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ON SOME INEQUALITIES

by

ALEXANDRU LUPAŞ

Cluj

1. In this paper we prove two general inequalities which contain as special cases some results of F. RIESZ [2] and M. BIERNACKI [1].

THEOREM 1. *Let* (a_n), $n = 1, 2, \ldots$, *be a sequence of positive numbers and* $\|p_{ij}\|$, $i \leqq j$, $j = 1, 2, \ldots$, *a triangular-matrix formed with positive numbers. If for each natural numbers* h, n, $h < n$, *there is a constant* K, $K \geqq 1$, *such that*

$$\frac{a_{h+1}p_{h+1,n} + a_{h+2}p_{h+2,n} + \cdots + a_n p_{n,n}}{p_{h+1,n} + p_{h+2,n} + \cdots + p_{n,n}} \leqq K a_h \tag{1}$$

then

$$a_n = O(\varepsilon_n), \quad n = 1, 2, \ldots,$$

where

$$\varepsilon_n = \left(\frac{p_{1,n} + p_{2,n} + \cdots + p_{n,n}}{p_{n,n}}\right)^{1-\frac{1}{K}}$$

Proof. We define the functions $f : (0, +\infty) \to R$ and $w_n : (0, +\infty) \to R$, $n = 1, 2, \ldots$, by

$$\left.\begin{aligned} f(t) &= a_k, & t \in (k-1, k], \\ w_n(t) &= p_{k,n}, & t \in (k-1, k] \end{aligned}\right\} k = 1, 2, \ldots$$

Likewise let $F_n : [0, n-1] \to R$ be defined as

$$F_n(x) = \int_x^n f(t) w_n(t)\, dt. \tag{2}$$

For $x \in [0, n-1]$, if we assume that $y < x < y+1$ where $y = [x]$, we have

$$a_{y+2}p_{y+2,n} + \ldots + a_n p_{n,n} \leq K a_{y+1}(p_{y+2,n} + \ldots + p_{n,n})$$

therefore

$$a_{y+1}p_{y+1,n}(y - x + 1) + a_{y+2}p_{y+2,n} + \ldots + a_n p_{n,n} \leq$$

$$\leq K a_{y+1}\left[\frac{p_{y+1,n}(y-x+1)}{K} + p_{y+2} + \ldots + p_{n,n}\right].$$

Consequently

$$a_{y+1}p_{y+1,n}(y - x + 1) + a_{y+2}p_{y+2,n} + \ldots + a_n p_{n,n} \leq$$

$$\leq K a_{y+1}[(y - x + 1)p_{y+1,n} + p_{y+2,n} + \ldots + p_{n,n}]$$

and by (2) we conclude with

$$F_n(x) \leq K f(x) \int_x^n w_n(t)\, dt. \tag{3}$$

If $y = x$, that is if x is an integer, then (3) follows from (1). Therefore, by (2) and (3)

$$\frac{F_n'(x)}{F_n(x)} \leq -\frac{w_n(x)}{K\int_x^n w_n(t)\,\mathrm{d}t}, \quad x \in [0, n-1],$$

which implies

$$\ln F_n(j+1) - \ln F_n(j) \leq \frac{1}{K}[\ln W_n(j+1) - \ln W_n(j)]$$

where

$$W_n(x) = \int_x^n w_n(t)\, dt, \quad x \in [0, n-1].$$

This inequality implies the following

$$\sum_{j=0}^{n-2} \ln F_n(j+1) - \sum_{j=0}^{n-2} \ln F_n(j) \leq \frac{1}{K}\left[\sum_{j=0}^{n-2} \ln W_n(j+1) - \sum_{j=0}^{n-2} \ln W_n(j)\right],$$

that is

$$F_n(n-1) \leq F_n(0)\left[\frac{W_n(n-1)}{W_n(0)}\right]^{\frac{1}{K}}, \quad n = 1, 2, \ldots. \tag{4}$$

From (1) with $h = 1$, we have

$$F_n(0) = a_1 p_{1,n} + \sum_{j=2}^n a_j p_{j,n} \leq a_1 p_{1,n} + K a_1 \sum_{j=2}^n p_{j,n} \leq K a_1 \sum_{j=1}^n p_{j,n}.$$

Because

$$W_n(n-1) = p_{n,n} \quad W_n(0) = \sum_{j=1}^n p_{j,n}, \quad F_n(n-1) = a_n p_{n,n},$$

by (4) we conclude that there is a constant C, such that

$$a_n \leq C\left[\frac{1}{p_{n,n}} \sum_{j=1}^n p_{j,n}\right]^{1-\frac{1}{K}}$$

and the proof of this theorem is complete.

THEOREM 2. *Let (a_n), (p_n), $n = 1, 2, \ldots$, are sequences of positive numbers. If there is a constant K, $K \in (0, 1]$, such that*

$$\frac{\sum_{j=1}^n a_j p_j}{\sum_{j=1}^n p_j} \geq K a_{n+1}, \quad n = 1, 2, \ldots, \tag{5}$$

then

$$\sum_{j=1}^n a_j p_j = O(\delta_n), \quad n = 1, 2, \ldots,$$

where

$$\delta_n = \left(\sum_{j=1}^n p_j\right)^{\frac{1}{K}}.$$

Proof. By defining the functions $g: [0, +\infty) \to R$ and $z: [0, +\infty) \to R$, as

$$\left.\begin{array}{ll} g(t) = a_k, & t \in [k-1, k) \\ z(t) = p_k, & t \in [k-1, k) \end{array}\right\}, \quad k = 1, 2, \ldots,$$

it is easy to observe that

$$G(x) \geqq Kg(x)\int_0^x z(t)\,dt, \quad x \in [1, +\infty), \tag{6}$$

where $G: [1, +\infty) \to R$ is defined by

$$G(x) = \int_0^x g(t)z(t)\,dt.$$

Indeed, let us suppose that x is an integer, namely $x = n$; in this case

$$g(n) = a_{n+1}, \quad \int_0^n z(t)\,dt = \sum_{j=1}^n p_j, \quad G(x) = \sum_{j=1}^n a_j p_j$$

and (6) follows from (5).

If $y < x < y + 1$, $y = [x]$, then (6) may be write in the following form

$$(x-y)a_{y+1}p_{y+1} + \sum_{j=1}^y a_j p_j \geqq K a_{y+1}\left[(x-y)p_{y+1} + \sum_{j=1}^y p_j\right]$$

or

$$\sum_{j=1}^y a_j p_j \geqq K a_{y+1}\left[(x-y)\left(1 - \frac{1}{K}\right) p_{y+1} + \sum_{j=1}^y p_j\right]$$

which is a consequence of (5), in accordance with the fact that $K \in (0, 1]$

Because $G'(x) = g(x)z(x)$, we have

$$\frac{G'(x)}{G(x)} \leqq \frac{1}{K} \cdot \frac{Z'(x)}{Z(x)}, \quad x \in [1, +\infty), \tag{7}$$

where

$$Z(x) = \int_0^x z(t)\,dt.$$

In the same manner as in the proof of theorem 1, we obtain

$$G(n) \leqq G(1)\left[\frac{Z(n)}{Z(1)}\right]^{\frac{1}{K}}, \quad n = 1, 2, \ldots. \tag{8}$$

Since

$$G(n) = \sum_{j=1}^n a_j p_j, \quad G(1) = a_1 p_1, \quad Z(n) = \sum_{j=1}^n p_j,$$

there is a constant C such that

$$\sum_{j=1}^n a_j p_j \leqq C\left(\sum_{j=1}^n p_j\right)^{\frac{1}{K}}, \quad n = 1, 2, \ldots,$$

and the theorem is proved.

Corollary 1. *If (a_n), $n = 1, 2, \ldots$, is a sequence of positive numbers and for each natural numbers h, n, $h < n$, there is a constant $K (K \geqq 1)$ such that*

$$\frac{1^p a_n + 2^p a_{n-1} + \ldots + (n-h)^p a_{h+1}}{1^p + 2^p + \ldots + (n-h)^p} \leqq K a_h$$

then

$$a_n = O(\varepsilon_{n,p}), \quad n = 1, 2, \ldots,$$

where

$$\varepsilon_{n,0} = n^{1-\frac{1}{K}}, \quad \varepsilon_{n,1} = n^{2-\frac{2}{K}}, \quad \varepsilon_{n,2} = n^{3-\frac{3}{K}}, \quad \varepsilon_{n,3} = n^{4-\frac{4}{K}}.$$

The orders $\varepsilon_{n,0}$ and $\varepsilon_{n,1}$ were discovered by M. BIERNACKI [1] which improved a result of F. RIESZ [2].

Corollary 2. *Let (a_n), $n = 1, 2, \ldots$, be a sequence of positive numbers. If there is a constant K, $K \in (0, 1]$, such that*

$$\frac{\sum_{k=1}^n k^p a_k}{\sum_{k=1}^n k^p} \geqq K\, a_{n+1}, \quad n = 1, 2, \ldots,$$

then

$$\sum_{k=1}^n k^p a_k = O(\delta_{n,p}), \quad n = 1, 2, \ldots,$$

where

$$\delta_{n,p} = n^{\frac{p+1}{K}}, \quad p = 0, 1, 2, 3.$$

Corollary 3. *Let $f: (0, +\infty) \to (0, +\infty)$, $g: (0, +\infty) \to (0, +\infty)$ are continuous functions on $(0, +\infty)$. If there is a constant K, $K \in (0, 1]$, such that*

$$\frac{\int_0^n f(x)g(x)\,dx}{\int_0^n g(x)\,dx} \geqq K \frac{\int_n^{n+1} f(x)g(x)\,dx}{\int_n^{n+1} g(x)\,dx}, \quad n = 1, 2, \ldots,$$

then there is a constant C such that

$$\int_0^n f(x)g(x)\,dx \leq C\left[\int_0^n g(x)\,dx\right]^{\frac{1}{K}},\quad n = 1, 2, \ldots$$

Proof. In the second theorem we put

$$p_j = \int_{j-1}^{j} g(x)\,dx,\quad j = 1, 2, \ldots,$$

and a_j are defined by

$$a_j\int_{j-1}^{j} g(x)\,dx = \int_{j-1}^{j} f(x)g(x)\,dx,\quad j = 1, 2, \ldots.$$

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SUR LA GÉOMÉTRIE DE LA REPRÉSENTATION CONFORME

par

PETRU T. MOCANU

à Cluj

1. Dans l'étude de l'aspect géométrique de la représentation conforme un rôle important est joué par l'étude de l'allure des images de quelques familles de courbes (lignes de niveau, etc.) par la transformation considérée.

On connaît les conditions nécessaires et suffisantes pour que l'image d'une courbe donnée soit convexe ou étoilée par rapport à un point. Dans la présente note nous voulons insister sur une propriété géométrique intermédiaire entre la convexité et l'étoilement qui les contient comme deux cas limite. D'une manière générale on peut définir cette propriété comme suit.

Soit $C_a : z = \varphi(t, a)$, $t \in [t_0, t_1]$, $a \in [0, +\infty)$, une famille de courbes de Jordan de classe C^2, qui sont isotopes et qui se contractent à l'origine, c'est à dire $\varphi(t, 0) = 0$, $\varphi(t_0, a) = \varphi(t_1, a)$. Considérons deux fonctions f et g holomorphes et univalentes dans un domaine qui contient ces courbes et notons $\Gamma_a^f = f(C_a)$, $\Gamma_a^g = g(C_a)$. Supposons que $f(0) = g(0) = 0$ et que pour tout a et α qui vérifient l'inégalité $\alpha \leqslant a$ on a $\Gamma_\alpha^g \subset (\Gamma_a^f)$, en désignant par (C) le domaine intérieur à la courbe de Jordan C.

Définition 1. *On dit que la courbe Γ_a^f est étoilée par rapport à Γ_α^g, où $\alpha \leq a$, si l'angle fait avec l'axe réel par chaque demitangente à la courbe Γ_α^g menée du point $f(z) \in \Gamma_a^f$, $z = \varphi(t, a)$, varie d'une manière monotone avec le paramètre t.*

Lorsque $f = g$ et les courbes C_a sont des cercles centrés à l'origine, cette définition a été donnée dans un travail antérieur [2].