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Received 16. III. 1970.

MATHEMATICA VOL. 12 (35), 2, 1970, pp. 237–251

ON UNIFORM APPROXIMATION BY FUNCTIONS HAVING RESTRICTED RANGES

by

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1. Introduction

It has recently been studied by A. C. BACOPOULOS [1], F. DEUTSCH [4], J. W. KAMMERER [5], P. J. LAURENT [6] and G. D. TAYLOR [8] some problems of uniform approximation of continuous functions by functions having restricted ranges.

In this note we shall examine a problem of uniform approximation of real-valued continuous functions on a compact metric space, which is a generalization of all problems studied in the above mentioned papers. Thus we shall discuss in detail the existence and the characterization of the best approximations to a given continuous function. Finally we give some results concerning the uniqueness of the best approximation.

2. Definitions and notations

Let T, U, L be compact sets, not necessarily disjoints, of a metric space $S = T \cup U \cup L$ and let $C(S)$ denote the linear space of all real-valued continuous functions defined on S.

Let $M = [\varphi_1, \varphi_2, \ldots, \varphi_n]$ be an n-dimensional linear subspace of $C(S)$, where the functions $\varphi_1, \varphi_2, \ldots, \varphi_n$ form a basis for M.

We shall fix away a couple (u, l) of extended real-valued functions $u : U \to \overline{R}$ and $l : L \to \overline{R}$ subject to the following restrictions:

(i) $\{s \in U : u(s) = -\infty\} = \emptyset$ but

$\{s \in U : u(s) = +\infty\} = U_{+\infty}$ may be not empty.

(ii) $\{s \in L: \ l(s) = +\infty\} = \emptyset$ but

$\{s \in L: \ l(s) = -\infty\} = L_{-\infty}$ may be not empty.

(iii) $U_{+\infty}$ is an open subset of U and $L_{-\infty}$ is an open subset of L.

(iv) u is continuous on $U^* = U - U_{+\infty}$ and l is continuous on $L^* = L - L_{-\infty}$.

(v) If $s \in U \cap L$ then $u(s) > l(s)$.

Then define the set of approximating functions:

$$\overline{M} = \{p \in M: \ l(s) \leq p(s), \ s \in L \text{ and } p(s) \leq u(s), \ s \in U\}.$$

Let

$$\|f\| = \max_{s \in T} |f(s)| \text{ for } f \in C(S).$$

Definition. *A function $p_0 \in \overline{M}$ is said to be a best approximation in $\overline{M}$ to $f \in C(S)$ if*

$$\rho(f) = \|f - p_0\| = \inf_{p \in \overline{M}} \|f - p\|.$$

Remarks. 1) If $T = U = L$ we obtain the approximation problem studied in [8].

2) If $U = \emptyset$ and $l = f \in C(S)$ respectively $L = \emptyset$ and $u = f \in C(S)$ we obtain the one-sided approximation problem of functions [5], [6] and a generalization if $l \neq f$ respectively $u \neq f$.

3) If $U = \{s_1, s_2, \ldots, s_k\}$, $k < n$, $L = \emptyset$ we get the approximation problems with interpolatory and ε-interpolatory constraints [4], [7], [1].

3. Existence of best approximations

We begin this section by proving an existence theorem.

THEOREM 3.1. *If $\overline{M}$ is nonempty for the couple (u, l) then there exists at least one best approximation $p_0 \in \overline{M}$ to $f \in C(S)$.*

Proof. We observe immediately that $\overline{M}$ is a closed convex subset of M. If $p^* \in \overline{M}$ is not a best approximation to f then:

$$\eta = \|f - p^*\| > \rho(f)$$

and for $p \in M$ such that $\|p - p^*\| > 2\eta$ we give:

$$\|f - p\| \geq \Big| \|p - p^*\| - \|p^* - f\| \Big| > \eta.$$

Therefore the infimum for $\|f - p\|$ may be attained only in the closed ball:

$$D = \{p \in M: \|p - p^*\| \leq 2\eta\}.$$

Hence

$$\rho(f) = \inf_{p \in \overline{D}} \|f - p\|, \quad \overline{D} = D \cap \overline{M},$$

the infimum being attained taking into account that $\overline{D}$ is compact.

It is possible to prove in an easy way the following statement:

THEOREM 3.2. *If $f \in C(S)$ then the set of the best approximations in $\overline{M}$ to f is a convex set.*

4. Characterizations of best approximations

To begin the characterization theorems of best approximations in $\overline{M}$ to the elements of $C(S)$ we define for $p_0 \in \overline{M}$ and $f \in C(S)$ the following critical points sets:

$$\gamma_0^+ = \{s \in T: f(s) - p_0(s) = \|f - p_0\|\}$$

$$\gamma_0^- = \{s \in T: f(s) - p_0(s) = -\|f - p_0\|\}$$

$$\gamma_0^u = \{s \in U: p_0(s) = u(s)\}$$

$$\gamma_0^l = \{s \in L: p_0(s) = l(s)\}$$

$$\Gamma_0^+ = \gamma_0^+ \cup \gamma_0^l, \quad \Gamma_0^- = \gamma_0^- \cup \gamma_0^u$$

$$\overline{\Gamma}_0 = \gamma_0^+ \cup \gamma_0^-, \quad \Gamma_0 = \Gamma_0^+ \cup \Gamma_0^-.$$

It is easy to prove

Lemma 4.1. *If*

$$\Gamma_0^+ \cap \Gamma_0^- \neq \emptyset$$

then $p_0 \in \overline{M}$ is a best approximation to $f \in C(S)$.

For the remainder of this note we assume that the function $f \in C(S)$ is taken such that the hypothesis:

$$(\mathbf{H}_1) \qquad \Gamma_0^+ \cap \Gamma_0^- = \emptyset \quad \textit{for all } p_0 \in \overline{M},$$

is satisfied.

Remark. If $f \in C(S)$, $f \notin \overline{M}$ and

$$u(s) \geq f(s), \ s \in U \ \text{and} \ f(s) \geq l(s), \quad s \in L,$$

then f satisfies $(\mathbf{H}_1)$.

THEOREM 4.2. *If M is a subspace with identity and $p_0 \in \overline{M}$ is a best approximation to $f \in C(S)$, then the sets Γ_0^+ and Γ_0^- are nonempty.*

Proof. We shall prove by contradiction only that $\Gamma_0^+ \neq \emptyset$. Suppose $\Gamma_0^+ = \gamma_0^+ \cup \gamma_0^l = \emptyset$. Because $\gamma_0^l = \emptyset$:

$$\alpha = \min_{s \in L^*} \{p_0(s) - l(s)\} > 0$$

and from $\gamma_0^+ = \emptyset$ we give:

$$f(s) = p_0(s) < \rho(f), \quad s \in T.$$

Then

$$\max_{s \in T} \{f(s) - p_0(s)\} = \rho(f) - 2\beta \leq \rho(f) - 2h$$

where

$$0 < h = \min \{\alpha, \beta\}.$$

Therefore for all $s \in T$ we have

$$-\rho(f) \leq f(s) - p_0(s) \leq \rho(f) - 2h$$

and immediately

$$-\rho(f) + h \leq f(s) - \{p_0(s) - h\} \leq \rho(f) - h.$$

As $p_0 - h \in \overline{M}$ and

$$|f(s) - \{p_0(s) - h\}| \leq \rho(f) - h, \quad s \in T$$

we obtain the desired contradiction.

THEOREM 4.3. *If for a function $p_0 \in \overline{M}$ there exists a function $p \in M$ satisfing*

$$(1) \qquad p(s) > 0 \quad \textit{for all } s \in \Gamma_0^+, \quad \textit{and}$$

$$(2) \qquad p(s) < 0 \quad \textit{for all } s \in \Gamma_0^-$$

then p_0 is not a best approximation to $f \in C(S)$.

Proof. We shall prove that for an $\varepsilon > 0$ the function $\tilde{p} = p_0 + \varepsilon \cdot p$ is from $\overline{M}$ and

$$\|f - \tilde{p}\| < \|f - p_0\|.$$

First we observe that $\Gamma_0, \gamma_0^l, \gamma_0^u$ are compact sets. If $\alpha = \min_{s \in \Gamma_0} |p(s)|$ then $\alpha > 0$.

For the remainder of this proof we denote $\sigma(s_0, \delta)$ an open ball with the radius δ in the metric space S.
Since p is a uniform continuous function on S there exists a $\delta > 0$ such that:

$$3) \qquad |p(s)| > \alpha \ \text{and} \ |f(s) - p_0(s)| > r\rho,$$

for all $s \in \sigma(s_0, \delta)$ and for all $s_0 \in \Gamma_0$, where $0 < r < 1$ and $\rho = \|f - p_0\|$,

$$(4) \qquad p(s) < 0 \ \text{for all } s \in \sigma(s_0, \delta) \ \text{and for all } s_0 \in \gamma_0^u,$$

$$(5) \qquad p(s) > 0 \ \text{for all } s \in \sigma(s_0, \delta) \ \text{and for all } s_0 \in \gamma_0^l.$$

Next we define

$$\Sigma_0 = \bigcup_{s_0 \in \Gamma_0} \sigma(s_0, \delta) \cap T \quad \text{and} \ \tilde{T} = T - \Sigma_0,$$

$$\Sigma_0^u = \bigcup_{s_0 \in \gamma_0^u} \sigma(s_0, \delta) \cap U \quad \text{and} \ \tilde{U} = U - \Sigma_0^u,$$

$$\Sigma_0^l = \bigcup_{s_0 \in \gamma_0^l} \sigma(s_0, \delta) \cap L \quad \text{and} \ \tilde{L} = L - \Sigma_0^l,$$

where $\tilde{T}$, $\tilde{U}$, $\tilde{L}$ are compact sets.
To determine the desired ε we consider the following cases:

a) Let $s \in \Sigma_0$. Then:

$$|f(s) - \tilde{p}(s)| = |f(s) - p_0(s) - \varepsilon p(s)| \leq |f(s) - p_0(s)| - \varepsilon |p(s)|,$$

if $|f(s) - p_0(s)| > \varepsilon |p(s)|$, that is $\varepsilon < r\rho m^{-1}$ with $m = \max_{s \in S} |p(s)|$. In this case $|f(s) - \tilde{p}(s)| \leq \rho - \varepsilon\alpha < \rho$.

b) Let $s \in \tilde{T}$. If denote $\tilde{\rho} = \max_{s \in \tilde{T}} |f(s) - p_0(s)|$ we certainly have $\tilde{\rho} < \rho$. Then

$$|f(s) - \tilde{p}(s)| \leq \tilde{\rho} + \varepsilon m < \rho \qquad \text{if } \varepsilon < (\rho - \tilde{\rho}) m^{-1}.$$

c) Let $s \in \Sigma_0^u$. Then

$u(s) - \tilde{p}(s) = u(s) - p_0(s) - \varepsilon p(s) > 0$ because $u(s) \geq p_0(s)$ and $-\varepsilon p(s) > 0$.

d) Let $s \in \tilde{U}$. Denote $\alpha_u = \min_{s \in \tilde{U}} \{u(s) - p_0(s)\} > 0$. Then

$$u(s) - \tilde{p}(s) \geq \alpha_u - \varepsilon m > 0 \qquad \text{if } \varepsilon < \alpha_u m^{-1}.$$

e) Let $s \in \Sigma_0^l$. Then

$\tilde{p}(s) - l(s) = p_0(s) - l(s) + \varepsilon p(s) > 0$ because $p_0(s) \geq l(s)$ and $\varepsilon p(s) > 0$.

f) Let $s \in \tilde{L}$. Denote $\alpha_l = \min_{s \in \tilde{L}} \{p_0(s) - l(s)\} > 0$.

Then $$\tilde{p}(s) - l(s) \geq \alpha_l - \varepsilon m > 0 \qquad \text{if } \varepsilon < \alpha_l m^{-1}.$$

Thus if

$$0 < \varepsilon < \min m^{-1} \{r\rho,\ \rho - \tilde{\rho},\ \alpha_u,\ \alpha_l\},$$

the function $\tilde{p} = p_0 + \varepsilon p$ is in $\overline{M}$ and

$$\|f - \tilde{p}\| < \|f - p_0\|.$$

Remark. By the Theorem 4.3., the Theorem 4.2. it may be proved without the additional hypothesis that M is a subspace with identity.

The next characterization theorem is a corolar of the Theorem 8 from [2]. *We shall give a direct proof in the case* $\gamma_0^u = \gamma_0^l = \emptyset$.

THEOREM 4.4. *The function $p_0 \in \overline{M}$ is a best approximation to $f \in C(S)$ if and only if for any function $p \in \overline{M}$ there exists a point $s' \in \Gamma_0^+$ such that*

$$p(s') \leq p_0(s') \tag{6}$$

or a point $s'' \in \Gamma_0^-$ such that

$$p(s'') \geq p_0(s''). \tag{7}$$

Proof. Sufficiency. If $p \in \overline{M}$ and $s' \in \gamma_0^+$ then

$$\|f - p_0\| = f(s') - p_0(s') \leq f(s') - p(s') \leq \|f - p\|.$$

If $s'' \in \gamma_0^-$ we obtain

$$\|f - p_0\| = p_0(s'') - f(s'') \leq p(s'') - f(s'') \leq \|f - p\|.$$

Necessity. By contradiction. Let $p_0 \in \overline{M}$ be a best approximation to $f \in C(S)$ and suppose that (6) and (7) are not satisfied. That is, there exists $p \in \overline{M}$ such that

$$p(s) > p_0(s) \text{ if } s \in \gamma_0^+ \text{ and } p(s) < p_0(s) \text{ if } s \in \gamma_0^-.$$

Also by continuity and compactness considerations there exists a $\delta > 0$ such that

$$p(s) - \delta \geq p_0(s) \text{ if } s \in \gamma_0^+ \text{ and } p(s) + \delta \leq p_0(s) \text{ if } s \in \gamma_0^-.$$

Now we take a number ε satisfing $0 < \varepsilon < \delta$ and consider the sets:

$$V^+ = \left\{s \in T : p(s) - \varepsilon > p_0(s) \text{ and } f(s) - p_0(s) > \frac{\rho(f)}{2}\right\},$$

$$V^- = \left\{s \in T : p(s) + \varepsilon < p_0(s) \text{ and } f(s) - p_0(s) < -\frac{\rho(f)}{2}\right\}.$$

It is straightforward to establish that $\gamma_0^+ \subseteq V^+$, $\gamma_0^- \subseteq V^-$, $V^+ \cap V^- = \emptyset$ and also by continuity considerations that V^+ and V^- are open sets in T. If we denote $Z = T - (V^+ \cup V^-)$ and $\bar{\rho} = \max_{s \in Z} |f(s) - p_0(s)|$ then $\bar{\rho} < \rho(f)$.

Next let

$$\mu_u = \min_{s \in U^*} |u(s) - p_0(s)| > 0, \quad \mu_l = \min_{s \in L^*} |l(s) - p_0(s)| > 0.$$

Now we select $\lambda > 0$ such that $\lambda\nu < \eta$ where $\nu = \max\limits_{s\in S} |p_0(s) - p(s)|$ and $\eta = \min\left\{\rho(f) - \bar{\rho}, \frac{\rho(f)}{2}, \mu_u, \mu_l\right\}$.

Finally we will show that $\tilde{p} = p_0 - \lambda(p_0 - p)$, certainly from $\overline{M}$, is the best approximation.

Letting $\tilde{\rho} = \max\limits_{s\in T} |f(s) - \tilde{p}(s)|$ the proof will be completed by showing that $\tilde{\rho} < \rho(f)$.

Indeed:

1) If $s \in Z$ we have

$$|f(s) - \tilde{p}(s)| \leq |f(s) - p_0(s)| + \lambda|p_0(s) - p(s)| < \bar{\rho} + \rho(f) - \bar{\rho} = \rho(f)$$

2) If $s \in V^+$ we obtain

$$|f(s) - \tilde{p}(s)| = f(s) - p_0(s) - \lambda\{p(s) - p_0(s)\} \leq \rho(f) - \lambda\varepsilon$$

using the definition of the set V^+ and the choice of λ.

3) If $s \in V^-$ then

$$|f(s) - \tilde{p}(s)| = p_0(s) - f(s) - \lambda\{p_0(s) - p(s)\} \leq \rho(f) - \lambda\varepsilon$$

using the definition of the set V^- and the choice of λ.

The necessity of the next theorem may be proved only if M verifies the hypothesis:

($\mathbf{H}_2$) *There exists at least one $p \in M$ such that the inequalities*

$$p(s) < u(s) \text{ if } s \in U \text{ and } p(s) > l(s) \text{ if } s \in L$$

are satisfied.

THEOREM 4.5. *Suppose that ($\mathbf{H}_2$) is verified. The function $p_0 \in \overline{M}$ is a best approximation to $f \in C(S)$ if and only if there exists k points ($k \leq n+1$): $s_1, s_2, \ldots, s_k$ in Γ_0 (at least one in $\overline{\Gamma}_0$) and a functional of the form:*

$$\Phi(p) = \sum_{i=1}^{k} \alpha_i p(s_i), \quad p \in C(S)$$

such that:

(8) $$\Phi(p) = 0 \text{ if } p \in M, \text{ and}$$

(9) $$\alpha_i > 0 \text{ if } s_i \in \Gamma_0^+ \text{ and } \alpha_i < 0 \text{ if } s_i \in \Gamma_0^-.$$

Proof. Sufficiency. Suppose that $p_0 \in \overline{M}$ verifies the conditions of the theorem but it is not a best approximation to $f \in C(S)$. Let $p_1 \in \overline{M}$ be a best approximation to $f \in C(S)$, that is

$$\rho(f) = \|f - p_1\| < \|f - p_0\|.$$

Letting $\tilde{p} = p_1 - p_0 = (f - p_0) - (f - p_1)$ we see that $\tilde{p} \in M$ and:

$$\tilde{p}(s_i) > 0 \text{ if } s_i \in \gamma_0^+ \text{ and } \alpha_i > 0,$$

$$\tilde{p}(s_i) < 0 \text{ if } s_i \in \gamma_0^- \text{ and } \alpha_i < 0,$$

$$\tilde{p}(s_i) \leq 0 \text{ if } s_i \in \gamma_0'' \text{ and } \alpha_i < 0,$$

$$\tilde{p}(s_i) \geq 0 \text{ if } s_i \in \gamma_0' \text{ and } \alpha_i > 0.$$

Because at least one of s_i is in $\overline{\Gamma}_0$ we have:

$$\Phi(\tilde{p}) = \sum_{i=1}^{k} \alpha_i \tilde{p}(s_i) > 0$$

that is in contradiction with (8).

Neccesity. Let $p_0 \in \overline{M}$ be a best approximation to $f \in C(S)$.

Denote

$$\mathcal{G} = \{\xi(s)\hat{s} : s \in \Gamma_0\} \subset R^n$$

where

$$\xi(s) = \begin{cases} +1 & \text{if} \quad s \in \Gamma_0^+, \\ -1 & \text{if} \quad s \in \Gamma_0^- \end{cases}$$

$$\hat{s} = (\varphi_1(s), \ldots, \varphi_n(s))$$

and R^n is the Euclidean n-dimensional space.

First we prove by contradiction that the origin θ of the R^n space belongs to the convex hull $\mathrm{Co}(\mathcal{G})$ of $\mathcal{G}$.

Indeed if $\theta \notin \mathrm{Co}(\mathcal{G})$, $\mathrm{Co}(\mathcal{G})$ being a compact set (because $\mathcal{G}$ is compact) there exists a support hyperplane. Thus there exists $\tau > 0$ and β_j, $j = 1, \ldots, n$ such that:

$$\sum_{j=1}^{n} \beta_j z_j \geq \tau \quad \text{for any } z = (z_1, \ldots, z_n) \in \mathrm{Co}(\mathcal{G}).$$

In particular for the points of $\mathcal{G}$ we have:

$$\sum_{j=1}^{n} \beta_j \varphi_j(s) \geqq \tau > 0 \quad \text{if} \quad s \in \Gamma_0^+$$

and

$$-\sum_{j=1}^{n} \beta_j \varphi_j(s) \geqq \tau > 0 \quad \text{if} \quad s \in \Gamma_0^-.$$

Letting $p = \sum_{j=1}^{n} \beta_j \varphi_j$ we get that $p \in M$. By Theorem 4.3 we obtain that p_0 is not a best approximation. Thus $\theta \in \mathrm{Co}(\mathcal{G})$. Consequently by Carathéodory's theorem on convex sets in R^n there exist k points ($k \leq n+1$) $z^{(1)}, \ldots, z^{(k)}$ in $\mathcal{G}$ such that

$$\theta \in \mathrm{Co}\,\{z^{(1)}, \ldots, z^{(k)}\}.$$

Let $s_1, \ldots, s_k$ be the points in S corresponding to $z^{(1)}, \ldots, z^{(k)}$.

Thus

$$\theta = \sum_{i=1}^{k} \delta_i z^{(i)}, \qquad \sum_{i=1}^{k} \delta_i = 1, \qquad \delta_i > 0, \quad i = 1, \ldots, k,$$

or in projection

$$0 = \sum_{i=1}^{k} \delta_i \varepsilon_i \varphi_j(s_i), \qquad j = 1, \ldots, n,$$

where

$$\varepsilon_i = \begin{cases} +1 & \text{if} \quad s_i \in \Gamma_0^+ \\ -1 & \text{if} \quad s_i \in \Gamma_0^- . \end{cases}$$

Taking $\alpha_i = \delta_i \cdot \varepsilon_i$ we obtain the functional $\Phi(p) = \sum_{i=1}^{k} \alpha_i p(s_i)$, which satisfies the conditions (8) and (9).

Now we shall prove that at least one of s_i belongs to $\overline{\Gamma}_0$. We have the cases:

a) If $s_i \in U - T$ then for any $p \in \overline{M}$ we get $p(s_i) = u(s_i)$. Indeed, if for an index ν suppose $p(s_\nu) < u(s_\nu) = p_0(s_\nu)$, then $\Phi(p_0 - p) < 0$ which is a contradiction.

b) If $s_i \in L - T$ then for any $p \in \overline{M}$ we get $p(s_i) = l(s_i)$, using the above way.

By the hypothesis (H_2) the proof is complete.

Now we use a last hypothesis:

$(\mathbf{H}_3)$ *M is an n-dimensional Haar subspace of $C(S)$. That is, M is an n-dimensional subspace of $C(S)$ such that the zero function is the only element of M which vanishes at n or more distinct points of S,*

to prove the following statement [8, Theorem 3.2].

THEOREM 4.6. *Let $S \subset [a, b]$, $f \in C(S)$, $p_0 \in \overline{M}$ and suppose* $(\mathbf{H}_3)$ *verified. The following three statements are equivalent.*

(10) *p_0 is a best approximation to f.*

(11) *The origin of R^n space belongs to the convex hull of $\{\xi(s)\hat{s}: s \in \Gamma_0\}$, where $\xi(s) = +1$ if $s \in \Gamma_0^+$, $\xi(s) = -1$ if $s \in \Gamma_0^-$ and $\hat{s} = (\varphi_1(s), \ldots, \varphi_n(s))$ is in R^n.*

(12) *There exist $n+1$ distinct points $s_1, \ldots, s_{n+1}$ in Γ_0 satisfying $\xi(s_i) = (-1)^{i+1}(s_1)$, $i = 2, \ldots, n+1$.*

Proof. (10) implies (11). This implication belongs to the proof of necesity of Theorem 4.5.

(11) implies (12). This may be found in [3, p. 74—75].

(12) implies (10). By contradiction. Let $s_1, \ldots, s_{n+1}$ be distinct points in Γ_0 satisfying $\xi(s_i) = (-1)^{i+1}\xi(s_1)$, $i = 2, \ldots, n+1$. We assume that there exists $p_1 \in \overline{M}$ such that $\|f - p_1\| < \|f - p_0\|$ and $\xi(s_1) = +1$. Thus we have either $f(s_1) - p_0(s_1) = \|f - p_0\|$ or $p_0(s_1) = l(s_1)$. In either case it follows that $p_0(s_1) \leq p_1(s_1)$. By a similar argument at the remaining points we find that:

$$(-1)^{i+1}(p_1(s_i) - p_0(s_i)) \geqq 0, \qquad i = 1, 2, \ldots, n+1.$$

Now using the hypothesis $(\mathbf{H}_3)$ and Theorem 4.4 we conclude that $p_0 \equiv p_1$.

5. Results concerning the uniqueness of the best approximation

If the hypothesis $(\mathbf{H}_3)$ is satisfied and $S \subset [a, b]$ then it may be easyly got a uniqueness theorem and a strong unicity theorem [3, p. 77].

THEOREM 5.1. *If $f \in C(S)$ and $p_0 \in \overline{M}$ is a best approximation to f then p_0 is unique.*

Proof. The proof given in (12) implies (10) can be used to show that p_0 is unique.

THEOREM 5.2. *Let $f \in C(S)$ and let $p_0 \in \overline{M}$ be the best approximation to f. Then there exists a constant $\gamma > 0$ depending only on f such that for any $p \in \overline{M}$:*

$$\|f - p\| \geqq \|f - p_0\| + \gamma \|p_0 - p\|.$$

Proof. By Theorem 4.6 there exist points $s_1, s_2, \ldots, s_k$ in Γ_0 and the signs $\xi_1, \ldots, \xi_k$ such that

$$\theta \in \text{Co}\ \{\xi_i(\varphi_1(s_i), \ldots, \varphi_n(s_i)),\quad i = 1, 2, \ldots, k\}.$$

By the hypothesis $(\mathbf{H}_3)$ and Carathéodory's theorem we have $k = n + 1$. Thus, in projection we get

$$(13)\quad 0 = \sum_{i=1}^{n+1} \alpha_i\, \xi_i\, \varphi_j(s_i),\quad j = 1, 2, \ldots, n \text{ and } \alpha_i > 0,\quad i = 1, 2, \ldots, n + 1.$$

If $p \in M$ then $p(s) = \sum_{j=1}^{n} \beta_j \varphi_j(s)$ and by (13) it follows that:

$$\sum_{i=1}^{n+1} \alpha_i\, \xi_i\, p(s_i) = \sum_{j=1}^{n} \beta_j \sum_{i=1}^{n+1} \alpha_i\, \xi_i\, \varphi_j(s_i) = 0.$$

By the hypothesis $(\mathbf{H}_3)$ at least one of $\xi_i p(s_i)$ is positiv. Because the functional $F(p) = \max_i \xi_i p(s_i)$ is continuous on the compact set $M^* = \{p \in M : ||p|| = 1\}$ we obtain:

$$\gamma = \min_{p \in M^*} F(p) > 0.$$

Now, if $p \in \overline{M}$, $p \neq p_0$, the function $\tilde{p} = (p_0 - p)||p_0 - p||^{-1}$ belongs to M^* and thus there exists an index ν, $1 \leq \nu \leq n + 1$ such that

$$\xi_\nu(p_0(s_\nu) - p(s_\nu)) \geq \gamma ||p_0 - p||.$$

We shall prove that $s_\nu \in \overline{\Gamma}_0$. Indeed:

(a) If $s_i \in \gamma_0'$ then $\xi_i = +1$ and $p_0(s_i) = l(s_i)$.

Therefore $\xi_i(p_0(s_i) - p(s_i)) \leq 0$.

(b) If $s_i \in \gamma_0''$ then $\xi_i = -1$ and $p_0(s_i) = u(s_i)$.

Hence $\xi_i(p_0(s_i) - p(s_i)) \leq 0$.

Thus

$$||f - p|| \geq \xi_\nu(f(s_\nu) - p(s_\nu)) = \xi_\nu(f(s_\nu) - p_0(s_\nu)) +$$
$$+ \xi_\nu(p_0(s_\nu) - p(s_\nu)) \geq ||f - p_0|| + \gamma||p_0 - p||,$$

and the proof is ended.

If we assume that only the hypothesis $(\mathbf{H}_2)$ is satisfied then we get the following theorem:

THEOREM 5.3. *Suppose that $U \cap L = \emptyset$. Then each function from $C(S)$ has only one best approximation in $\overline{M}$ if and only if there does no exist a functional of the form*

$$\Psi(p) = \sum_{i=1}^{k} \alpha_i\, p(s_i),\quad p \in C(S),$$

with $k \leq n$, $s_i \in S$, $i = 1, \ldots, k$ and such that:

(14) *If $p \in M$ then $\Psi(p) = 0$.*

(15) *At least one s_i belongs to T.*

(16) *$\alpha_i > 0$ if $s_i \in L - T$ and $\alpha_i < 0$ if $s_i \in U - T$.*

Proof. Sufficiency. If the conditions of theorem are verified then for the functional Φ in the Theorem 4.5 we have $k = n + 1$. Denote p_0 and p_1 two best approximations in $\overline{M}$ to $f \in C(S)$. By the Theorem 3.2. the element $p_2 = (p_0 + p_1)2^{-1}$ is also a best approximation in $\overline{M}$ to f. Let $s_1, s_2, \ldots, s_{n+1}$ be points in Γ_2. It immediately follows that these points belong to Γ_0 and Γ_1 and also they have the same character.

If $h = p_0 - p_1 = \sum_{j=1}^{n} \eta_j \varphi_j$, then $h(s_i) = 0$, $i = 1, 2, \ldots, n$.

We shall prove that $h \equiv 0$, that is $\eta_j = 0$, $j = 1, 2, \ldots, n$.
Because $\overline{\Gamma}_2 \neq \emptyset$, at least one s_i, $i = 1, \ldots, n + 1$ belongs to T. More precisely $s_\nu \in T$. The coefficients η_j are the solution of the linear homogeneous system:

$$\sum_{j=1}^{n} \eta_j\, \varphi_j(s_i) = 0,\quad i = 1, \ldots, n + 1,\ i \neq \nu.$$

We shall prove that $D = \det\, [\varphi_j(s_i)] = 0$, $j = 1, \ldots, n$, $i = 1, \ldots, n + 1$, $i \neq \nu$, which implies $\eta_j = 0$, $j = 1, \ldots, n$.
Denote α_i, $i = 1, \ldots, n + 1$ the coefficients of the characterization functional Φ for p_2. If suppose $D = 0$ then the system

$$\sum_{\substack{i=1 \\ i \neq \nu}}^{n+1} \alpha_i\, \varphi_j(s_i) = -\alpha_\nu\, \varphi_j(s_\nu),\qquad j = 1, \ldots, n,$$

has not a unique solution. May be selected an index μ and a solution α_i', $i = 1, \ldots, n + 1$ such that

$$\alpha_i\, \alpha_i' > 0,\quad i = 1, \ldots, n + 1,\ i \neq \mu \text{ and}$$
$$\alpha_\mu' = 0.$$

The functional $\Psi(p) = \sum_{\substack{i=1 \\ i \neq \mu}}^{n+1} \alpha_i' p(s_i)$ verifies the conditions (14), (15) and (16), which is a contradiction.

Neccesity. Suppose that there exists a functional Ψ such that the conditions (14), (15), (16) are verified with $k = n$.
The coefficients α_i, $i = 1, \ldots, n$ form a non-trivial solution of the system

$$\sum_{i=1}^{n} \alpha_i \varphi_j(s_i) = 0, \qquad j = 1, \ldots, n.$$

Thus $\Delta = \det [\varphi_j(s_i)] = 0, \quad i, j = 1, \ldots, n.$

It immediately follows that there exists $p = \sum_{j=1}^{n} \beta_j \varphi_j$ with non-zero coefficients β_j, $j = 1, \ldots, n$ such that

$$p(s_i) = \sum_{j=1}^{n} \beta_j \varphi_j(s_i) = 0, \qquad i = 1, \ldots, n.$$

May be chosen p with $\max_{s \in S} |p(s)| < 1$ and $\bar{f} \in C(S)$ such that:

$$1°. \ \bar{f}(s_i) = \begin{cases} +1 & \text{if} \quad s_i \in T \\ -1 & \text{if} \quad \alpha_i < 0, \ s_i \in T - U \ \text{ or } \ \alpha_i > 0, \ s_i \in T - L \\ 0 & \text{if} \quad \alpha_i < 0, \ s_i \in U \ \text{ or } \ \alpha_i > 0, \ s_i \in L \end{cases}$$

$2°. \ \|\bar{f}\| = 1.$

Now letting $f = \bar{f}(1 - |p|)$ and $p_0 = \varepsilon \cdot p$ we shall prove that for each $\varepsilon \in [0, \delta]$, $0 < \delta \leq 1$, p_0 is a best approximation in $\overline{M}$ to f. First we get that $\|f - p_0\| \leq 1$. Indeed, for all $s \in T$ and each $\varepsilon \in [0, 1]$ by the condition 2° we have:

$$|f(s) - p_0(s)| = |f(s) - \varepsilon p(s)| \leq |f(s)| + \varepsilon |p(s)| =$$

$$= |\bar{f}(s)|(1 - |p(s)|) + \varepsilon |p(s)| \leq 1 - (1 - \varepsilon)|p(s)| \leq 1.$$

It is obvious that it may be chosen a $\delta' > 0$ such that p_0 belongs to $\overline{M}$ for each $\varepsilon \in [0, \delta']$. Set $\delta = \min \{1, \delta'\}$.
Now it immediately follows that

$$f(s_i) - p_0(s_i) = \bar{f}(s_i), \qquad i = 1, \ldots, n,$$

and by the condition 1° we observe a concordance between the character of the points s_i and the signs of the coefficients α_i. By the Theorem 4.5, p_0 is a best approximation in $\overline{M}$ to f for each $\varepsilon \in [0, \delta]$. Thus the unicity of the best approximation does not hold.

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Received 6. II. 1970.