und

$$\partial F(x_0) = \{x^* \in \partial F(\theta_X) : x^*(x_0) = F(x_0)\}$$

für alle $x_0 \in X \setminus \{\theta_X\}$. Beachtet man dieses Ergebnis und Hilfssatz 2.1, so erhält man

SATZ 3.2. *Es sei X ein lokalkonvexer topologischer Vektorraum und $F: X \to [0, +\infty[$ eine stetige Halbnorm. Eine nichtleere Teilmenge V von X ist genau dann eine Sonne bezüglich F, wenn für jedes $x_0 \in X$ gilt:*

Ist W eine nichtleere Teilmenge von $\mathfrak{M}_F[x_0; V]$, dann gibt es zu jedem Element v aus V ein Funktional $x_v^* \in Ep(\partial F(\theta_X))$, so dass

$$x_v^*(w - x_0) = F(w - x_0) \text{ und } \operatorname{Re} x_v^*(w - v) \leq 0 \tag{18}$$

für alle $w \in W$ gilt.

Korollar 3.2. *Es sei X ein lokalkonvexer topologischer Vektorraum, $F: X \to [0, +\infty[$ eine stetige Halbnorm und $V \subseteq X$ eine Sonne bezüglich F. Eine nichtleere Teilmenge W von V ist genau dann eine Menge von Minimallösungen bezüglich $(V; F)$ für ein Element x_0 aus X, wenn es zu jedem Element v aus V ein Funktional $x_v^* \in Ep(\partial F(\theta_X))$ gibt, so dass (18) für alle $w \in W$ gilt.*

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ON A PROBLEM OF RESTRICTIVE UNIFORM APPROXIMATION

by

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1. Introduction. Recently, various results, in the theory of approximating with a class of functions having restricted ranges has been obtained by F. DEUTSCH [4], P. J. LAURENT [5], G. D. TAYLOR [6] and the author of this note [3].

The purpose of this paper is to study a problem of restrictive uniform approximation. Thus we shall discuss the existence, the characterization and the uniqueness of the best approximation to a given continuous function. Some other results concerning the studied problem are given.

2. Definitions and notations. Let T be a compact metric space and let $C[T]$ denote the Banach space of all real-valued continuous functions defined on T with norm:

$$||f|| = \max_{t \in T} |f(t)|.$$

Definition 2.1. *M is an n-dimensional Haar subspace of $C[T]$ if the zero function is the only element of M which vanishes at n or more distinct points of T.*

Let P, Q be two finit-dimensional Haar subspaces of $C[T]$ and let $\mathfrak{U}$, $\mathfrak{V}$ denote two operators:

$$\mathfrak{U}: P \to Q \text{ and } \mathfrak{V}: Q \to P.$$

For the remainder of this note we assume that $\mathfrak{U}$ is an onto and linear operator.

We shall fix away the functions $f \in C[T] \setminus \bar{P}$ and $g \in C[T] \setminus \bar{Q}$ and define the following numbers:

$$\rho_0 = \|f - p_0\| = \inf_{p \in P} \|f - p\|, \; p_0 \in P,$$

$$\eta_0^* = \|g - \mathfrak{U}(p_0)\|,$$

$$\eta_0 = \|g - q_0\| = \inf_{q \in Q} \|g - q\|, \quad q_0 \in Q,$$

$$\rho_0^* = \|f - \mathfrak{V}(q_0)\|.$$

We observe that:

$$\rho_0 < \rho_0^* \quad \text{and} \quad \eta_0 < \eta_0^*$$

For a fixed $\eta \in [\eta_0, \eta_0^*]$ we define the set of approximating functions:

$$P_\eta = \{p \in P : \|g - \mathfrak{U}(p)\| \leq \eta\}.$$

Definition 2.2. *An element $p^* \in P_\eta$ is said to be a best approximation in P_η to f if:*

$$\rho(\eta) = \|f - p^*\| = \inf_{p \in P_\eta} \|f - p\|. \tag{1}$$

This section is ended with:

Definition 2.3. *The operator $\mathfrak{U} : P \to Q$ is said positive if it satisfies the condition:*

$$\text{If } f \in P, \; f \geq 0 \text{ then } \mathfrak{U}(f) \geq 0, \; \mathfrak{U}(f) \in Q. \tag{2}$$

3. Existence of best approximations and other properties. We begin this section by proving some lemmas:

Lemma 3.1. *For a fixed $\eta \in [\eta_0, \eta_0^*]$ the set P_η is nonempty.*

Proof. We shall prove that the element:

$$p_2 = p_0' + (\eta_0 - \eta)\, p_1,$$

where $\mathfrak{U}(p_0') = q_0$, $p_1 \in P$ and $\|\mathfrak{U}(p_1)\| \leq 1$, is from P_η.

Indeed:

$$\|g - \mathfrak{U}(p_2)\| = \|g - \mathfrak{U}(p_0') + (\eta - \eta_0)\, \mathfrak{U}(p_1)\| \leq$$
$$\leq \|g - \mathfrak{U}(p_0')\| + (\eta - \eta_0)\, \|\mathfrak{U}(p_1)\| \leq \eta.$$

Lemma 3.2. *P_η is a closed convex set.*

Proof. If $p_1 \in P_\eta$, $p_2 \in P_\eta$, $\lambda \geq 0$, $\mu \geq 0$, $\lambda + \mu = 1$ then:

$$\|g - \mathfrak{U}(\lambda p_1 + \mu p_2)\| \leq \lambda \|g - \mathfrak{U}(p_1)\| + \mu \|g - \mathfrak{U}(p_2)\| \leq (\lambda + \mu)\, \eta = \eta.$$

Now we prove an existence theorem:

THEOREM 3.3. *For a fixed $\eta \in [\eta_0, \eta_0^*]$, there exist at least one best approximation $p^* \in P_\eta$ to f.*

Proof. Using the Lemma 3.2. and the fact that P is a finite-dimensional subspace we conclude that P_η is a compact set.

If $p_1 \in P_\eta$ is not a best approximation to f then:

$$\varepsilon = \|f - p_1\| > \rho(\eta)$$

and for $p \in P$ such that $\|p - p_1\| > 2\varepsilon$ we have:

$$\|f - p\| > \big|\, \|p - p_1\| - \|p_1 - f\| \,\big| > \varepsilon.$$

Letting:

$$D = \{p \in P : \|p - p_1\| \leq 2\varepsilon\},$$

we have:

$$\rho(\eta) = \inf_{p \in D_\eta} \|f - p\|, \; D_\eta = D \cap P_\eta,$$

the infimum being attained taking into account that D_η is compact.

It is possible to prove in an easy way the following statements:

THEOREM 3.4. *The set of the best approximations in P_η to f is a convex set.*

THEOREM 3.5. *$\rho(\eta)$ is a strictly decreasing and convex function on $[\eta_0, \eta_0^*]$.*

4. Characterizations of best approximations. To begin the characterization theorems of best approximations in P_η to f we define for $p^* \in P_\eta$ and f the following point sets:

$$\Gamma_+^* = \{t \in T : f(t) - p^*(t) = \|f - p^*\|\},$$

$$\Gamma_-^* = \{t \in T : f(t) - p^*(t) = -\|f - p^*\|\},$$

$$\Lambda_+^* = \{t \in T : \mathfrak{U}(p^*)(t) + \eta = g(t)\},$$

$$\Lambda_-^* = \{t \in T : \mathfrak{U}(p^*)(t) - \eta = g(t)\},$$

$$\Gamma^* = \Gamma_+^* \cup \Gamma_-^*, \; \Lambda = \Lambda_+^* \cup \Lambda_-^*,$$

$$\Sigma_+^* = \Gamma_+^* \cup \Lambda_+^*,$$

$$\Sigma_-^* = \Gamma_-^* \cup \Lambda_-^*.$$

Now we shall begin with two lemmas:

Lemma 4.1. *If*

$$\Sigma_+^* \cap \Sigma_-^* \neq \emptyset$$

and $\mathfrak{U}$ *is a positive operator then* p^* *is a best approximation to* f.

Proof. We have the cases $\Gamma_+^* \cap \Lambda_-^* \neq \emptyset$ and $\Gamma_+^* \cap \Lambda_-^* \neq \emptyset$. We shall study only the first case.

If $t_0 \in \Gamma_+^* \cap \Lambda_-^*$, $p_1 \in P_\eta$ and $p_1(t_0) > p^*(t_0)$ then $\mathfrak{U}(p_1)(t_0) \geq \mathfrak{U}(p^*)(t_0) = g(t_0) + \eta$. Thus $p_1 \notin P_\eta$ which is a contradiction.

Lemma 4.2. *If for a function* $p^* \in P_\eta$ *there exists a function* $p \in P$ *satisfing*:

$$p(t) > 0 \quad \text{for all} \quad t \in \Gamma_+^* \tag{3}$$

$$p(t) < 0 \quad \text{for all} \quad t \in \Gamma_-^* \tag{4}$$

$$\mathfrak{U}(p)(t) > 0 \quad \text{for all} \quad t \in \Lambda_+^* \tag{5}$$

$$\mathfrak{U}(p)(t) < 0 \quad \text{for all} \quad t \in \Lambda_-^* \tag{6}$$

then p^* *is not a best approximation to* f.

Proof. We shall prove that for an $\varepsilon > 0$ the element

$$\tilde{p} = p^* + \varepsilon p \text{ is from } P_\eta \quad \text{and} \quad ||f - \tilde{p}|| < ||f - p^*|| = \rho.$$

First we observe that Γ^*, Λ_+^*, Λ_-^* are compact sets and if $\alpha = \min\limits_{t \in \Gamma^*} |p(t)|$ then $\alpha > 0$.

In the followings we denote $\sigma(t_0, \delta)$ an open ball with the radius δ in the metric space T.

Since p and $\mathfrak{U}(p)$ are two uniform continuous functions on T there exists a $\delta > 0$ such that:

— $|p(t)| > \alpha$ and $|f(t) - p^*(t)| > r\rho$ for all $t \in \sigma(t_0, \delta)$ and for all $t_0 \in \Gamma^*$, where $0 < r < 1$.

— $\mathfrak{U}(p)(t) > 0$ for all $t \in \sigma(t_0, \delta)$ and for all $t_0 \in \Lambda_+^*$.

— $\mathfrak{U}(p)(t) < 0$ for all $t \in \sigma(t_0, \delta)$ and for all $t_0 \in \Lambda_-^*$.

Now we define:

$$S_\Gamma^* = \bigcup_{t_0 \in \Gamma^*} \sigma(t_0, \delta) \cap T \quad \text{and} \quad \tilde{S}_\Gamma = T - S_\Gamma^*$$

$$S_+^* = \bigcup_{t_0 \in \Lambda_+^*} \sigma(t_0, \delta) \cap T \quad \text{and} \quad \tilde{S}_+ = T - S_+^*$$

$$S_-^* = \bigcup_{t_0 \in \Lambda_-^*} \sigma(t_0, \delta) \cap T \quad \text{and} \quad \tilde{S}_- = T - S_-^*$$

where $\tilde{S}_\Gamma$, $\tilde{S}_+$, $\tilde{S}_-$ are compact sets.

For the determination of the desired ε we shall consider the following cases:

1) Let $t \in S_\Gamma^*$. Then:

$$|f(t) - \tilde{p}(t)| = |f(t) - p^*(t) - \varepsilon p(t)| \leq$$
$$\leq |f(t) - p^*(t)| - \varepsilon|p(t)|$$

if $|f(t) - p^*(t)| > \varepsilon|p(t)|$, that is $\varepsilon < r\rho||p||^{-1}$.

2) Let $t \in \tilde{S}_\Gamma$. If we denote $\tilde{\rho} = \max\limits_{t \in \tilde{S}_\Gamma} |f(t) - p^*(t)|$, we certainly have $\tilde{\rho} < \rho$. Then:

$$|f(t) - \tilde{p}(t)| \leq \tilde{\rho} + \varepsilon||p|| < \rho, \text{ if } \varepsilon < (\rho - \tilde{\rho})||p||^{-1}.$$

3) Let $t \in S_-^*$. Then:

$$\mathfrak{U}(\tilde{p})(t) - \eta - g(t) = \mathfrak{U}(p^*)(t) - \eta - g(t) + \varepsilon\mathfrak{U}(p)(t) \leq 0$$

because $\mathfrak{U}(p^*)(t) - \eta \leq g(t)$ and $\mathfrak{U}(p)(t) < 0$.

4) Let $t \in S_+^*$. Then:

$$\mathfrak{U}(\tilde{p})(t) + \eta - g(t) = \mathfrak{U}(p^*)(t) + \eta - g(t) + \varepsilon\mathfrak{U}(p)(t) \geq 0$$

because $\mathfrak{U}(p^*)(t) + \eta \geq g(t)$ and $\mathfrak{U}(p)(t) > 0$.

5) Let $t \in \tilde{S}_-$ and denote:

$$\alpha^- = \min_{t \in \tilde{S}_-} \{g(t) + \eta - \mathfrak{U}(p^*)(t)\} > 0.$$

Then:

$$g(t) + \eta - \mathfrak{U}(\tilde{p})(t) = g(t) + \eta - \mathfrak{U}(p^*)(t) - \varepsilon\mathfrak{U}(p)(t) \geq$$
$$\geq \alpha^- - \varepsilon\, \mathfrak{U}(p)(t) \geq \alpha^- - \varepsilon||\mathfrak{U}(p)|| \geq 0$$

if $\varepsilon < \alpha^-||\mathfrak{U}(p)||^{-1}$.

6) Let $t \in \tilde{S}_+$ and denote:

$$\alpha+ = \min_{t \in \tilde{S}_+} \{\mathfrak{U}(p^*)(t) - g(t) + \eta\} > 0.$$

Then:

$$\mathfrak{U}(\tilde{p})(t) - g(t) + \eta = \mathfrak{U}(p^*)(t) - g(t) + \eta + \varepsilon\mathfrak{U}(p)(t) \geqq$$
$$\geqq \alpha^+ - \varepsilon||\mathfrak{U}(p)|| \geqq 0$$

if $\varepsilon < \alpha^+||\mathfrak{U}(p)||^{-1}$.

Thus if:

$$0 < \varepsilon < \min \{r\rho||p||^{-1}, (\rho - \tilde{\rho})||p||^{-1}, \alpha^-||\mathfrak{U}(p)||^{-1}, \alpha^+||\mathfrak{U}(p)||^{-1}\}$$

the element $\tilde{p}$ is from P_η and $||f - \tilde{p}|| < \rho$.

By the Lemma 4.2. may be easy proved the following theorem:

THEOREM 4.3. *If $\mathfrak{U}$ is a positive operator and $p^* \in P_\eta$ is a best approximation to f then the sets Σ^*_+ and Σ^*_- are nonempty.*

A first characterization theorem, by Lemma 3.2., is a corolar of Theorem 8. from [1]. We give:

THEOREM 4.4. *The element $p^* \in P_\eta$ is a best approximation to f if and only if for any function $p \in P_\eta$ there exists a point $t' \in \Gamma^*_+$ such that:*

(7) $$p(t') \leqq p^*(t')$$

or a point $t'' \in \Gamma^*_-$ such that:

(8) $$p(t'') \geqq p^*(t'').$$

Remark. If $\mathfrak{U}$ is a positive operator the Theorem 4.4., in the case $\Sigma^*_+ \cap \Sigma^*_- = \emptyset$ holds, by replacing of Γ^*_+ by Σ^*_+ and Γ^*_- by Σ^*_-.

We shall state the following theorem, too:

THEOREM 4.5. *Suppose that:*

(9) *P is an n-dimensional subspace of $C[T]$.*

(10) *$\mathfrak{U}$ is a positive operator.*

(11) *There exists at least one $p \in P$ such that:*

$$||g - \mathfrak{U}(p)|| < \eta.$$

The element $p^* \in P_\eta$ is a best approximation to f if and only if there exists k points $(k \leqq n + 1)$: $t_1, t_2, \ldots, t_k$ in $\Gamma^* \cup \Lambda^*$ (at least one in Γ^*) and a functional of the form:

$$\Phi(p) = \sum_{t_i \in \Gamma^*} \alpha_i p(t_i) + \sum_{t_i \in \Lambda^*} \alpha_i \mathfrak{U}(p)(t_i), \quad p \in C[T]$$

such that:

(12) $$\Phi(p) = 0 \text{ if } p \in P, \text{ and}$$

(13) $$\alpha_i > 0 \text{ if } t_i \in \Sigma^*_+ \text{ and } \alpha_i < 0 \text{ if } t_i \in \Sigma^*_-.$$

Proof. Sufficiency. Suppose that $p^* \in P$ verifies the conditions of the theorem but is not a best approximation to f. Let $p_1 \in P$ be a best approximation to f; that is:

$$\rho = ||f - p_1|| < ||f - p^*||.$$

Letting $\tilde{p} = p_1 - p^* = (f - p^*) - (f - p_1)$ we see that $\tilde{p} \in P$ and:

$$\tilde{p}(t_i) > 0 \text{ if } t_i \in \Gamma^*_+ \text{ and } \alpha_i > 0,$$
$$\tilde{p}(t_i) < 0 \text{ if } t_i \in \Gamma^*_- \text{ and } \alpha_i < 0,$$
$$\mathfrak{U}(\tilde{p})(t_i) \geqq 0 \text{ if } t_i \in \Lambda^*_+ \text{ and } \alpha_i > 0$$
$$\mathfrak{U}(\tilde{p})(t_i) \geqq 0 \text{ if } t_i \in \Lambda^*_- \text{ and } \alpha_i < 0.$$

Because at least one of t_i is in Γ^* we have $\Phi(\tilde{p}) > 0$ that is a contradiction with (12).

Necessity. We shall assume throughout this proof that the functions $\varphi_1, \ldots, \varphi_n$ form a basis for P. Let $p^* \in P_\eta$ be a best approximation to f.

Denote:

$$\mathcal{G} = \{\xi(t)\,\hat{t}\colon\ t \in \Gamma^* \cup \Lambda^*\} \subset R^n$$

where

$$\xi(t) = \begin{cases} +1 & \text{if} \quad t \in \Sigma^*_+ \\ -1 & \text{if} \quad t \in \Sigma^*_- \end{cases},$$

$$\hat{t} = (\varphi_1(t), \ldots, \varphi_n(t))$$

and R^n is the Euclidian n-dimensional space.

Now we prove by contradiction that the origin θ of the R^n space belongs to the convex hull $\mathrm{Co}(\mathcal{G})$ of $\mathcal{G}$.

Indeed if $\theta \notin \mathrm{Co}(\mathcal{G})$, $\mathrm{Co}(\mathcal{G})$ being compact set (because $\mathcal{G}$ is compact) there exists a support hyperplane. That is there exists $\tau > 0$ and β_j, $j = 1, \ldots, n$ such that:

$$\sum_{j=1}^{n} \beta_j z_j \geqq \tau \text{ for any } z = (z_1, \ldots, z_n) \in \mathrm{Co}(\mathcal{G}).$$

In particular for points from $\mathcal{G}$ we have:

$$\sum_{j=1}^{n} \beta_j \varphi_j(t) \geqq \tau > 0 \text{ if } t \in \Sigma_+^*$$

$$-\sum_{j=1}^{n} \beta_j \varphi_j(t) \geqq \tau > 0 \text{ if } t \in \Sigma_-^*.$$

Letting $p = \sum_{j=1}^{n} \beta_j \varphi_j$ we get that $p \in P$, but by Lemma 4.2. we obtain that p^* is not a best approximation. Thus $\theta \in \mathrm{Co}(\mathcal{G})$.

Consequently by Carathéodory's theorem on convex sets in R^n there exists k points $(k \leqq n+1)$: $z^{(1)}, \ldots, z^{(k)}$ in $\mathcal{G}$ such that

$$\theta \in \mathrm{Co}\,\{z^{(1)}, \ldots, z^{(k)}\}.$$

Let $t_1, \ldots, t_k$ be the points in T corresponding to $z^{(1)}, \ldots, z^{(k)}$. Thus $0 = \sum_{i=1}^{k} \delta_i z^{(i)}$, $\sum_{i=1}^{k} \delta_i = 1$, $\delta_i > 0$, $i = 1, \ldots, k$ or in projection:

$$0 = \sum_{i=1}^{k} \delta_i \varepsilon_i \varphi_j(t_i), \quad j = 1, \ldots, n,$$

where

$$\varepsilon_i = \begin{cases} +1 & \text{if} \quad t_i \in \Sigma_+^* \\ -1 & \text{if} \quad t_i \in \Sigma_-^* \end{cases}.$$

Taking $\alpha_i = \delta_i \varepsilon_i$ we obtain the functional which satisfies the conditions (12) and (13).

Now we shall prove that at least one of t_i belongs to Γ^*. If all $t_i \in \Lambda^*$ we obtain immediately that $\Phi(p^* - p) < 0$ for any $p \in P_n$ which is a contradiction.

5. Uniqueness of best approximation. If the hypothesis (9) for Theorem 5.1. and (9) and (10) for Theorem 5.2., from Theorem 4.5. are satisfied, then may be get easy, see [6], an uniqueness theorem and a strong unicity theorem [2, p. 77].

THEOREM 5.1. *If $p^* \in P_n$ is a best approximation to f then p^* is unique.*

THEOREM 5.2. *Let $p^* \in P_n$ be the best approximation to f. Then there exists a constant $\gamma > 0$ such that for any $p \in P_n$:*

$$\|f - p\| \geqq \|f - p^*\| + \gamma\|p^* - p\|.$$

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