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Programmed motion in the presence of homogeneity

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In the framework of the inverse problem of dynamics, we face the following question with reference to the motion of one material point: Given a region $T_{\rm orb}$ of the xy plane, described by the inequality $g(x,y) \leq c_0$, are there potentials $V = V(x,y)$ which can produce monoparametric families of orbits $f(x,y) = c$ (also to be found) lying exclusively in the region $T_{\rm orb}$? As the relevant PDEs are nonlinear, an answer to this question (generally affirmative, but not with assurance) can be given by the procedure of the determination of certain constants specifying the pertinent functions. In this paper we ease the mathematics involved by making certain simplifying assumptions referring to the homogeneity of both the function $g(x,y)$ (describing the boundary of $T_{\rm orb}$) and of the slope function $\gamma(x,y) = f_y/f_x$ (representing the required family $f(x,y) = c$). We develop the method to treat the so formulated problem and we show that, even under these restrictive assumptions, an affirmative answer is guaranteed provided that two algebraic equations have in common at least one solution.

1 Introduction

The inverse problem of dynamics – finding the forces which give rise to a family of orbits – has been of interest since Newton discovered the inverse square law for the motion of the planets. In modern days the problem has received special attention after the publication by Szebehely (1974) of the partial differential equation for the potential generating a given family of orbits. Szebehely's equation was aimed to be used for the determination of the potential of the Earth by means of satellite observations. The form of Szebehely's equation allowed Bozis & Ichtiaroglou (1994) to state that the orbits of the family are actually traced only in a region which is limited by the family boundary curves (FBC). Later on, Bozis (1996) and Anisiu & Bozis (2000) considered the following related problem: given a planar region, find the potential and the families of curves described by a particle precisely in the given region (programmed to remain trapped into it).

In this paper we approach the problem of the programmed motion for a general planar region and families of curves with homogeneous slope, which contain as a special case the orbits $\gamma = -x/(4y)$ found by Bozis et al. (1997) for the Hénon-Heiles (1964) potential $V(x,y) = 1/2\,x^2 + 8y + x^2y + 16/3\,y^3$ in the region $(x^2 + 8y^2 + 12y)y \leq 0$.

The system of the second order differential equations for the motion of one material point allows both for real and nonreal orbits to result from a real or complex force field. In particular, a planar orbit traced by one material point in the presence of a two-dimensional force field (conservative or not, in an inertial or in a rotating frame Oxy) may be escaping or trapped in a region of the plane depending on the initial conditions given to the orbit. For conservative force fields, the requirement that, at any point along the orbit, the total energy E of the moving particle must be greater than the potential $V(x,y)$ at this point, determines the well known zero velocity curves (ZVC). Each ZVC, with equation $E - V(x,y) = 0$, is associated with *one* specific orbit and with all possible orbits having *the same energy* E (e.g. Szebehely 1967).

On the other hand, in the framework of the inverse problem, Szebehely's (1974) partial differential equation, as modified later by Bozis (1983), relates, in an inertial frame, slope functions $\gamma(x,y) = f_y/f_x$ of monoparametric families of orbits $f(x,y) = \text{const.}$ to potentials $V(x,y)$ which can produce these families (for adequately chosen initial conditions, of course) and to the energy dependence function $E = E(f(x,y))$. This PDE also allows both for real and complex pairs $(V(x,y), \gamma(x,y))$, accompanied by complex values of the total energy E as well (Contopoulos & Bozis 2000). Naturally, families of complex orbits are of pure mathematical interest, whereas real orbits are actually observed and, so or otherwise, good to have in Physics. Out of these real orbits, those which are trapped in the interior of a certain finite region $T_{\rm orb}$ of the plane are of interest in many physical situations. Our possibility of managing to have orbits of this nature partly answers the more general question of *programmed motion in mechanics* (Galiullin 1984).

To succeed to program motion in the above sense, we are helped by the so-called *family boundary curves* (FBC).

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Those are curves defining regions $B(x,y) \geq 0$ where motion of the particle may take place (as the particle moves on various members of the same family), *with total energy generally varying from member to member* (Bozis & Ichtiaroglou 1994). We know that to each pair (V,γ) of a potential and a family there corresponds a specific FBC. An essential difficulty which arises is due to the fact that infinitely many pairs (V,γ) may lead to the same geometrical entity representing the region $T_{\rm orb}$ which we have preassigned.

In the present study we face this problem aided by some homogeneity assumptions referring both to the form of the slope function $\gamma(x,y)$ and to the form of the equation $g(x,y) \leq c_0$ representing $T_{\rm orb}$. In Sect. 2 we remind the reader of the definition of the boundary function $B(x,y)$ and we put the problem in its general prospect. In Sect. 3 we derive differential relations involving the boundary function $B(x,y)$ and the slope function $\gamma(x,y)$, respectively $B(x,y)$ and the potential $V(x,y)$. We find also a formula giving explicitly the slope function $\gamma(x,y)$ in terms of $B(x,y)$ and $V(x,y)$. In Sect. 4 we describe some aspects of the problem and we focus on what we define as the *basic programmed-motion problem* which then we study in detail in Sect. 5 with homogeneous functions. In Sect. 6 we offer an example for the basic programmed motion. In Sect. 7 we make some general concluding remarks.

2 The boundary function

Monoparametric families of orbits $f(x,y) = c$, which are produced by a given potential $V(x,y)$ and which have the 'slope function' $\gamma(x,y) = f_y/f_x$, satisfy the second order nonlinear PDE (Bozis 1995)

$$\gamma^2\gamma_{xx} - 2\gamma\gamma_{xy} + \gamma_{yy} = h, \tag{1}$$

where

$$h = \frac{\gamma_y - \gamma\gamma_x}{V_x + \gamma V_y} \times \left(\gamma_x V_x - (2\gamma\gamma_x - 3\gamma_y)V_y - \gamma(V_{xx} - V_{yy}) - (\gamma^2 - 1)V_{xy}\right). \tag{2}$$

Having in mind a pair $(V(x,y), \gamma(x,y))$ satisfying the above PDE (1), let us denote by $B(x,y)$ the 'boundary function'

$$B(x,y) = \frac{V_x + \gamma V_y}{\gamma_y - \gamma\gamma_x}. \tag{3}$$

Families of straight lines for which $\gamma\gamma_x - \gamma_y = 0$ and $V_x + \gamma V_y = 0$ (Bozis & Anisiu 2001) are excluded from our study, as both h and B are indeterminate.

As we know (Bozis & Ichtiaroglou 1994), in general, the inequality

$$B(x,y) \geq 0 \tag{4}$$

determines the region $T_{\rm orb}$ of the xy plane where the potential $V(x,y)$ creates *real* orbits or *real parts* of the orbits belonging to the family with slope function $\gamma(x,y)$. On these grounds, we shall refer to $B(x,y)$ as the 'boundary function' for the compatible pair $(V(x,y), \gamma(x,y))$. Apparently, the pair $(-V(x,y), \gamma(x,y))$ is compatible outside the region $T_{\rm orb}$.

Conversely, if we have in mind only a specific region $T_{\rm orb}$ of the xy plane which *we want to make the exclusive allowed region* for certain unknown families created by an unknown potential, the hidden function $B(x,y)$ is also not known to us. In fact, the boundary function $B(x,y)$ is not known even if *one* of the functions $\gamma(x,y)$ or $V(x,y)$ is not given.

We restrict ourselves to regions which are described by one inequality, say

$$b(x,y) \geq 0, \tag{5}$$

and impose the condition that the function $B(x,y)$ corresponding to a pair $(V(x,y),\ \gamma(x,y))$ defines the same region (4) as the inequality (5) does. We interpret this by stating that there must exist a *nonvanishing* function $\Theta(x,y)$, in a region T_0 broader from the region $T_{\rm orb}$, such that

$$B(x,y) = b(x,y)\Theta(x,y) \tag{6}$$

with

$$\Theta(x,y) \geq 0, (x,y) \in T_0, \tilde{\Theta}(x,y) \neq \infty, \tag{7}$$

where $\tilde{\Theta}(x,y)$ denotes the (one-variable) function $\Theta(x,y)$ evaluated at the points of the curve $b(x,y) = 0$.

For the special case of the families $f(x,y) = y - H(x)$, Anisiu & Bozis (2000) solved the problem of programmed motion by taking $B = b$, i.e. $\Theta = 1$. The generalizing step made in the present study refers to this multiplier $\Theta(x,y)$ which is now allowed to be *any function* satisfying the conditions (7).

3 PDEs relating the boundary functions to families of orbits and to potentials

We now obtain a PDE (Eq. (11) and its counterpart Eq. (13) below) relating slope functions $\gamma(x,y)$ and boundary functions $B(x,y)$ as follows: Solving Eq. (3) for $\gamma\gamma_x - \gamma_y$ and taking derivatives in x and y we obtain respectively

$$\gamma\gamma_{xx} - \gamma_{xy} = h_1, \quad \gamma\gamma_{xy} - \gamma_{yy} = h_2, \tag{8}$$

where h_1 and h_2 are functions of γ and B and first order partial derivatives of them, and also of first and second order derivatives of V. We now see that the algebraic system of the three Eqs. (1) and (8) in γ_{xx}, γ_{xy}, γ_{yy} is indeterminate and this implies that $h - \gamma h_1 + h_2 = 0$ or, after some algebra,

$$\gamma = \frac{B_y + 2V_y}{B_x}. \tag{9}$$

We solve Eqs. (3) and (9) for V_x, V_y and we find

$$V_x = -B(\gamma\gamma_x - \gamma_y) + \tfrac{1}{2}\gamma(B_y - \gamma B_x), \quad V_y = -\tfrac{1}{2}(B_y - \gamma B_x). \tag{10}$$

The above formulae (10) serve to determine the potential, when a compatible pair $(B(x,y),\ \gamma(x,y))$ is given. This compatibility is ensured by the requirement that $V_{xy} = V_{yx}$ and implies that the PDE

$$-B_{xx} + k^* B_{xy} + B_{yy} = \lambda^* B_x + \mu^* B_y + \nu^* B, \tag{11}$$

where

$$\begin{aligned} &k^* = \tfrac{1-\gamma^2}{\gamma}, \ \ \lambda^* = \tfrac{\gamma_x + 2\gamma\gamma_y}{\gamma}, \\ &\mu^* = \tfrac{2\gamma\gamma_x - 3\gamma_y}{\gamma}, \\ &\nu^* = \tfrac{2(\gamma_x\gamma_y - \gamma_{yy} + \gamma\gamma_{xy})}{\gamma} \end{aligned} \tag{12}$$

is satisfied. Equation (11) can also be derived directly from Szebehely's equation (Anisiu 2003).

An alternative form of formula (11) is

$$\begin{aligned} &2B(\gamma\gamma_{xy} - \gamma_{yy} + \gamma_x\gamma_y) + (B_x + 2\gamma B_y)\gamma_x \\ &\quad + (2\gamma B_x - 3B_y)\gamma_y + B_{xy}\gamma^2 \\ &\quad + (B_{xx} - B_{yy})\gamma - B_{xy} = 0. \end{aligned} \tag{13}$$

This is better suited for finding γ when $B(x, y)$ is given.

Inserting in Eq. (1) the function $\gamma(x, y)$, as given by (9), we find the PDE

$$\begin{aligned} &B\left[(B_y - 2V_y)^2 B_{xx} - 2B_x(B_y + 2V_y)B_{xy}\right. \\ &\quad + B_x^2 B_{yy} + 2V_{yy}B_x^2 - 2V_{xy}B_xB_y \\ &\quad \left.- 4V_yV_{xy}B_x\right] = B_x^2(V_xB_x + V_yB_y + 2V_y^2), \end{aligned} \tag{14}$$

which relates potentials V and boundary functions B, and which is nonlinear in both variables V and B. Equation (14) can also be derived from the corresponding equation given by Bozis (1996) for nonconservative fields.

3.1 Remarks

1. If we insert (10) into (1), we obtain again the same PDE (11), relating families $\gamma(x, y)$ to boundary functions $B(x, y)$.
2. If (11) is satisfied by the pair (γ, B) which is associated with the potential V, it is also satisfied by the pair (γ, k_0B), now associated with the potential k_0V, where k_0 is a constant. This remark also applies to Eq. (14).
3. As seen from (11), for a certain family γ there may exist, in general, as many boundary functions B as two arbitrary functions allow. Each of these B's specifies which members of γ are lying inside which region. Each B, of course, is associated with one (except for an additive constant) of the potentials given by (10) which can generate the given family.
4. If γ is homogeneous of degree zero, then so is k^*, whereas λ^*, μ^* are of degree -1 and ν^* is of degree -2. If, in addition to that, B is homogeneous of degree n, all terms in (11) are of degree $n - 2$.
   Moreover, due to the linearity of (11), if the function $B(x, y)$ is weighted homogeneous of degrees e.g. n_1 and n_2 (i.e. B is the sum of two homogeneous expressions of degrees n_1 and n_2), then the entire Eq. (11) will lead to a weighted homogeneous expression of degrees $n_1 - 2$ and $n_2 - 2$. We shall make use of this remark in Sect. 5.

3.2 Example

Consider the family of concentric circles $\gamma = y/x$ produced by all potentials

$$V(r, \theta) = g(r) + \frac{1}{r^2}h(\theta), \tag{15}$$

where r, θ are polar coordinates and $g(r)$, $h(\theta)$ arbitrary functions (Broucke & Lass 1977). The corresponding boundary function is found from (3):

$$B(r, \theta) = \frac{\cos^2\theta}{r^2}(r^3\frac{dg}{dr} - 2h). \tag{16}$$

All formulae (9) to (14) can be verified for this particular example. The presence of the arbitrary functions $g(r)$ and $h(\theta)$ is helpful in conceiving the meaning of the formulae and also of the above Remark 3.

4 Some aspects of the programmed-motion problem

a The usual situation a physicist is confronted with is to have to deal with a *specific potential (direct problem)*. In this case one may think of the following two aspects of programmed motion:

- **a1** One may ask to find families of orbits inside (or outside) a preassigned region of the xy plane described by the inequality (5). Clearly, such a requirement may not admit of an affirmative answer. Indeed, there is an 'immense' but 'specific' set of compatible pairs $(V(x, y), \gamma(x, y))$ corresponding to the given V. According to (3), each of these pairs is accompanied by a certain function B which, once V and b are given, cannot generally be forced to be of the prescribed form (6)–(7), basically because of the restriction (7).
- **a2** As seen from (14), for the given potential, there may exist, in principle, infinitely many functions B (in fact, as many as two arbitrary functions allow) and, consequently, infinitely many allowed regions. To each compatible pair $(B(x, y),\ V(x, y))$ there corresponds one monoparametric family $\gamma(x, y)$, given by (9). Thus, e.g. to a certain potential of the form (15) (i.e. for a specific selection of the functions $g(r)$ and $h(\vartheta)$) there correspond infinitely many functions $B(x, y)$. Only one of these is given by (16) and this corresponds to the family of circles produced by (15).

b Let us now disregard the assumption that the potential is known and suppose that only the region is given in advance by the unique inequality (5). We shall refer to this as the *basic programmed-motion problem*. The question is: What families can be created in the given region and which are the potentials generating these families?
To answer this question we introduce the function $B(x, y)$, as given by (6), into the Eq. (11) and we obtain a linear PDE with coefficients which are functions

of the known function $b(x,y)$ and of the unknown function $\gamma(x,y)$. This equation reads

$$b\left(-\Theta_{xx}+K\Theta_{xy}+\Theta_{yy}\right) - L\Theta_x - M\Theta_y - N\Theta = 0, \tag{17}$$

where

$$K=k^*,\quad L=\lambda^* b+2b_x-k^*b_y,$$
$$M=b\mu^*-k^*b_x-2b_y, \tag{18}$$
$$N=\nu^*b+\lambda^*b_x+\mu^*b_y+b_{xx}-k^*b_{xy}-b_{yy}$$

and where $k^*,\lambda^*,\mu^*,\nu^*$, given by (12), depend merely on γ and its derivatives.

At first sight, it seems as if, except for b, we can also give in advance the family γ and ask for solutions $\Theta(x,y)$ of the linear PDE (17) with the provision that the inequalities (7) are also satisfied. However, this last requirement implies that such solutions Θ of (17) may not exist *for any preassigned* γ. Indeed, the fact that b is fixed a priori reduces the set of boundary functions $B(x,y)$ (as Eq. 6 suggests) and, according to (11), does not allow for any functions γ to belong to this set.

5 Basic programmed motion with homogeneous functions

In this section we outline the procedure to be followed in order to face the basic programmed-motion problem, as we defined it in Sect. 4 (**b**). In order to ease the mathematics, we put some additional assumptions regarding the homogeneity of the functions involved. Specifically we assume the following:

(i) The allowed region is given in the form

$$g(x,y)\le c_0, \tag{19}$$

where g is homogeneous of degree $m\neq 0$, and c_0 is a nonzero constant. Comparing (19) to (5) we take

$$b=c_0-x^m b_0(z),\quad z=\frac{y}{x}, \tag{20}$$

where $b_0\neq 0$.

(ii) The required slope functions γ are homogeneous of degree zero. i.e.

$$\gamma=\gamma(z). \tag{21}$$

(iii) The functions Θ to be determined are also homogeneous of degree k, i.e.

$$\Theta(x,y)=x^k\Theta_0(z), \tag{22}$$

where $\Theta_0\neq 0$.

Inserting (20), (21) and (22) into (17), we rewrite (as implied by Remark 4 of Sect. 3) this equation so that its left hand side is a weighted homogeneous expression, i. e.

$$R_1x^k+R_2x^{m+k}=0, \tag{23}$$

where R_1 and R_2 are functions of z.

Both R_1 and R_2 must vanish identically, and this leads to the two equations (the dots representing differentiation with respect to z)

$$2\Theta_0(z\gamma+1)\ddot\gamma+2\Theta_0 z\dot\gamma^2+k_1\dot\gamma+k_0=0, \tag{24}$$

$$2b_0\Theta_0(z\gamma+1)\ddot\gamma+2b_0\Theta_0 z\dot\gamma^2+m_1\dot\gamma+m_0=0, \tag{25}$$

where (arranged in $\Theta_0,\dot\Theta_0,\ddot\Theta_0$)

$$k_1=k_{10}\Theta_0+k_{1d}\dot\Theta_0,\ k_0=k_{00}\Theta_0+k_{0d}\dot\Theta_0+k_{0dd}\ddot\Theta_0 \tag{26}$$

with

$$k_{10}=2(1-k)\gamma+kz,\ k_{1d}=4z\gamma-z^2+3,$$
$$k_{00}=k(1-k)\gamma,\ k_{0d}=(1-k)(\gamma^2-2z\gamma-1), \tag{27}$$
$$k_{0dd}=z\gamma^2+(1-z^2)\gamma-z,$$

and

$$m_1=m_{10}\Theta_0+m_{1d}\dot\Theta_0,$$
$$m_0=m_{00}\Theta_0+m_{0d}\dot\Theta_0+m_{0dd}\ddot\Theta_{00} \tag{28}$$

with (arranged in $b_0,\dot b_0,\ddot b_0$)

$$m_{10}=-b_0(2(k+m-1)\gamma-(k+m)z) + \dot b_0(4z\gamma-z^2+3),\quad m_{1d}=b_0(4z\gamma-z^2+3), \tag{29}$$

$$\begin{aligned} m_{00} = {} & -b_0\,(k+m)\,(k+m-1)\,\gamma \\ & -\dot b_0(k+m-1)(\gamma^2-2z\gamma-1) \\ & +\ddot b_0\,(\gamma-z)\,(z\gamma+1)\,, \\ m_{0d} = {} & -b_0(k+m-1)(\gamma^2-2z\gamma-1) \\ & +2\dot b_0(\gamma-z)(z\gamma+1), \\ m_{0dd} = {} & \ b_0(\gamma-z)(z\gamma+1). \end{aligned} \tag{30}$$

Our hypotheses ($b_0\neq 0$, $\Theta_0\neq 0$ and straight lines excluded) assure that the coefficient of $\ddot\gamma$ in (24) and (25) is different from zero.

One can see easily that one of the two Eqs. (24) and (25), say (25), may be replaced by the simpler equation

$$(m_1-b_0k_1)\dot\gamma+m_0-b_0k_0=0. \tag{31}$$

If we set

$$\dot\Theta_0=w\Theta_0,\quad \dot b_0=rb_0 \tag{32}$$

we also have

$$\ddot\Theta_0=(\dot w+w^2)\Theta_0,\quad \ddot b_0=(\dot r+r^2)b_0. \tag{33}$$

From (31) we then obtain (for $m_1-b_0k_1\neq 0$)

$$\dot\gamma=\frac{\Gamma_2\gamma^2+\Gamma_1\gamma+\Gamma_0}{\Delta_1\gamma+\Delta_0}, \tag{34}$$

where

$$\Gamma_2=\Gamma_{00}+\Gamma_{01}w,\ \Gamma_1=\Gamma_{10}+\Gamma_{11}w,\ \Gamma_0=-\Gamma_2, \tag{35}$$

$$\Gamma_{00}=(1-k-m)r+z(\dot r+r^2),\ \Gamma_{01}=2zr-m$$
$$\Gamma_{10}=m(1-2k-m)-2(1-k-m)zr + (1-z^2)(\dot r+r^2) \tag{36}$$
$$\Gamma_{11}=2(r+mz-rz^2),$$

and

$$\Delta_1=2(m-2rz),\quad \Delta_0=rz^2-mz-3r. \tag{37}$$

Also, the coefficients in the two Eqs. (24) and (25) may be freed from Θ_0 and be expressed in terms of the ratio $w=\dot\Theta_0/\Theta_0$, given by (32). Besides that (and although b_0 is known) the calculations suggest that, instead of b_0, it is simpler to use $r = \dot{b}_0/b_0$, also given by (32). We take these remarks into account and, as we are interested only in formula (24), we rewrite it here as

$$2(1+\gamma z)\ddot{\gamma} + 2z\dot{\gamma}^2 + K_1\dot{\gamma} + K_0 = 0, \tag{38}$$

with the coefficients arranged in powers of γ as follows

$$K_1 = K_{11}\gamma + K_{10}, \quad K_0 = K_{02}\gamma^2 + K_{01}\gamma + K_{00}, \tag{39}$$

where

$$K_{11} = 4zw + 2(1-k), \quad K_{10} = -(z^2-3)w + kz \tag{40}$$

and

$$\begin{aligned} K_{02} &= (1-k)w + z(\dot{w} + w^2), \quad K_{00} = -K_{02} \\ K_{01} &= k(1-k) - 2z\,(1-k)\,w \\ &\quad + (1-z^2)(\dot{w} + w^2). \end{aligned} \tag{41}$$

So, now we have to deal with the two Eqs. (34) and (38), where the coefficients K_1, K_0 in (38) are given by Eqs. (39)–(41) and the coefficients in the fraction (34) are given by Eqs. (35)–(37).

We consider m, $r = \dot{b}_0/b_0$, c_0 (i.e. the boundary function b given by Eq. 20) as known and we try to find appropriate γ's satisfying the Eqs. (34) and (38). To this end we prepare $\ddot{\gamma}$ from (34) and insert into (38). In so doing, we obtain the quintic in γ algebraic equation

$$\alpha_5\gamma^5 + \alpha_4\gamma^4 + \alpha_3\gamma^3 + \alpha_2\gamma^2 + \alpha_1\gamma + \alpha_0 = 0, \tag{42}$$

where the coefficients $\alpha_5, \alpha_4, ..., \alpha_0$ are functions of z and w and its derivative of the first order.

We now differentiate (42) in z and we obtain $\dot{\gamma}$ which we equate to $\dot{\gamma}$ given by (34). In so doing we obtain the sixth order algebraic equation

$$\beta_6\gamma^6 + \beta_5\gamma^5 + \beta_4\gamma^4 + \beta_3\gamma^3 + \beta_2\gamma^2 + \beta_1\gamma + \beta_0 = 0 \tag{43}$$

with the coefficients $\beta_6, \beta_5, ..., \beta_0$ functions of z and of w, $\dot{w}, \ddot{w}$. We are interested in the common roots of the Eqs. (42) and (43) and this leads us to the eleventh order Sylvester determinant which is an ODE in w of the second order. A solution w of this equation (for a suitable k) allows us to solve the problem to completion.

A special case arises from the fact that the above reasoning was made under the hypothesis $m_1 - b_0k_1 \neq 0$, imposed from formula (34) on. Therefore we have to analyze also the case when $m_1 - b_0k_1 = 0$, i. e.

$$\Delta_1\gamma + \Delta_0 = 0. \tag{44}$$

We express γ from Eq. (44) and substitute it in $m_0 - b_0k_0 = 0$, i. e. $\Gamma_2\gamma^2 + \Gamma_1\gamma + \Gamma_0 = 0$. We obtain then the value of w, and substitute γ and w in Eq. (38); if there is any k for which the expression becomes identically null, we have a solution for the problem.

6 Example

As an example of the basic problem of programmed motion let us try to find families of orbits and the corresponding potentials creating them in the region

$$\frac{3}{8}x^2 + y^2 + y \leq 0, \tag{45}$$

which represents the interior of an ellipse. Apparently, the region (45) lies in the negative y's, in fact it is tangent to the x-axis at the origin, its center is at the point $(0, -1/2)$ and its semi-axes (parallel to the coordinate axes x and y) are $\sqrt{2/3}$ and $1/2$, respectively.

At first we write (45) in the form (19) as

$$g(x,y) = -\left(\frac{3x^2}{8y} + y\right) = -x\left(\frac{3}{8z} + z\right), \quad c_0 = 1. \tag{46}$$

Therefore

$$m = 1, b_0(z) = -\left(\frac{3}{8z} + z\right), \quad c_0 = 1, \tag{47}$$

and, in view of (20),

$$b(x,y) = \frac{3x^2}{8y} + y + 1\,. \tag{48}$$

Aided by a Maple program, we formed, for the case at hand, the two Eqs. (42) and (43) and we checked that, for $k = 3$, their Sylvester determinant vanishes when $w = 1/z$. So, in view of the Eq. (32) and the inequality (7) we obtain

$$\Theta_0(z) = -z, \tag{49}$$

meaning that, for the case at hand and according to (22), it is

$$\Theta(x,y) = -x^2y\,. \tag{50}$$

The function Θ satisfies the condition (7) with $T_0 = \{(x,y) : y \leq 0\}$. According to (6), (48) and (50), we find

$$B(x,y) = -\left(\frac{3}{8}x^4 + x^2y^2 + x^2y\right). \tag{51}$$

With the function $w(z) = 1/z$, we write the Eqs. (42) and (43) and we look for a common solution of the form $\gamma = \gamma(z)$, $z = y/x$, which is

$$\gamma = -\frac{2}{3}z. \tag{52}$$

Finally, with the aid of (51) and (52), we find from (10) the potential

$$\begin{aligned} V(x,y) &= -\tfrac{1}{48}(3x^4 + 36x^2y^2 + 8y^4) \\ &\quad - \tfrac{1}{18}y(9x^2 + 4y^2). \end{aligned} \tag{53}$$

7 General comments

In Sect. 4 we discussed some aspects of the question of the programmed planar motion. We were led to focus attention on what appeared to us as the most meaningful version of the problem which we called *basic programmed-motion problem*. We treated this problem, having at our disposal (i) the Eq. (9), offering explicitly the slope function γ in terms of B and V and (ii) the two nonlinear PDEs (13) and (14), relating γ, B, and V, B, respectively. Due to the complexity of these tools we were led to study the problem in some detail under certain additional assumptions regarding the homogeneity of the functions involved. Specifically we considered the case of an allowed region of the form (19), combined with a family of homogeneous orbits (21). In addition to that and in order to facilitate the calculations, we restricted ourselves to consider multiplying functions $\Theta(x, y)$ which are also homogeneous.

The example in Sect. 6 has indicated that an affirmative answer to the programmed-motion problem can be obtained in spite of all the above restrictive assumptions. Yet, this is not generally what one expects. For another preassigned region of the form (19) the two algebraic equations (analogous to Eqs. 42 and 43) could be such as not to provide a common solution.

So, then, what is it that we generally expect, if we free ourselves from some or all the homogeneity assumptions?

We argued in Sect. 4 (**b**) that Eq. (17) would generally have no 'adequate' solution $\Theta(x, y)$ for any $\gamma(x, y)$. In fact, the finding of such pairs (V, γ) is our objective. (Once γ and Θ are found, we also have $B = b\Theta$.) Of course, in view of (14), to any potential $V(x, y)$ producing a family $\gamma(x, y)$ which we may suspect as possibly lying in the given region $b(x, y) \geq 0$, there correspond infinitely many but definite functions $B(x, y)$. However, none of these functions is obliged to provide (as the inequality (3) suggests) the same information with $b(x, y) \geq 0$.

On the other hand, the PDE (13) is more promising in offering an affirmative answer. Given the function $b(x, y)$, *we can select* appropriate functions $\Theta(x, y)$ (or, even better, select forms of functions $\Theta(x, y)$, introduced by certain constants) and consider $B = b\Theta$. Then, introduce this B in (13) and try to obtain solutions of (13) for $\gamma(x, y)$ *again of a certain form* by determining the constants.

Thus, e.g. for $b(x, y) = -x^2 + 3x + y$, let us try $\Theta(x, y) = x^2 + 2\Theta_1 xy + \Theta_2 y^2$ (which is a non-negative definite quadratic expression for $\Theta_1^2 \leq \Theta_2$). With $B(x, y) = b\Theta$, let us now search for solutions γ of the form $\gamma(x, y) = \gamma_0 + \gamma_1 y/x$, with $\gamma_1 \neq 0$ (no straight lines). The formula (13) leads to a polynomial in x, y including fifth and fourth degree terms which becomes identically equal to zero if and only if: $\Theta_1 = \Theta_2 = 0$ and $\gamma_0 = 1$, $\gamma_1 = -2$.

Therefore the family with $\gamma = 1 - 2y/x$, whose members are lying 'in one side' of the parabola $y \geq x^2 - 3x$ is $f(x, y) = 4y^3 - x^3 - 3x^2 y$ and, as can be found from (10), is produced by the potential $V(x, y) = 3x^4 - 6x^3 y + 6x^2 y^2 - 10x^3 + 12x^2 y - 12xy^2 - 2y^3$.

Notice that the function $b = -x^2 + 3x + y$, used in the above example, can be put in the form (20). Thus, the same result can be found by the straightforward method developed in Sect. 5.

We remark that, for slope functions of the form $\gamma = \gamma(y/x)$, the allowed region was found to be of the form (20) for potentials of the Hénon-Heiles type by Bozis et al. (1997), and for quartic perturbations of a harmonic oscillator by Anisiu (2007).

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