Yet it is a very rich source of solutions $\gamma = \gamma(x, y)$ of (15), containing, e.g., all forms of solutions found in section 3.

Indeed, let us consider the constants $\gamma_0, \gamma_1, \gamma_2, \delta_0, \delta_1, \delta_2$ and the arbitrary function $G = G(\gamma)$ and let us choose as

$$A(\gamma) = \frac{\delta_2 G - \gamma_2}{\delta_1 G - \gamma_1}, \qquad B(\gamma) = \frac{-\delta_0 G + \gamma_0}{\delta_1 G - \gamma_1}.$$

Then (38) leads to (34), which, as already mentioned, covers (22) and (23).

**Remark 4.** For a compatible pair $(\gamma(x, y), V(x, y))$ let us call $s$ the common ratio

$$\frac{\gamma_{01}}{\gamma_{10}} = \frac{V_{01}}{V_{10}} = s. \tag{41}$$

Then (36) is written as $s_y = s s_x$, i.e., $\left(\frac{V_{01}}{V_{10}}\right)_y = \frac{V_{01}}{V_{10}}\left(\frac{V_{01}}{V_{10}}\right)_x$ or

$$V_{01}^2 V_{20} - 2V_{10}V_{01}V_{11} + V_{10}^2 V_{02} = 0. \tag{42}$$

In conclusion, if the family $\gamma = \gamma(x, y)$ satisfies (35), the corresponding $V(x, y) = v(\gamma(x, y))$ satisfies (42).

**Remark 5.** Up to this point, we worked with $\Gamma \neq 0$. As seen from equation (5), $\Gamma = 0$ is associated with

$$V_x + \gamma V_y = 0 \tag{43}$$

and with a family of straight lines in the plane (Bozis and Anisiu 2001). As we seek solutions of the form (7), equation (43) gives $(\gamma_x + \gamma\gamma_y)v' = 0$, meaning that, either the potential $v(\gamma)$ must be constant or

$$\gamma_x + \gamma\gamma_y = 0. \tag{44}$$

Condition (44) coincides with (11(ii)) and implies the presence of the families (12). At any rate (44) and $\Gamma = 0$, i.e.,

$$\gamma\gamma_x - \gamma_y = 0 \tag{45}$$

are incompatible.

**Example 1.** For $A = \gamma^2$, $B = 2\gamma$, equation (38) gives

$$\gamma = \frac{1 \pm \sqrt{1 - xy}}{y} \tag{46}$$

and equation (39) becomes

$$r = \frac{-3\gamma^4 + 4\gamma^3 + 2\gamma - 1}{\gamma(\gamma - 1)(\gamma^3 + 1)}. \tag{47}$$

Then, from (17) and (47) we obtain (except for a multiplicative and an additive constant)

$$v(\gamma) = \frac{1 + \gamma^2}{(1 + \gamma^3)^{2/3}} \tag{48}$$

and from (7) and (46)

$$V(x, y) = \frac{xy - y^2 - 2(1 \pm \sqrt{1 - xy})}{(y^3 + (1 \pm \sqrt{1 - xy})^3)^{2/3}}. \tag{49}$$

The compatibility of (46) and (49) as far as equation (5) is concerned can be checked by direct computations.

## 5. Discussion

One basic result of the present study is condition (15). It provides a very rich tank of adequate families $\gamma = \gamma(x, y)$ for which we can solve the (generally not solvable) partial differential equation (5) of the inverse problem of dynamics, i.e., we can find by quadratures potentials of the form $V(x, y) = v(\gamma(x, y))$.

The idea for looking for such potentials emerged from the fact that potentials of this type appeared in certain examples, as $V(x, y) = (x^4 + y^4)/(x - y)^4$ and $\gamma(x, y) = y^2/x^2$ in the paper of Borghero and Bozis (2002). Dealing with the direct problem, Bozis *et al* (2000) found the compatible pair $V(x, y) = -1/x^2$ and $\gamma(x, y) = \pm(k_1 - k_0/x^2)^{1/2}$. In addition, the motivation for selecting this particular form (7) for the unknown potential $V(x, y)$ was also of mathematical nature, i.e., we did so in order to ease the algebra by shifting from the *partial* differential equation (5) to the *ordinary* differential equation (8). Due to the linearity of (5) in $V$, it was expected that (8) would be solvable by quadratures. But, of course, this happens for 'adequate' families $\gamma(x, y)$. Naturally, then our central interest was directed to the study of the differential condition (15).

Seen as a partial differential equation in $\gamma$, equation (15) is nonlinear of the third order whose general solution would be desirable. As the task of finding such a solution appeared impossible to us, we treated equation (15) by proceeding from relatively simple to more complicated forms of solutions. The solutions of the forms (22) and (23) lead to the potentials (21) and (26), respectively. As already mentioned in section 3(c), an analogous result can be established for the slope functions of the more general form (34). Formulae like (21) and (26) are useful per se and, for this reason, we studied these cases separately.

The potentials of the form (21) are integrable. A further study is needed for the integrability of potentials of the form (26).

Another basic result is equation (38) with the two arbitrary functions $A(\gamma)$ and $B(\gamma)$. It stands for the general solution of (35) and gives a very rich set of functions $\gamma = \gamma(x, y)$ which make both sides of (15) zero and for which the pertinent potential $v = v(\gamma)$ can be found by quadratures from

$$\frac{v''(\gamma)}{v'(\gamma)} = r(\gamma) \tag{50}$$

with $r(\gamma)$ given by (39).

In section 4, we indicated how the subset of functions $\gamma$ presented in section 3 can be obtained from (38). The totality of pairs $(\gamma(x, y), V(x, y))$ which we established as solutions to our problem described in section 1 satisfy both equations (35) and (42). These equations are solved to completion.

In view of all the examples presented in this study, we now direct our attention to the fact that all pairs $(\gamma, v(\gamma))$ correspond to families $\gamma$ traced isoenergetically in the presence of the potential $v(\gamma)$. This is actually a general fact for families $\gamma$ which satisfy equation (35) and it was proved in section 4.

Finally we note that, in all three cases of section 3, the ordinary differential equation $\mathrm{d}y/\mathrm{d}x = -1/\gamma$ can be solved by quadratures and the pertinent monoparametric family can be found explicitly in the form $f(x, y) = c$. Yet this is not generally the case for functions $\gamma(x, y)$ given by (38).

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