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AN INTERPOLATION PROBLEM FOR THE SOLUTIONS OF A CLASS OF LINEAR DIFFERENTIAL EQUATIONS

by

ALEXANDRU LUPAŞ

Cluj

§ 1. Introduction

Let us consider the differential operator

$$L_{n,p} = \frac{d^n}{dx^n} + \sum_{k=1}^{n} a_k(x) \frac{d^{n-k}}{dx^{n-k}}$$

where $a_k \in L^p\,[a,\ a+h]$, $k = 1, 2, \ldots, n$, $h > 0$, $1 \leq p \leq +\infty$. In what follows, for $k = 1, 2, \ldots, n$ we use the notation

$$\|a_k\|_p = \begin{cases} \left[\int_a^{a+h} |a_k(x)|^p\,dx\right]^{\frac{1}{p}}, & \text{if } p < +\infty \\ \text{ess. sup } |a_k(x)|, & \text{if } p = +\infty, \end{cases}$$

and

$$M_p\,(a_k, h) = h^{-1/p}\,\|a_k\|, \quad \text{if } \ p < +\infty.$$

If we consider the linear differential equation

$$L_{n,p}[y] = 0 \tag{1}$$

we denote by Y_n the set of the solutions of the equation (1), that is the set of functions $y:[a,\ a+h] \to \mathbf{R}$ with the properties

i) y has a derivative of order $(n-1)$ which is absolutely continuous on $[a, a+h]$,

ii) y verifies (1) almost everywhere on $[a, a+h]$.

Definition 1. *The differential operator $L_{n,p}$ has the interpolation property $I_n[a, a+h)$ if for each*

$$P = (x_1, x_2, \ldots, x_n) \in [a, a+h)^n, \quad x_i \neq x_j \text{ for } i \neq j,$$

and for each

$$V = (y_1, y_2, \ldots, y_n) \in \mathbf{R}^n$$

there is an unique solution $y^* = y^*(P, V; L_{n,p}) \in Y_n$ such that

$$y^*(x_i) = y_i, \quad i = 1, 2, \ldots, n.$$

Definition 2. *A differential operator $L_{n,p}$ has the interpolation property $H_n[a, a+h)$ if for each*

$$P = (x_1, x_1, \ldots, x_1, x_2) \in [a, a+h)^n, \quad x_1 \neq x_2,$$

and for each

$$V = (y_1, y_2, \ldots, y_n) \in \mathbf{R}^n$$

here is an unique solution $\bar{y} = \bar{y}(P, V; L_{n,p}) \in Y_n$ such that

$$\bar{y}(x_1) = y_1, \ \bar{y}'(x_1) = y_2, \ldots, \bar{y}^{(n-2)}(x_1) = y_{n-1}, \ \bar{y}^{(w)}(x_2) = y_n,$$

where w is one the numbers $0, 1, \ldots, n-2$.

The purpose of this paper is to give some upper bounds for h, that is for the length of the interval of interpolation. These upper bounds are expressed in the form of some nonlinear inequalities, namely

$$\sum_{k=1}^{n} R_{n-k+1} h^{n-k+1} M_p(a_{n-k+1}, h) \leq 1, \quad \text{if } p < +\infty$$

or

$$\sum_{k=1}^{n} T_{n-k+1} h^{n-k+1} \|a_{n-k+1}\|_\infty \leq 1, \quad \text{if } p = +\infty,$$

where R_j, T_j, $j = 1, 2, \ldots, n$, are constants.

Obtaining upper bounds of this kind has a long history: see for instance O. ARAMĂ and D. RIPIANU [2] — [3], PH. HARTMAN and A. WINTNER [9], Z. NEHARI [10] and Z. OPIAL [11].

In the case $n = 2$, $p = +\infty$, Z. OPIAL [11] obtained the following result: if

$$\frac{1}{\pi^2} h^2 \|a_2\|_\infty + \frac{2}{\pi^2} h \|a_1\|_\infty \leq 1, \tag{2}$$

then the operator $L_{2\infty}$, has the interpolation property $I_2[a, a+h)$.

For $n = 3$, $p = 2$, O. ARAMĂ and D. RIPIANU [3] showed that if

$$\frac{1}{\pi^2\sqrt{2}} h^3 M_2(a_3, h) + \frac{\sqrt{2}}{\pi} h^2 M_2(a_2, h) + \frac{1}{\sqrt{2}} h M_2(a_1, h) \leq 1 \tag{3}$$

then $L_{3,2}$ has the property $I_3[a, a+h)$. Likewise in [1] O. ARAMĂ proves the inequality

(4)

$$\frac{1}{\pi^2\sqrt{6}} h^4 M_2(a_4, h) + \frac{1}{\pi^2\sqrt{2}} h^3 M_2(a_3, h) + \frac{\sqrt{2}}{\pi} h^2 M_2(a_2, h) + \frac{1}{\sqrt{2}} h M_2(a_1, h) \leq 1$$

in order that the differential operator $L_{4,2}$ to have the property $H_4[a, a+h)$. We note and the paper of D. BOBROWSKI [6] which obtained the following inequality, as a sufficient condition for the property $H_4[a, a+h)$ of the operator $L_{4,\infty}$

$$\frac{1}{2\pi^2} h^4 \|a_4\|_\infty + \frac{1}{2\pi^2} h^3 \|a_3\|_\infty + \frac{1}{\pi^2} h^2 \|a_2\|_\infty + \frac{1}{4} h \|a_1\|_\infty \leq 1. \tag{5}$$

In this paper we give some inequalities of this type and in the same time we improve the inequalities (3), (4) and (5). We need the following inequalities:

a) *Wirtinger's inequality* (see [8]); Let $y: [a, b] \to \mathbf{R}$ an absolutely continuous function on $[a, b]$. Let ξ_1 and ξ_2 real numbers such that $a \leq \xi_1 \leq \xi_2 \leq b$ and $y(\xi_1) = y(\xi_2)$. Then

(6)

$$\int_a^b |y(x) - y(\xi_1)|^2 \, dx \leq \frac{4}{\pi^2} \max\left[(\xi_1 - a)^2, (b - \xi_2)^2, \left(\frac{\xi_2 - \xi_1}{2}\right)^2\right] \cdot \int_a^b |y'(x)|^2 dx$$

b) *Opial's inequality* (see [1], [2] and [11]); If $y:[a, b] \to \mathbf{R}$ is absolutely continuous on $[a, b]$ and if $y(a) = 0$, then

$$\int_a^b |y(x)y'(x)|\,dx \leq \frac{b-a}{2} \int_a^b |y'(x)|^2\,dx \tag{7}$$

the equality hold if and only if $y(x) = kx$ for some constant k.

c) *The inequality of Boyd* ([7]). If $y:[a, b] \to \mathbf{R}$ is absolutely continuous on $[a, b]$ and $y(a) = 0$ then for $r \geqq 1$, $p > 0, 0 \leqq q \leqq r$ hold the inequality

$$\int_a^b y(x)|^p\,|y'(x)|^q\,dx \leq (b-a)^{\frac{rp-p-q+r}{r}} K(p, q, r)\left\{\int_a^b |y'(x)|^r\,dx\right\}^{\frac{p+q}{r}}. \tag{8}$$

The best constants $K(p, q, r)$ are given by the following expressions:

c₁) if $p > 0$, $r > 1$, $0 \leqq q < r$, then

$$K(p, q, r) = \frac{(r-q)p^P}{(r-1)(p+q)}\, \beta^{p+q-r}\, I(p, q, r)^{-p} \tag{8.1}$$

where

$$\beta = \left\{\frac{p(r-1) + (r-q)}{(r-1)(p+q)}\right\}^{\frac{1}{r}}$$

and

$$I(p, q, r) = \int_0^1 \left\{1 + \frac{r(q-1)}{r-q}\, t\right\}^{\frac{-(q+p+rp)}{rp}} \{1 + (q-1)\,t\}\, t^{\frac{1}{p}-1}\,dt.$$

c₂) If $r = 1$, then

$$K(p, q, 1) = \begin{cases} q^q(p+q)^{-q}, & \text{if } q > 0 \\ 1 & \text{, if } q = 0. \end{cases} \tag{8.2}$$

c₃) If $q = r$, then

$$K(p, r, r) = \frac{rp^p}{p+r}\left(\frac{p}{p+r}\right)^{\frac{p}{r}} B\left(\frac{1}{r} + 1, \frac{1}{p}\right)^{-p}. \tag{8.3}$$

If $r = 1$, $q = 0$ there is strictly inequality for every $y \not\equiv 0$, while for all other cases there is equality if and only if $y(x) = a_0 y_0(p,q,r,x)$ where a_0 is a constant and $y_0 \in C^\infty(a, b)$ and y_0 is concave if $0 \leqq q < 1$, convex for $q > 1$ and linear for $q = 1$.

§ 2. Some integral inequalities

THEOREM 2.1. *If $z:[a, b] \to \mathbf{R}$ is absolutely continuous on $[a, b]$ and if x_1, x_2 are numbers such that $a \leqq x_1 \leqq x_2 \leqq b$, $z(x_1) = z(x_2)$, then for $p > 0$, $q \geqq 0$, $p + q \geqq 1$, hold the inequality*

$$\int_a^b |z(x) - z(x_1)|^p\,|z'(x)|^q\,dx \leq \tag{9}$$

$$\leq \max\left[(x_1 - a)^p,\ (b - x_2)^p,\ \left(\frac{x_2 - x_1}{2}\right)^p\right] K(p, q, p+q) \int_a^b |z'(x)|^{p+q}\,dx$$

where $K(p, q, p+q)$ is defined by (8.1) *or* (8.2). *In the case $p+q>1, q>0$,*

$$K(p, q, p+q) = q(p+q)^{p-1}\{pL(p, q) + q\}^{-p},$$

$$L(p, q) = \int_0^1 \frac{dx}{1 - kx^p},\quad k = \frac{(p+q)(q-1)}{q(p+q-1)}.$$

Proof. For $r = p + q$ the inequality (8) yields with $y(x) = z(x) - z(\alpha)$, $\alpha, \beta \in [a, b]$.

$$\int_\alpha^\beta |z(x) - z(\alpha)|^p\,|z'(x)|^q\,dx \leq (\beta - \alpha)^p K(p, q, p+q) \int_\alpha^\beta |z'(x)|^{p+q}\,dx. \tag{10}$$

If we put $w(t) = z(\alpha + \beta - t) - z(\beta)$, we observe that $w(\alpha) = 0$ and therefore

$$\int_\alpha^\beta |z(\alpha + \beta - t) - z(\beta)|^p\,|z'(\alpha + \beta - t)|^q\,dt \leq$$

$$\leq (\beta - \alpha)^p K(p, q, p+q) \int_\alpha^\beta |z'(\alpha + \beta - t)|^{p+q}\,dt$$

and making the substitution $x = \alpha + \beta - t$ we obtain

(11) $$\int_{\alpha}^{\beta} |z(x) - z(\beta)|^p |z'(x)|^q dx \leq (\beta - \alpha)^p K(p, q, p+q) \int_{\alpha}^{\beta} |z'(x)|^{p+q} dx.$$

Now let $\xi \in [a, b]$; then

(12) $$\int_a^b |z(x) - z(\xi)|^p |z'(x)|^q dx \leq$$

$$\leq \max\ [(\xi - a)^p, (b - \xi)^p] K(p, q, p+q) \int_a^b |z'(x)|^{p+q} dx.$$

Indeed, if we take $\xi = a$ or $\xi = b$ then (12) follow from (10) or from (11) with $\alpha = a$, $\beta = b$; let assume $\xi \in (a, b)$. In this case by using (11) on the interval $[a, \xi]$ and next (10) on the interval $[\xi, b]$ we have the inequalities

(12.1) $$\int_a^{\xi} |z(x) - z(\xi)|^p |z'(x)|^q dx \leq$$

$$\leq (\xi - a)^p K(p, q, p+q) \int_a^{\xi} |z'(x)|^{p+q} dx$$

and

(12.2) $$\int_{\xi}^b |z(x) - z(\xi)|^p |z'(x)|^q dx \leq$$

$$\leq (b - \xi)^p K(p, q, p+q) \int_{\xi}^b |z'(x)|^{p+q} dx.$$

Adding (12.1) and (12.2) we give (12).

In order to obtain (9) we can proceed as follows: consider (11) on the interval $[a, x_1]$, and (10) on the subinterval $\left[x_1, \frac{x_1 + x_2}{2}\right]$, then (11) on $\left[\frac{x_1 + x_2}{2}, x_2\right]$ and finally we write the inequality (10) on $[x_2, b]$. Taking into

account that $z(x_1) = z(x_2)$ if we add the above fourth inequalities we obtain (9). The form of the constant $K(p, q, p+q)$ results from (8.1) or (8.2).

From (12) we conclude that the following statement is valid.

Corollary 1. *If $y: [a, b] \to \mathbf{R}$ is absolutely continuous on $[a, b]$, $x_0 \in [a, b]$, $y(x_0) = 0$, $p > 0$, $q \geq 0$, $p + q \geq 1$, then*

(13) $$\int_a^b |y(x)|^p |y'(x)|^q dx \leq (b - a)^p K(p, q, p+q) \int_a^b |y'(x)|^{p+q} dx.$$

Also from (9) follows the inequality of the corollary 2;

Corollary 2. *If $y: [a, b] \to \mathbf{R}$ is absolutely continuous on $[a, b]$, $y(a) = y(b) = 0$ $p > 0$, $q \geq 0$, $p + q \geq 1$, then*

(14) $$\int_a^b |y(x)|^p |y'(x)|^q dx \leq \frac{(b-a)^p}{2^p} K(p, q, p+q) \int_a^b |y'(x)|^{p+q} dx.$$

However in this paper we not study the equality cases, but this study is similar to the one from the paper of J. B. DIAZ and F. T. METCALF [8].

THEOREM 2.2. *If $y: [a, b] \to \mathbf{R}$ is absolutely continuous on $[a, b]$ and if $y(a) = 0$, then the following inequalities are valid*

(15) $$\int_a^b |y(x) y'(x)|^2 dx \leq \frac{4}{\pi^2} (b - a) \left[\int_a^b |y'(x)|^2 dx\right]^2$$

(16) $$\int_a^b |y(x)|^4 dx \leq \frac{3 \cdot 2^8 \pi^2}{\left[\Gamma\left(\frac{1}{4}\right)\right]^8} (b - a)^3 \left[\int_a^b |y'(x)|^2 dx\right]^2.$$

If moreover, $y: [a, b] \to \mathbf{R}$ has a derivative which is absolutely continuous on $[a, b]$ and if $y(a) = y'(a) = y''(a) = 0$, $y''(b) = 0$, then

(17) $$\int_a^b |y(x) y''(x)| dx \leq \frac{4}{\pi^4} (b - a)^4 \int_a^b |y'''(x)|^2 dx.$$

Proof. From (8) we can write with $p = q = r = 2$,

$$\int_a^b |y(x) y'(x)|^2 dx \leq K(2, 2, 2)(b - a) \left[\int_a^b |y'(x)|^2 dx\right]^2.$$

On the other hand from (8.3), we have

$$K(2,\ 2,\ 2) = B\left(\frac{3}{2},\ \frac{1}{2}\right)^{-2} = \frac{4}{\pi^2}.$$

If $p = 4$, $q = 0$, $r = 2$, the inequality (8) implies

$$\int_a^b |y(x)|^4\, dx \leq K(4,\ 0,\ 2)(b-a)^3 \left[\int_a^b |y'(x)|^2\, dx\right]^2$$

and (8.1) furnishes

$$K(4,\ 0,\ 2) = \frac{3 \cdot 2^8 \pi^2}{\left[\Gamma\left(\frac{1}{4}\right)\right]^8}.$$

Now let $y(a) = 0$; according to Wirtinger's inequality we give

$$\int_a^b |y(x)|^2\, dx \leq \frac{4}{\pi^2}(b-a)^2 \int_a^b |y'(x)|^2\, dx$$

and because $y'(a) = 0$ we have

$$\int_a^b |y(x)|^2\, dx \leq \frac{16}{\pi^4}(b-a)^2 \int_a^b |y''(x)|^2\, dx.$$

If $y''(a) = y''(b) = 0$ then

$$\int_a^b |y(x)|^2\, dx \leq \frac{16}{\pi^6}(b-a)^6 \int_a^b |y'''(x)|^2\, dx. \tag{18}$$

The Schwarz's inequality yields

$$\int_a^b |y(x)\, y''(x)|\, dx \leq \left\{\int_a^b |y(x)|^2\, dx\right\}^{\frac{1}{2}} \left\{\int_a^b |y''(x)|^2\, dx\right\}^{\frac{1}{2}},$$

and from (6) and (18)

$$\int_a^b |y(x)\, y''(x)|\, dx \leq \left\{\frac{16}{\pi^6}(b-a)^6 \int_a^b |y'''(x)|^2\, dx\right\}^{\frac{1}{2}} \left\{\frac{(b-a)^2}{\pi^2} \int_a^b |y'''(x)|^2\, dx\right\}^{\frac{1}{2}}$$

that is

$$\int_a^b |y(x)\, y''(x)|\, dx \leq \frac{4}{\pi^4}(b-a)^4 \int_a^b |y'''(x)|^2\, dx.$$

§ 3. The cases $n = 2$, $p = +\infty$ and $n = 2$, $p = 2$

THEOREM 3.1. *If*

$$\frac{4}{3\pi^2} h^2 ||a_2||_\infty + \frac{1}{6} h ||a_1||_\infty \leq 1 \tag{19}$$

then the operator $L_{2,\infty}$ has the interpolation property $L_2[a,\ a+h)$.

Proof. If we assume that $L_{2,\infty}$ has not the interpolatory property $L_2[a, a+h)$ then there is at least a nontrivial solution of the equation

$$y'' + a_1(x)\, y' + a_2(x)\, y = 0,\ a_k \in L^\infty[a,\ a+h],\ k = 1,\ 2,$$

vanishing at the given points x_1 and x_2. Let $y_0 \in Y_2$ such that $y_0(x_1) = y_0(x_2) = 0$, $x_1 < x_2$, $x_1, x_2 \in [a,\ a+h]$.

Because

$$(y_0^3 y_0')' = 3y_0^2 y_0'^2 + y_0^3 y_0'' = 3y_0^2 y_0'^2 - y_0^3(a_1 y_0' + a_2 y_0)$$

we give

$$3\int_{x_1}^{x_2} |y_0(x)\, y_0'(x)|^2\, dx \leq ||a_1||_\infty \int_{x_1}^{x_2} |y_0(x)|^3\, |y_0'(x)|\, dx + ||a_2||_\infty \int_{x_1}^{x_2} |y_0(x)|^4\, dx. \tag{20}$$

Now let $\varphi\colon [a,\ a+h] \to \mathbf{R}$ be defined by

$$\varphi(x) = \frac{1}{2} y_0^2(x),\ x \in [a,\ a+h],\ \varphi(x_1) = \varphi(x_2) = 0.$$

In this case (20) implies

$$3\int_{x_1}^{x_2}|\varphi'(x)|^2\,dx \leq 2\|a_1\|_\infty\int_{x_1}^{x_2}|\varphi(x)\,\varphi'(x)|\,dx + 4\|a_2\|_\infty\int_{x_1}^{x_2}|\varphi(x)|^2\,dx.$$

By using (14) with $p = q = 1$, and then with $p = 2$, $q = 0$, we have

$$3\int_{x_1}^{x_2}|\varphi'(x)|^2\,dx \leq \frac{1}{2}\|a_1\|_\infty(x_2 - x_1)\int_{x_1}^{x_2}|\varphi'(x)|^2\,dx +$$

$$+\frac{4}{\pi^2}\|a_2\|_\infty(x_2 - x_1)^2\int_{x_1}^{x_2}|\varphi'(x)|^2\,dx.$$

This inequality yields

$$1 < \frac{1}{6}h\|a_1\|_\infty + \frac{4}{3\pi^2}h^2\|a_2\|_\infty \tag{21}$$

by taking into account that $x_2 - x_1 < h$ and that $\int_{x_1}^{x_2}|\varphi'(x)|^2\,dx > 0$. Thus (21) is a contradiction with (19) and therefore the theorem is proved.

THEOREM 3.2. *If*

$$\frac{16\pi\sqrt{3}}{\left[\Gamma\left(\frac{1}{4}\right)\right]^4}h^2M_2(a_2,\ h) + \frac{2}{\pi}hM_2(a_1,\ h) \leq 1 \tag{22}$$

*then the differential operator $L_{2,2}$ has the interpolation property $I_2[a, a + h)$**).

Proof. If $L_{2,2}$ has not property $I_2[a,\ a + h)$ then there is a nontrivial solution y_0 such that $y_0(x_1) = y_0(x_2) = 0$ where x_1 and x_2 are prescribed points in $[a,\ a + h)$.

On $[a,\ a + h]$ the following relations

$$(y_0y_0')' = y_0'^2 + y_0y_0'' = y_0'^2 - y_0[a_1y_0' + a_2y_0]$$

*) $\dfrac{16\pi\sqrt{3}}{\left[\Gamma\left(\frac{1}{4}\right)\right]^4} = 0{,}2063\ldots$

are valid almost everywhere, and we conclude with

$$\int_{x_1}^{x_2}|y_0(x)|^2\,dx \leq \left\{\int_{x_1}^{x_2}|y_0(x)\,y_0(x)|^2\,dx\right\}^{\frac{1}{2}}\left\{\int_{x_1}^{x_2}|a_1(x)|^2\,dx\right\}^{\frac{1}{2}} +$$

$$+\left\{\int_{x_1}^{x_2}y_0(x)|^4\,dx\right\}^{\frac{1}{2}}\left\{\int_{x_1}^{x_2}|a_2(x)|^2\,dx\right\}^{\frac{1}{2}}.$$

Note that $\left\{\int_{x_1}^{x_2}|a_k(x)|^2\,dx\right\}^{\frac{1}{2}} < \sqrt{h}\,M_2(a_k,\ h)$, $k = 1,\ 2$. Therefore from (15) and (16), we have

$$\int_{x_1}^{x_2}|y_0'(x)|^2\,dx < \frac{2}{\pi}hM_2(a_1,\ h)\int_{x_1}^{x_2}|y_0'(x)|^2\,dx + \frac{16\pi\sqrt{3}}{\left[\Gamma\left(\frac{1}{4}\right)\right]^4}h^2M_2(a_2,\ h)\int_{x_1}^{x_2}|y_0(x)|^2\,dx$$

or

$$1 < \frac{2}{\pi}hM_2(a_1,\ h) + \frac{16\pi\sqrt{3}}{\left[\Gamma\left(\frac{1}{4}\right)\right]^4}h^2M_2(a_2,\ h)$$

which is a contradiction.

§ 4. The case $n = 3$ and $2 \leq p < +\infty$

THEOREM 4.1. *If for* $\frac{1}{p} + \frac{1}{q} = 1$, $p \in [2,\ +\infty)$ *holds the inequality*

$$\frac{K(q,\ q,\ 2)^{1/q}}{\pi^2}h^3M_p(a_3,\ h) + K(2q,\ 0,\ 2)^{1/q}\,h^2M_p(a_2,\ h) + K(q,\ q,\ 2)^{1/q}\,hM_p(a_1,\ h) \leq 1 \tag{23}$$

then the differential operator $L_{3,p}$ has property $I_3[a,\ a + h)$. The constants $K(q,\ q,\ 2)$ and $K(2q,\ 0,\ 2)$ are defined by (8.1).

Proof. Let us suppose that $L_{3,p}$ has not property $I_3[a,\ a + h)$. In this case there is at least a nontrivial solution y_0 of the equation

$$y''' + a_1(x)y'' + a_2(x)y' + a_3(x)y = 0,\quad a_k \in L^p\,[a,\ a + h],\ (k = 1,\ 2,\ 3)$$

such that for $x_1 < x_2 < x_3$, y_0 verifies

$$y_0(x_1) = y_0(x_2) = y_0(x_3) = 0.$$

According to Rolle's theorem there is at least a pair of points (ξ_1, ξ_2), $x_1 < \xi_1 < x_2 < \xi_2 < x_3$, such that

$$y_0'(\xi_1) = y_0'(\xi_2) = 0.$$

In this case

$$\int_{\xi_1}^{\xi_2} [y_0''(x)]^2 dx = \int_{\xi_1}^{\xi_2} y_0'(x)\,[a_1(x)y_0''(x) + a_2(x)y_0'(x) + a_3(x)y_0(x)]\,dx$$

and by using the Hölder inequality with $\frac{1}{p} + \frac{1}{q} = 1$ we give

$$\int_{\xi_1}^{\xi_2} |y_0''(x)|^2\,dx \leq \left\{\int_{\xi_1}^{\xi_2} |y_0'(x)y_0''(x)|^q\,dx\right\}^{\frac{1}{q}} ||a_1||_p + \tag{24}$$

$$+ \left\{\int_{\xi_1}^{\xi_2} |y_0'(x)|^{2q}\,dx\right\}^{\frac{1}{q}} ||a_2||_p + \left\{\int_{\xi_1}^{\xi_2} |y_0(x)y_0'(x)|^q\,dx\right\}^{\frac{1}{q}} ||a_3||_p.$$

From (8), we obtain

$$\int_{\xi_1}^{\xi_2} |y_0'(x)\,y_0''(x)|^q\,dx \leq (\xi_2 - \xi_1)\,K(q, q, 2)\left\{\int_{\xi_1}^{\xi_2} |y_0''(x)|^2\,dx\right\}^q \tag{24.1}$$

$$\int_{\xi_1}^{\xi_2} |y_0'(x)|^{2q}\,dx \leq (\xi_2 - \xi_1)^{q+1}\,K(2q, 0, 2)\left\{\int_{\xi_1}^{\xi_2} |y_0''(x)|^2\,dx\right\}^q \tag{24.2}$$

$$\int_{\xi_1}^{\xi_2} |y_0(x)y_0'(x)|^q\,dx \leq (\xi_2 - \xi_1)\,K(q, q, 2)\left\{\int_{\xi_1}^{\xi_2} |y_0'(x)|^2\,dx\right\}^q \leq \tag{24.3}$$

$$\leq \frac{K(q, q, 2)}{\pi^{2q}}\,(\xi_2 - \xi_1)^{2q+1}\left\{\int_{\xi_2}^{\xi_2} |y_0''(x)|^2\,dx\right\}^q.$$

Substituting (24.1) — (24.3) in (24) and taking into account that $\xi_2 - \xi_1 < h$ and that $||a_k||_p = h^{\frac{1}{p}}\,M_p(a_k, h)$, $k = 1, 2, 3$ we conclude with

$$1 < K(q, q, 2)^{1/q}\,hM_p(a_1, h) + K(2q, 0, 2)^{1/q}\,h^2M_p(a_2, h) +$$

$$+ \frac{K(q, q, 2)^{1/q}}{\pi^2}\,h^3M_p(a_3, h).$$

This contradiction completes the proof of this theorem.

We can carry out some special cases of (23). For $p = q = 2$

$$K(2, 2, 2)^{1/2} = \frac{2}{\pi}$$

and

$$K(4, 0, 2)^{1/2} = \frac{16\pi\sqrt{3}}{\left[\Gamma\left(\frac{1}{4}\right)\right]^4}$$

Thus there is true the following statement.

THEOREM 4.2. *If*

$$\frac{2}{\pi^3}\,h^3M_2(a_3, h) + \frac{16\pi\sqrt{3}}{\left[\Gamma\left(\frac{1}{4}\right)\right]^4}\,h^2M_2(a_2, h) + \frac{2}{\pi}\,hM_2(a_1, h) \leq 1 \tag{25}$$

then the operator $L_{3,2}$ *has property* $I_3[a, a + h)$.

The inequality (25) may be regarded as an improvment of (3).

§ 5. The cases $n = 4$, $p = +\infty$, and $n = 4$, $p = 2$.

THEOREM 5.1. *If*

$$\frac{4}{\pi^4}\,h^4||a_4||_\infty + \frac{1}{2\pi^2}\,h^3||a_3||_\infty + \frac{1}{\pi^2}\,h^2||a_2||_\infty + \frac{1}{4}\,h||a_1||_\infty \leq 1 \tag{26}$$

then the operator $L_{3,\infty}$ *has property* $H_4[a, a + h)$.

Proof. If $L_{4,\infty}$ has not property $H_4[a, a+h)$ then for each points $x_1 < x_2$ there is at least a nontrivial solution y_0 of the equation

$$y^{(IV)} + a_1(x)\,y''' + a_2(x)\,y'' + a_3(x)\,y' + a_4(x)\,y = 0$$

$$a_k \in L^\infty[a,\ a+h],\quad k = 1,\ \ldots,\ 4,$$

with the property

$$y_0(x) = y_0'(x_1) = y_0''(x_1) = 0$$

and

$$y_0^{(s)}(x_2) = 0.$$

where s is one of the numbers 0, 1, 2.

If $s = 0$ there are the points ξ_1, ξ_2

$$x_1 < \xi_2 < \xi_1 < x_2$$

such that $y_0(\xi_1) = 0$ and $y_0''(\xi_2) = 0$.

In this case we select $\alpha = x_1$ and $\beta = \xi_2$

If $s = 1$ there is at least a point $\eta_2 \in (x_1,\ x_2)$ such that $y_0''(\eta_2) = 0$ and we set $\alpha = x_1$ and $\beta = \eta_2$. If $s = 2$ we assume that $\alpha = x_1$ and $\beta = x_2$. Because

$$(y_0''y_0''')' = y_0'''^2 - y_0''[a_1(x)\,y_0''' + a_2(x)\,y_0'' + a_3(x)\,y_0 + a_4(x)\,y_2]$$

with

$$\int_\alpha^\beta |y_0'''(x)|^2\,dx = \Omega$$

we have

$$\Omega \leq ||a_1||_\infty \int_\alpha^\beta |y_0''(x)\,y_0'''(x)|\ dx + ||a_2||_\infty \int_\alpha^\beta |y_0''(x)|^2\,dx +$$

$$+ ||a_3||_\infty \int_\alpha^\beta |y_0'(x)\,y_0''(x)|\,dx + ||a_4||_\infty \int_\alpha^\beta |y_0(x)\,y_0''(x)|\,dx.$$

We observe that in all cases $y_0(\alpha) = y_0'(\alpha) = y_0''(\alpha) = 0$ and $y_0''(\beta) = 0$ From Opial's inequality

$$\int_\alpha^\beta |y_0''(x)\,y_0'''(x)|\,dx < \frac{h}{4}\,\Omega$$

and from Wirtinger's inequality

$$\int_\alpha^\beta |y_0''(x)|^2\,dx < \frac{h^2}{\pi^2}\,\Omega.$$

Likewise

$$\int_\alpha^\beta |y_0'(x)\ y_0''(x)|\,dx < \frac{h}{2}\int_\alpha^\beta |y_0''(x)|^2\,dx < \frac{h^3}{2\pi^2}\,\Omega$$

and from Theorem 2.2 (inequality (17)) we have

$$\int_\alpha^\beta |y_0(x)\ y_0''(x)|\,dx < \frac{4}{\pi^4}\,h^4\Omega.$$

Thus we see that

$$1 < \frac{h}{4}\,||a_1||_\infty + \frac{h^2}{\pi^2}\,||a_2||_\infty + \frac{h^3}{2\pi^2}\,||a_3||_\infty + \frac{4}{\pi^4}\,h^4||a_4||_\infty$$

which is a contradiction.

Remark. The inequality (26) is an improvment of the inequality (5).

THEOREM 5.2. *If h satisfies the following inequality*

$$\frac{1}{\pi^2\sqrt{6}}\,h^4M_2(a_4,\ h) + \frac{2}{\pi^3}\,h^3M_2(a_3,\ h) + \frac{16\pi\sqrt{3}}{\left[\Gamma\left(\frac{1}{4}\right)\right]^4}\,h^2M_2(a_2,\ h) + \tag{27}$$

$$+ \frac{2}{\pi}\,hM_2(a_1,\ h) \leq 1$$

then $L_{4,2}$ has property $H_4[a,\ a+h)$.

Proof. As in the proof of theorem 5.1 let us suppose that y_0 is a nontrivial solution from Y_4 such that

$$y_0(\alpha) = y_0'(\alpha) = y_0''(\alpha) = 0, \; y_0''(\beta) = 0.$$

Let

$$J = \int_\alpha^\beta |y_0'''(x)|^2 \, dx.$$

In [1] O. ARAMĂ proves the inequality

$$\int_\alpha^\beta |y_0(x) y_0''(x)|^2 \, dx < \frac{(\beta - \alpha)^7}{6\pi^4} J^2. \tag{28}$$

Also from (15) and (16) we have

$$\int_\alpha^\beta |y_0''(x) \, y_0'''(x)|^2 \, dx \leq \frac{4}{\pi^2} (\beta - \alpha) \, J^2 \tag{29}$$

$$\int_\alpha^\beta |y_0''(x)|^4 \, dx \leq \frac{3 \cdot 2^8 \cdot \pi^2}{\left[\Gamma\left(\frac{1}{4}\right)\right]^8} (\beta - \alpha)^3 J^2. \tag{30}$$

In the same manner

$$\int_\alpha^\beta |y_0'(x) y_0''(x)|^2 \, dx \leq \frac{4}{\pi^2} (\beta - \alpha) \left\{ \int_\alpha^\beta | \, y_0''(x)|^2 \; d \right\}^2 \leq \tag{31}$$

$$\leq \frac{4}{\pi^6} (\beta - \alpha)^5 J^2.$$

From

$$J = \int_\alpha^\beta a_1(x) \, y_0''(x) \, y_0'''(x) \, dx + \int_\alpha^\beta a_2(x) \, [y_0''(x)]^2 \, dx +$$

$$+ \int_\alpha^\beta a_3(x) \, y_0'(x) \, y_0''(x) \, dx + \int_\alpha^\beta a_4(x) \, y_0(x) \, y_0''(x) \, dx$$

we give

$$J \leq ||a_1||_2 \left\{ \int_\alpha^\beta |y_0''(x) \, y_0'''(x)|^2 \, dx \right\}^{\frac{1}{2}} + ||a_2||_2 \left\{ \int_\alpha^\beta |y_0''(x)|^4 \, dx \right\}^{\frac{1}{2}} +$$

$$+ ||a_3||_2 \left\{ \int_\alpha^\beta |y_0'(x) \, y_0''(x)|^2 \, dx \right\}^{\frac{1}{2}} + ||a_4||_2 \left\{ \int_\alpha^\beta |y_0(x) \, y_0''(x)|^2 \, dx \right\}^{\frac{1}{2}}$$

and using (28)—(31) we conclude with

$$J < \frac{2}{\pi} \sqrt{h} \, ||a_1||_2 \cdot J + \frac{16\pi\sqrt{3}}{\left[\Gamma\left(\frac{1}{4}\right)\right]^4} h \sqrt{h} \, ||a_2||_2 J +$$

$$+ \frac{2}{\pi^3} h^2 \sqrt{h} \, ||a_3||_2 J + \frac{1}{\pi^2\sqrt{6}} h^3 \sqrt{h} \, ||a_4||_2 \cdot J$$

Dividing the inequality with J, $J > 0$, we give a contradiction, namely

$$1 < \frac{2}{\pi} h M_2(a_1, \; h) + \frac{16\pi\sqrt{3}}{\left[\Gamma\left(\frac{1}{4}\right)\right]^4} h^2 M_2(a_2, \; h) +$$

$$+ \frac{2}{\pi^3} h^3 M_2(a_3, \; h) + \frac{1}{\pi^2\sqrt{6}} h^4 M_2(a_4, \; h).$$

Remark. The inequality (27) may be compared with (4).

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ON STARSHAPEDNESS PRESERVING PROPERTIES OF A CLASS OF LINEAR POSITIVE OPERATORS

by

LUCIANA LUPAŞ
Cluj

1. In this paper we study the behavior of the sequence of linear positive operators, which were introduced by I. I. HIRSCHMAN and D. V. WIDDER [1], on the class of starshaped functions. Likewise it is our purpose and to discuss the behavior of the sequence of approximation operators constructed by D. LEVIATAN [2]. As a special case will result the starshapedness-preserving property of the S. N. Bernstein's operators [5].

2. A function $f:I \to \mathbf{R}$ is called *starshaped* on the interval I, if for every $a \in [0, 1]$ hold the inequality

$$f(ax) \leq af(x), \quad x \in I. \tag{1}$$

In the following we denote by $\mathcal{S}[0,1]$ the class of the starshaped functions on $[0, 1]$ and which have the following properties:

(a) $f(0) = 0$,
(b) f is nonnegative on $[0, 1]$.
(c) $f \in C[0, 1]$.

Lemma 1. *A necessary and sufficient condition for a continuous differentiable function f which has the properties (a)–(c) on the interval $[0, 1]$, to be starshaped on $[0, 1]$ is that*

$$f'(x) \geq \frac{f(x)}{x} \tag{2}$$

whatever $x \in (0, 1]$.