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Two-dimension potentials which generate spatial families of orbits

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We consider the following case of the 3D inverse problem of dynamics: Given a spatial two-parametric family of curves $f(x, y, z) = c_1$, $g(x, y, z) = c_2$, find possibly existing two-dimension potentials under whose action the curves of the family are trajectories for a unit mass particle. First we establish the conditions which must be fulfilled by the family so that potentials of the form $w(y, z)$ give rise to the curves of the family, and we present some applications. Then we examine briefly the existence of potentials depending on (x, z), respectively (x, y), which are compatible with the given family.

1 Introduction

The three-dimensional version of the inverse problem of dynamics has already a history of more than 25 years. Following Szebehely's (1974) presentation of the problem in two dimensions, Érdi (1982) was the first to address the 3D problem by considering a *monoparametric* family of spatial orbits. Seen from this viewpoint, the treatment of the problem is essentially similar to the study of the motion of a particle describing a monoparametric family of (spatial) orbits on a *fixed* surface (Mertens 1981). There followed studies which put the problem in a more accurate perspective: A *two-parametric* family of spatial curves is given and the potential giving rise to these curves is required (Bozis 1983; Váradi & Érdi 1983; Bozis & Nakhla 1986; Shorokhov 1988; Puel 1992). At almost the same period, the problem was generalized to account for holonomic systems with n degrees of freedom (Melis & Borghero 1986; Borghero & Melis 1990).

Assuming that a two-parametric set of orbits

$$f(x, y, z) = c_1, \quad g(x, y, z) = c_2, \tag{1}$$

in the $Oxyz$ space can be traced by a material point in the presence of an unknown potential $V = V(x, y, z)$ with energy dependence function $E = E(f, g)$ which is considered to be known, one aims to finding the potential. From its mathematical viewpoint, the problem consists in solving a linear system of two PDEs in the unique unknown function $V = V(x, y, z)$. It was soon made clear that this problem has no solution unless the 'given' energy function $E = E(c_1, c_2)$ satisfies certain conditions depending on the given functions $f(x, y, z)$ and $g(x, y, z)$ (Bozis & Nakhla 1986; Shorokhov 1988). In fact, if some specific conditions are satisfied, the problem admits of a unique solution $V = V(x, y, z)$, up to an additive and a multiplicative constant.

In a manner analogous to that used in the 2D case, Bozis & Kotoulas (2005) and also Anisiu (2005) eliminated the energy and produced a system of two linear in $V(x, y, z)$ PDEs, one of the first and one of the second order (Eqs. (5) and (6) below). If this system is compatible and can be solved for a given family of orbits (1), then the corresponding energy with which each member of the family is being traced can be found (Eq. (7) below).

In this paper we shall deal with these two energy-free PDEs, taking into account that, in spite of their linearity, the above system cannot be treated in a straightforward manner. Further assumptions, regarding either the form of the given orbits or the form of the required potential or both may simplify the problem. This practice was followed by Bozis & Kotoulas (2005) and by Kotoulas & Bozis (2006) and will be followed in the present paper also: We look for *two-dimension potentials* which can possibly give rise to 3D families of orbits (1), given in advance. We study in detail the case of potentials of the form $V = w(y, z)$.

In Sect. 2 we give the general energy-free PDEs of the inverse problem. In Sect. 3 we write down these equations for the case at hand and we pursue their solution by establishing the necessary and sufficient conditions on the 'given' family so that such solutions $w = w(y, z)$ do exist. If *all types of 2D potentials* creating a given family are required, the cases $w = w(x, z)$ and $w = w(x, y)$ must be studied separately. This can be done, of course, in a manner quite similar to that followed here. In Sect. 4 we examine some special cases exempted at a first step, so that the analysis could continue. In Sect. 5 we apply the theory to two rather broad sets of spatial curves and enrich the library of the ex-

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isting examples. Finally, in Sect. 6 we discuss shortly the case $w = w(x, z)$ of 2D potentials and we comment on the case $w = w(x, y)$.

2 The energy-free equations of the spatial inverse problem of dynamics

In a three-dimensional frame we deal with two-parametric families of orbits written in the form (1), which are in an one-to-one correspondence with a pair (α, β) of 'slope functions' defined by

$$\alpha = \frac{f_z g_x - f_x g_z}{f_y g_z - f_z g_y}, \quad \beta = \frac{f_x g_y - f_y g_x}{f_y g_z - f_z g_y}. \tag{2}$$

The indices denote partial derivatives. Equations (1) then constitute the general solution of the ODE system

$$\frac{\mathrm{d}\, y}{\mathrm{d}\, x} = \alpha\left(x, y, z\right), \quad \frac{\mathrm{d}\, z}{\mathrm{d}\, x} = \beta\left(x, y, z\right). \tag{3}$$

There exist in this case two energy-free PDEs (one of the first and one of the second order) relating families (α, β) and potentials $V(x, y, z)$ (Bozis & Kotoulas 2005; Anisiu 2005).

Let us assume that $\alpha_0 \neq 0$ and adopt the notation

$$\begin{aligned} &\bar{\epsilon} = (1, \alpha, \beta), \quad \alpha_0 = \bar{\epsilon}\, \mathrm{grad}\, \alpha, \quad \beta_0 = \bar{\epsilon}\, \mathrm{grad}\, \beta, \\ &\Theta = 1 + \alpha^2 + \beta^2, \quad n = \frac{\Theta}{\alpha_0}, \quad n_0 = \bar{\epsilon}\, \mathrm{grad}\, n. \end{aligned} \tag{4}$$

The two equations satisfied by the potential in the 3D inverse problem are

$$\left(\alpha\beta_0 - \alpha_0\beta\right) V_x - \beta_0 V_y + \alpha_0 V_z = 0, \tag{5}$$

$$\begin{aligned} &\alpha V_{xx} + \left(\alpha^2 - 1\right) V_{xy} + \alpha\beta V_{xz} - \alpha V_{yy} - \beta V_{yz} + \\ &\frac{1}{n}\left(\left(2 + \alpha n_0 + \alpha_0 n\right) V_x + \left(2\alpha - n_0\right) V_y + 2\beta V_z\right) = 0. \end{aligned} \tag{6}$$

For any compatible pair of potential $V(x, y, z)$ and orbit (α, β) the energy is given by

$$E(f, g) = V + \Theta\left(\alpha V_x - V_y\right) / 2\alpha_0, \tag{7}$$

and real motion is allowed in the region

$$\frac{\alpha V_x - V_y}{\alpha_0} \geq 0. \tag{8}$$

The potentials producing families of straight lines, for which both α_0 and β_0 are identically zero, must satisfy (Bozis & Kotoulas 2004) the equations

$$\alpha V_x - V_y = 0, \quad \beta V_x - V_z = 0. \tag{9}$$

If $\alpha_0 = 0$ but $\beta_0 \neq 0$ equation (6) is replaced by a second-order differential equation presented in Anisiu (2005) and Kotoulas & Bozis (2006).

3 Potentials $V = w(y, z)$ which generate spatial families of orbits

We shall consider exclusively potentials of the form

$$V = w(y, z). \tag{10}$$

The material point moving under the action of such a potential will have a uniform motion in x, i. e. $\dot{x}(t) = \mathrm{const.}$ As a consequence, the orbits cannot be closed. For V given in (10) the system of the two PDEs (5) and (6) becomes

$$w_z = G w_y \tag{11}$$

and

$$\Theta(\alpha + \beta G) w_{yy} + \Psi w_y = 0, \tag{12}$$

where

$$G = \frac{\beta_0}{\alpha_0} \tag{13}$$

and

$$\Psi = \beta\Theta G_y + \alpha_0\left(n_0 - 2(\alpha + \beta G)\right). \tag{14}$$

Equation (11) comes directly from (5) but, to obtain (12) from (6), we used (11) and, with its help, we expressed w_{yz} and w_{zz} in terms of w_y and w_{yy}.

It is seen from Eq. (11) that the function G must be independent of x, i. e.

$$G_x = 0. \tag{15}$$

We assume that $\alpha + \beta G \neq 0$ and we put

$$H = \frac{\Psi}{\Theta(\alpha + \beta G)}. \tag{16}$$

For the PDE (12), now written as

$$w_{yy} + H w_y = 0, \tag{17}$$

to have a solution of the form (10) it must be

$$H_x = 0. \tag{18}$$

Solving Eq. (12) for w_y we obtain

$$w_y = D(z) \exp(-\int^y H(u, z)\, \mathrm{d}\, u) \tag{19}$$

(with $D(z)$ momentarily arbitrary function of z), and from Eq. (11) we obtain w_z. The compatibility condition ($w_{yz} = w_{zy}$) for w_y and w_z, as these are given by Eqs. (19) and (11) respectively, is

$$D'(z) - D(z) J = 0, \tag{20}$$

where the function J is given by

$$J = G_y(y, z) - G(y, z) H(y, z) + \int^y H_z(u, z)\, \mathrm{d}\, u. \tag{21}$$

The function J must depend merely on z, i. e. it must be

$$J_y = 0. \tag{22}$$

This last condition (22) can also be written as

$$G_{yy} - G_y H - G H_y + H_z = 0. \tag{23}$$

For a given family (1) the 'integration function' $D(z)$ is specified from (20). It is

$$D(z) = d \exp(\int J(z)\,\mathrm{d}\,z), \tag{24}$$

where $J(z)$ is given by (21) and d is a constant. From Eqs. (7) and (8) we conclude that real motion is allowed in the region defined by the inequality

$$\frac{w_y}{\alpha_0} \leq 0, \tag{25}$$

and the particle moves with total energy

$$E = w(y,z) - \frac{\Theta w_y}{2\alpha_0}. \tag{26}$$

We summarize by the following

Proposition 1. *For $\alpha_0 \neq 0$, $\alpha + \beta G \neq 0$ and for any family (α, β) satisfying the conditions (15), (18) and (22) (equivalently 23), there exists a two-dimension compatible potential $w = w(y,z)$. The potential is given by the (compatible) Eqs. (19) and (11) with $D(z)$ given by (24). The family is lying in the region (25) and is traced with total energy given by Eq. (26).*

4 Special cases

a. $\alpha_0 = 0,\ \beta_0 = 0$
The case is trivial. Only if the given orbits are families of straight lines we are led to $\alpha_0 = \beta_0 = 0$ and the corresponding potential obtained from (9) is $w(y,z) = \mathrm{const}$.
b. $\alpha_0 = 0,\ \beta_0 \neq 0$
The Eq. (5) is still valid but Eq. (6) must be replaced by another second order PDE, as mentioned at the end of Sect. 2. Here we write down directly the pertinent system of the Eqs. (5) and (6) for 2D potentials of the form $w = w(y,z)$. This system reads

$$w_y = 0,\quad w_{zz} = \frac{2\beta - \widetilde{n}_0}{\widetilde{n}\beta} w_z, \tag{27}$$

where

$$\widetilde{n} = \frac{\Theta}{\beta_0},\quad \widetilde{n}_0 = \widetilde{n}_x + \alpha \widetilde{n}_y + \beta \widetilde{n}_z. \tag{28}$$

The system (27) has as solution a 1D potential $w = w(z)$, provided that the ratio $(2\beta - \widetilde{n}_0) / (\widetilde{n}\beta)$ is independent of x and y.
c. $\alpha_0 \neq 0,\ \alpha + \beta G = 0,\ \Psi \neq 0$
The case is trivial. From (12) we obtain $w_y = 0$, and from (11) $w_z = 0$ i. e. $w = \mathrm{const}$.

d. $\alpha_0 \neq 0,\ \alpha + \beta G = 0,\ \Psi = 0$
Then Eq. (12) is satisfied identically. The potential is found from (11), provided that the condition (15) holds, and it will not be uniquely determined.

Although at first sight questionable, the equations

$$G_x = 0,\quad \alpha + \beta G = 0,\quad n\beta G_y + n_0 = 0 \tag{29}$$

may hold simultaneously; for example, for the family $f = y \sin x + z \cos x = c_1,\ g = -z \sin x + y \cos x = c_2$ with the slope functions $\alpha = z$ and $\beta = -y$, it is $\alpha\alpha_0 + \beta\beta_0 = 0$ and $\Psi = 0$. Equation (11) can be easily integrated and gives $V = F(y^2 + z^2)$, with F an arbitrary function. The energy is $E = \left(1 + f^2 + g^2\right) F'(y^2 + z^2) + F(y^2 + z^2)$, and the allowed region $F'(y^2 + z^2) \geq 0$.

Proposition 2. *If one of the conditions **a** or **c** from above holds, the problem has only the trivial solution $w(y,z) = const$.*

*If condition **b** is fulfilled and the ratio $(2\beta - \widetilde{n}_0) / (\widetilde{n}\beta)$ is independent of x and y, the potential will depend only on the z-variable, otherwise no potential can be found.*

*Finally, if the given family satisfies condition **d** and $G_x = 0$, we shall obtain a family of potentials $w(y,z)$; if $G_x \neq 0$ there will be no such potential.*

5 Applications

Application 1 Let f_1, f_2, g_1, g_2 be constants such that

$$f_1(1 + g_2) = g_1(1 + f_2) \tag{30}$$

and

$$g_1(f_2 g_2 - 1)(1 + g_1 + g_2) \neq 0. \tag{31}$$

For any specific values of these constants (satisfying Eq. 30) we consider the family of curves in the inertial frame $Oxyz$

$$f = x^{f_1} y z^{f_2} = c_1,\quad g = x^{g_1} y^{g_2} z = c_2, \tag{32}$$

where c_1, c_2 are parameters. It can be checked that (32) constitutes a spatial two-parametric set of orbits produced by the 2D potential field

$$w(y,z) = d_1 (y^2 + z^2)^{\frac{1+g_1+g_2}{g_1}} \tag{33}$$

traced with total energy

$$E = -d_1 \left(\frac{1+g_2}{g_1}\right)^2 \left(f^{k_0 g_2} g^{-k_0} + f^{-k_0} g^{k_0 f_2}\right)^{\frac{1+g_2}{g_1}}, \tag{34}$$

where

$$k_0 = \frac{2}{f_2 g_2 - 1}, \tag{35}$$

and d_1 is an arbitrary constant.

Remarks:

1. In the formulae (32), the variable y in f and the variable z in g were taken to be raised to the power 1. Obviously, if the nonzero exponents are not so, they can always be done equal to one.
2. The condition $g_1 \neq 0$ does not reduce the generality of the family (32), because the x-variable must appear at least in one of f and g, hence we can suppose that g_1 is nonzero (otherwise we may perform the change $f \to g \to f$, $y \to z \to y$).
3. The condition $f_2 g_2 - 1 \neq 0$ ensures the fact that α, β, α_0, β_0 and k_0 are meaningfull, that is the denominator of these expressions is nonzero. It also guarantees that the two surfaces in (32) do not coincide.
4. For the family (32) we get (taking into account condition 31)

$$\alpha_0 = r(r-1)\frac{y}{x^2}, \quad \beta_0 = r(r-1)\frac{z}{x^2}, \tag{36}$$

where

$$r = \frac{g_1 f_2 - f_1}{1 - f_2 g_2}. \tag{37}$$

It is $r \neq 0$ because of $g_1(f_2 g_2 - 1) \neq 0$ (if $g_1 f_2 - f_1 = 0$, then $f_2 g_2 - 1 = (f_1 g_2 - g_1)/g_1 = (g_1 f_2 - f_1)/g_1 = 0$, contradiction). It is also $r \neq 1$ because of $g_1(1 + g_1 + g_2) \neq 0$, since in view of (31) $f_2(g_1 + g_2) - (f_1 + 1) = (g_1 f_2 - f_1)(1 + g_1 + g_2)/g_1$.
5. The case $f_1 = 0$ is not excluded, and in view of (30) it must be $f_2 = -1$. The pertinent family $f = y\, z^{-1} = c_1$, $g = x^{g_{1_1}}\, y^{g_2}\, z = c_2$ leads to potential and energy found directly from (33) and (34) for $f_1 = 0$, $f_2 = -1$.
6. The exponents f_1, f_2 do not appear in the potential (33), so, in view of (30) also, we understand that the family (32) is essentially three-parametric.
7. As seen from (25), real motion is allowed all over the space for negative d_1 and nowhere for positive d_1.

Application 2 It can be checked that, for any values of the constants f_1, g_1, g_2 such that not both f_1 and g_1 are zero, the family

$$f = f_1 x^2 + y^2 + z^2 = c_1, \quad g = g_1 x + g_2 y + z = c_2 \tag{38}$$

is produced by the potential

$$w\,(y,z) = \frac{d_2}{(y - g_2 z)^2} \tag{39}$$

and is traced with energy

$$E = \frac{d_2\,(f_1 - 1)}{c_1 g_1^2 + f_1(c_1 + c_1 g_2^2 - c_2^2)}, \tag{40}$$

d_2 being an arbitrary constant.

Remarks:

1. The first of Eqs. (38) is a family of surfaces of revolution around the x-axis. (For $f_1 > 0,\ c_1 > 0$ we have an ellipsoid of revolution, for $f_1 < 0$ we have a hyperboloid of one or two sheets, depending on whether $c_1 > 0$ or $c_1 < 0$ respectively.)
2. The (integrable) potential (39) is independent of f_1 and g_1, so, the family (38) is essentially four-parametric. Notice, however, that, besides c_1 and c_2, the total energy depends on the constants g_1 and g_2.
3. Depending on the values of the constants and the parameters involved and also on the sign in front of the potential (39), real motion of the massive particle takes place either everywhere (nowhere) in the 3D space or inside (outside) a cylinder whose generatrice is parallel to the x-axis. This remark is valid for all potentials of the form $w = w(y,z)$ which we study here.

6 Concluding remarks

For any family of spatial curves (1), we established the conditions which, if fulfilled, allow for the existence of a compatible 2D potential $V = w(y,z)$ of the form (10). These are

$$G_x = 0, \quad H_x = 0, \quad J_y = 0 \tag{41}$$

with the functions G, H, J given by (13), (16), (21) respectively, in terms of the given family.

For a specific family (1), the above conditions (41) are not expected to be satisfied, of course, meaning that no compatible potential of the form (10) exists. Yet, for that specific family, a compatible 2D potential may exist e. g. of the form

$$V = w^*(x,z). \tag{42}$$

Working with (42) and following the same steps as in Sect. 3, we come up with the following three conditions for the family (1):

$$G^*_y = 0, \quad H^*_y = 0, \quad J^*_x = 0, \tag{43}$$

where

$$G^* = \beta - \alpha\frac{\beta_0}{\alpha_0},$$
$$H^* = \frac{\Theta\alpha\beta G^*_x + \alpha_0(2 + \alpha n_0 + \alpha_0 n + 2\beta G^*)}{\alpha\Theta(1 + \beta G^*)}, \tag{44}$$
$$J^* = G^*_x - G^* H^* + \int^x H^*_z(u,z)\,\mathrm{d}\,u.$$

The two PDEs (5) and (6) become

$$w^*_z = G^* w^*_x, \tag{45}$$

$$\alpha\Theta(1 + \beta G^*)w^*_{xx} + (\Theta\alpha\beta G^*_x + \alpha_0(2 + \alpha n_0 + \alpha_0 n + 2\beta G^*))\,w^*_x = 0 \tag{46}$$

Let us apply the above reasoning to the same set of orbits (32) for which we found the potential (33), under the restriction (30) for the four exponents f_1, f_2, g_1, g_2. In place of (30) we now put the restriction

$$(f_1 + f_2)\, g_2 = 1 + g_1. \tag{47}$$

The corresponding potential is

$$w^*(x, z) = d_3(x^2 + z^2)^{f_1+f_2+1}, \tag{48}$$

and each orbit is traced with total energy

$$E = -d_3(f_1 + f_2)^2 \left(c_1^{\ell_0} c_2^{-\ell_0 f_2} + c_1^{-\ell_0 g_1} c_2^{\ell_0 f_1}\right)^{f_1+f_2}, \tag{49}$$

where

$$\ell_0 = \frac{2}{f_1 - f_2 g_1}, \tag{50}$$

and d_3 is an arbitrary constant.

Let us check now if the family (32), this time under the restriction (30), which can be described in the presence of the potential $w(y, z)$ given by (33), is compatible with a potential $w^*(x, z)$. It can be easily seen that $G^* = 0$ and the one-variable potential

$$w^*(x) = d_4 x^{\frac{2(1+g_1+g_2)}{1+g_2}} \tag{51}$$

will be compatible with the given family, traced with the energy

$$E = -d_4 \left(\frac{g_1}{1 + g_2}\right)^2 \left(f^{k_0 g_2} g^{-k_0} + f^{-k_0} g^{k_0 f_2}\right), \tag{52}$$

with k_0 given by (35). Examples of one-dimension potentials producing planar families of curves are to be found in Anisiu & Bozis (2007).

With the aid of the criteria (41) and (43) we can decide whether a given family (α, β) is or is not derived from a 2D potential of the form (10) or (42), respectively. If we want to find all the two-dimension potentials, there remains to us the obligation to check whether the family is derived from a potential of the form

$$V = w^{**}(x, y). \tag{53}$$

To face the question, we recall that the transformation $x \to x, y \to z \to y$ implies that $\alpha \to \beta \to \alpha$ and $\alpha_0 \to \beta_0 \to \alpha_0$. So, instead of checking $(\alpha(x, y, z), \beta(x, y, z))$, we check the family

$$(\beta(x, z, y), \alpha(x, z, y))\,. \tag{54}$$

If the conditions (43) are fulfilled for (54), we understand that there exists a potential $V = w^{**}(x, y)$ creating the given family. The above trick may be applied also in case that the given family leads to $\alpha_0 = 0$, $\beta_0 \neq 0$. If we look for a potential as (53) for the family (32) under the restriction (30), we find again (51).

As for application 2, it can be checked that for $f_1 = 1$, beside the potential $w(z, y)$ given by (39), the problem admits of solutions

$$\begin{aligned} w^*(x, z) &= \tfrac{d_5}{(g_1 z - x)^2}, \\ w^{**}(x, y) &= \tfrac{d_6}{(g_2 x - g_1 y)^2}, \end{aligned} \tag{55}$$

the orbits being traced isoenergetically.

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