The morphism v_e is an injection into the coproduct $H_1(C) = \oplus H_1(C(e))$ over $e \in E$. The family of morphisms $w_e\theta^{-1}_{e,e'}: H_1(C(e)) \to H_2(C * C'(e'), C)$, $e \in E$ induces a unique morphism $\theta'_{e'}: H_1(C) \to H_2(C * C'(e'), C)$ such that $\theta'_{e'}v_e = w_e\theta^{-1}_{e,e'}$ for every $e \in E$. Hence $\theta_{e'}\theta'_{e'}v_e = \theta_{e'}, w_e\theta^{-1}_{e,e'} = v_e\theta_{e,e'}\theta^{-1}_{e,e'} = v_e$ for every injection v_e, and therefore $\theta_{e'}, \theta'_{e'} =$ identity on $H_1(C)$, that is the middle row of the first diagram is split exact. It follows that the bottom row of the same diagram is split exact. So is also the bottom row of the second diagram. Thus we have following isomorphisms.

4. $H_2(C * C'(e'), C'(e')) \cong H_2(C * C'(e')) \oplus H_1(C'(e'))$.

5. $H_2(C * C'(e'), C) \cong H_2(C * C'(e')) \oplus H_1(C)$.

6. $H_2(C * C', C) \cong H_2(C * C') \oplus H_1(C)$.

Now, $H_2(C * C'(e'), C(e')) = \oplus H_2(C(e) * C'(e'), C'(e'))$ over $e \in E$. Since $H_2(C(e) * C'(e'), C'(e')) \cong H_1(C'(e'))$, we have by 4, $H_2(C * C'(e'))$ is isomorphic to the coproduct of $|E| - 1$ copies of $H_1(C'(e'))$. Hence, by 5, $H_2(C * C'(e'), C)$ is isomorphic to the coproduct of $|E| - 1$ copies of $H_1(C'(e'))$ with $H_1(C)$. Thus, $H_2(C * C', C) = \oplus H_2(C * C'(e'), C)$ over $e' \in E'$, and $H_1(C') = \oplus H_1(C'(e'))$ over $e' \in E'$, imply that $H_2(C * C', C)$ is isomorphic to the coproduct of $|E| - 1$ copies of $H_1(C')$ and $|E'|$ copies of $H_1(C)$. But then, 6 implies that $H_2(C * C')$ is isomorphic to the coproduct of $|E| - 1$ copies of $H_1(C')$ and $|E'| - 1$ copies of $H_1(C)$. This proves (iii).

In particular (ii) shows that a converse of lemma 2(i) holds. For if $A \neq 0$, then $H_0(C) \neq 0$ for every category $\underline{C}$. Simple examples show that a converse of Lemma 2(ii) does not hold.

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ON THE PRODUCT OF RELATIONS

by

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1. In the present paper we consider the so-called (i, j) product of some relations in a set $M^n = M \times M \times \ldots \times M$ and give some algebraic and geometric aspects of this product. The geometry of the relations in the set $\mathbf{R}^2$ has been considered in works [1] and [2].

The idea to study such problems has been suggested by ŞT. N. BERȚI.

2. Let M be a nonempty set. We consider two relations in M^n: $S = (G, M^n)$ and $T = (H, M^n)$ where $G \subset M^n$ and $H \subset M^n$ are the graphs of the relations S and T, M^n being the cartesian product $M \times M \times \ldots \times M$. We introduce following

D e f i n i t i o n. *The (i, j) product of relations S and T is defined by the set*

$$\{(x_1, x_2, \ldots, x_n) \in M^n \mid \exists Z \in M :$$

$$(x_1, \ldots, x_{i-1}, Z, x_{i+1}, \ldots, x_n) \in S \wedge (x_1, \ldots, x_{j-1}, Z, x_{j+1}, \ldots, x_n) \in T\},$$

where the numbers i and j are fixed and belong to the set $\{1, 2, \ldots, n\}$.

Let us denote the (i, j) product by $S(i) \cdot T(j)$.

First we suppose that $M = \mathbf{R}$, where $\mathbf{R}$ is the set of all real numbers. Then the hyperplanes of the n-dimensional space

$$S: a_1x_1 + \ldots + a_nx_n + b = 0$$

$$T: A_1x_1 + \ldots + A_nx_n + B = 0$$

can considered as the graphs of some relations in the set $\mathbf{R}^n$. We suppose that $i \neq j$. The numeration being irrelevant, let be $i > j$.

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Case a. If $a_i \neq 0$ and $A_j \neq 0$, from the definition of the (i, j) product it follows directly that $(x_1, \ldots, x_n) \in S(i) . T(j)$ if and only if there exists $Z \in \mathbf{R}$ such that

$$a_1x_1 + \ldots + a_{i-1}x_{i-1} + a_iZ + a_{i+1}x_{i+1} + \ldots + a_nx_n + b = 0$$
$$A_1x_1 + \ldots + A_{j-1}x_{j-1} + A_jZ + a_{j+1}x_{j+1} + \ldots + A_nx_n + B = 0$$

therefore by the elimination of Z we have

$$\sum_{u=1}^{j-1}(a_u A_j - a_iA_u)\, x_u + a_jA_jx_j + \sum_{d=j+1}^{i-1}(a_d A_j - a_i A_d)\, x_d - a_iA_ix_i + \sum_{t=i+1}^{n}(a_t A_j - a_i A_t)\, x_t + bA_j - a_i B = 0$$

which represents a hyperplane of n-dimensional space $\mathbf{R}^n$. Further we use the following notations:

$$H(d_1, d_2, \ldots, d_n, e) \equiv d_1x_1 + d_2x_2 + \ldots + d_n x_n + e = 0$$

and „." for the (i, j) product, where i, j are fixed. Hence

$$H(a_1, \ldots, a_n, b) \,.\, H(A_1, \ldots, A_n, B) = H(\ldots, a_u A_j - a_i A_u, \ldots, a_jA_j, \ldots, a_d A_j - a_i A_d, \ldots, -a_i A_i, \ldots, a_t A_j - a_i A_t, \ldots, bA_j - a_i B),$$

therefore we infer that the product of two hyperplanes of $\mathbf{R}^n$ is a hyperplane of $\mathbf{R}^n$.

Case b. If $a_i = 0$ and $A_j \neq 0$ we have

$$H(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n, b) \cdot H(A_1, \ldots, A_n, B) = H(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n, b),$$

which is a certain hyperplane of $\mathbf{R}^n$, parallel with the x_i axis.

Case c. For $a_i \neq 0$ and $A_j = 0$ we obtain a hyperplane which is parallel with the x_j axis.

Case d. For $a_i = A_j = 0$ we have that

$$(x_1, \ldots, x_n) \in H(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n, b) \cdot H(A_1, \ldots, A_{j-1}, 0, A_{j+1}, \ldots, A_n, B)$$

if there exists $Z \in \mathbf{R}$ such that

$$a_1x_1 + \ldots + a_{i-1}x_{i-1} + a_{i+1}x_{i+1} + \ldots + a_nx_n + b = 0$$
$$A_1x_1 + \ldots + A_{j-1}x_{j-1} + A_{j+1}x_{j+1} + \ldots + A_nx_n + B = 0$$

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which represents the intersection of two hyperplanes. It is obvious that for $a_k = 0$ $(k = 1, \ldots, n)$ and $A_p = 0$ $(p = 1, 2, \ldots, n)$ this set degenerates either in $\mathbf{R}^n$ (when $b = B = 0$) or in the empty set Φ (when $b \neq 0$, $B \neq 0$). In the other cases we are led to the same degenerations.

If $i = j$, then the (i, j) product of the hyperplanes $H(a_1, \ldots, a_n, b)$ and $H(A_1, \ldots, A_n, B)$ amount to

$$H(a_1A_i - a_iA_1, \ldots, a_{i-1}A_i - a_iA_{i-1}, 0, a_{i+1}A_i - a_iA_{i+1}, \ldots, a_nA_i - a_iA_n, bA_i - a_iB)$$

where $a_i \neq 0$ and $A_i \neq 0$, which is a hyperplane parallel with the x_i axis. In the other cases we can make an analogous discussion.

3. Further we assume that $a_i \neq 0$, $a_j \neq 0$, where $i \neq j$. Let us denote by $\mathcal{H}$ the set of all hyperplanes of the set $\mathbf{R}^n$ with this assumption and by „." the (i, j) product of two arbitrary elements of the set $\mathcal{H}$. Relating to the structure of the set $\mathcal{H}$ there occurs the following theorem:

THEOREM 1.

For any i, j the set $\mathcal{H}$ with the (i, j) product „." generates a group. We denote this group by $\mathcal{G} = (\mathcal{H}, \cdot)$.

Proof.

From the definition of the (i, j) product we can easy verify that for any $H_1, H_2, H_3 \in \mathcal{H}$ we have $(H_1 \cdot H_2) \cdot H_3 = H_1 \cdot (H_2 \cdot H_3)$. We obtain also from the system

$$\begin{cases} a_jA_j = a_j \\ -a_iA_i = a_i \\ a_pA_j - a_iA_p = a_p \quad (p = 1, \ldots, j-1, j+1, \ldots, i-1, i+1, \ldots, n) \\ bA_j - a_iB = b \end{cases}$$

the coefficients of the unit hyperplane:

$$A_j = 1 \quad A_i = -1 \quad A_p = B = 0.$$

It results, that there exists $U \in \mathcal{H}$ such that for any $H \in \mathcal{H}$ we have $H \cdot U = U \cdot H = H$, where U is the unit-hyperplane. Similarly, from the system

$$\begin{cases} a_jA_j = a_iA_i = 1 \\ a_pA_j - a_iA_p = bA_j - a_iB = 0 \end{cases}$$

we find the coefficients of the inverse hyperplane

$$H^{-1}(a_1, \ldots, a_n, b) = H(A_1, \ldots, A_n, B),$$

where

$$A_j = \frac{1}{a_j} \qquad A_i = \frac{1}{a_i} \qquad A_p = \frac{a_p}{a_i a_j} \qquad B = \frac{b}{a_i a_j}.$$

Therefore for any $H \in \mathcal{H}$ there exists $H^{-1} \in \mathcal{H}$ such that

$$H(a_1, \ldots, a_n, b) \cdot H^{-1}(a_1, \ldots, a_n, b) = H^{-1} \cdot H = U.$$

This completes the proof of our theorem.

Obviously the group G is not an abelian group. The conditions for $H(a_1, \ldots, a_n, b) \cdot H(A_1, \ldots, A_n, B) = H(A_1, \ldots, A_n, B) \cdot H(a_1, \ldots, a_n, b)$ are given by the system

$$\frac{a_1}{A_1} = \ldots = \frac{a_{j-1}}{A_{j-1}} = \frac{a_{j+1}}{A_{j+1}} = \ldots = \frac{a_{i-1}}{A_{i-1}} = \frac{a_{i+1}}{A_{i+1}} = \ldots = \frac{a_n}{A_n} = \frac{b}{B} = \frac{a_i + a_j}{A_i + A_j}.$$

The powers of a hyperplane $H \in \mathcal{H}$ are defined recurrently, by the formula $H^k = H^{k-1} \cdot H$, where $k = 2, 3, \ldots$. Thus, for $k = 2, 3$ we deduce that

$$H^2(a_1, \ldots, a_n, b) = H[\ldots, (a_j - a_i)\, a_u, \ldots, a_j^2, \ldots, (a_j - a_i)\, a_d, \ldots,$$
$$, - a_i^2, \ldots, (a_j - a_i)\, a_t, \ldots, (a_j - a_i)\, b]$$

$$H^3(a_1, \ldots, a_n, b) = H[\ldots, (a_j^2 - a_i a_j + a_i^2)\, a_u, \ldots, a_j^3, \ldots, (a_j^2 - a_i a_j + a_i^2)\, a_d,$$
$$, \ldots, a_i^3, \ldots, (a_j^2 - a_i a_j + a_i^2)\, a_t, \ldots, (a_j^2 - a_i a_j + a_i^2)\, b]$$

$$u = 1, \ldots, j - 1\,;\; d = j + 1, \ldots, i - 1\,;\; t = i + 1, \ldots, n.$$

It is obviously that for $a_i = a_j$ we have $H^2(a_1, \ldots, a_n, b) = U$, thus the order of the group $(\mathcal{H}, \cdot)$ is equal with 2.

4. We consider now an arbitrary hyperplane $H(a_1, \ldots, a_n, b) \in \mathcal{H}$. We notice that for any $s \neq 0$ we have $H(a_1 s, \ldots, a_n s, bs) = H(a_1, \ldots, a_n, b)$. Then we can assume always that $a_1^2 + \ldots + a_n^2 = 1$. In fact, if $a_1^2 + \ldots + a_n^2 = K^2 \neq 1$, then by taking $s = \frac{1}{K}$ we have satisfyed this condition. In this assumption the connection with the notion of perpendicularity is given by the following theorem:

THEOREM 2. *If H is a hyperplane of the set $\mathcal{H}$, with $|a_i - a_j| = 1$, then this hyperplane is a perpendicular on his inverse H^{-1}, where H^{-1} is determined with the (i, j) product.*

Proof.
In fact, the inverse of the hyperplane $H\,(a_1, \ldots, a_n, b) \in \mathcal{H}$ being

$$H^{-1} = H\left(\ldots, \frac{a_u}{a_i a_j}, \ldots, \frac{1}{a_j}, \ldots, \frac{a_d}{a_i a_j}, \ldots, \frac{1}{a_i}, \ldots, \frac{a_t}{a_i a_j}, \ldots, \frac{b}{a_i a_j}\right),$$

we infer that the condition of perpendicularity is given by

$$\sum_{u=1}^{j-1} \frac{a_u^2}{a_i a_j} + 1 + \sum_{d=j+1}^{i-1} \frac{a_d^2}{a_i a_j} + 1 + \sum_{t=i+1}^{n} \frac{a_t^2}{a_i a_j} = 0$$

which is equivalent with $|a_i - a_j| = 1$ and the theorem is proved.

5. In connection with the subgroups of the group $\mathcal{G} = (\mathcal{H}, \cdot)$ we have following results:

THEOREM 3. a) *The subset $\mathcal{H}_1$ of the set $\mathcal{H}$ with $b = 0$ forms a subgroup $\mathcal{G}_1 = (\mathcal{H}_1, \cdot)$ of the group $\mathcal{G} = (\mathcal{H}, \cdot)$.*

b) *The subset $\mathcal{H}_2$ of the set $\mathcal{H}$ with $a_i = - a_j$ forms a subgroup $\mathcal{G}_2 = (\mathcal{H}_2, \cdot)$ of the group $\mathcal{G} = (\mathcal{H}, \cdot)$.*

Proof. a) For any $H_1, H_2 \in \mathcal{H}_1$, from $b' = bA_j - a_i B = 0$, we conclude that $H_1 \cdot H_2 \in \mathcal{H}_1$. Also $U \in \mathcal{H}_1$ and $H_1^{-1} \in \mathcal{H}_1$, since $b' = \frac{b}{a_i a_j} = 0$.

b) Suppose H_1, H_2 to belong to $\mathcal{H}_2$. We assume that $a_i = - a_j = A_i = - A_j = 1$. From the definition of the (i, j) product it follows directly that

$$H_1(a_1, \ldots, a_n, b) \cdot H(A_1, \ldots, A_n, B) = H(\ldots, a_u + A_u, \ldots, - 1,$$
$$, \ldots, a_d + A_d, \ldots, 1, \ldots, a_t + A_t, \ldots, b + B) \in \mathcal{H}_2.$$

Also we have that $U \in \mathcal{H}_2$, since we may take $s = -1$. Similarly, for the invers plane we conclude that

$$H_1^{-1}(a_1, \ldots, a_n, b) = H_1(a_1, \ldots, a_n, b) \in \mathcal{H}_2.$$

Thus the theorem is proved.

Now we consider the set $\mathcal{H}_{1,2}$ defined by $\mathcal{H}_{1,2} = \mathcal{H}_1 \cap \mathcal{H}_2$. It is obviously that $\mathcal{H}_{1,2}$ is a subgroup of the group $\mathcal{H}$. For any $H \in \mathcal{H}$ and $h \in \mathcal{H}_{1,2}$ we form the following product: $H^{-1} \cdot h \cdot H$. This product has the form

$$- \sum_{u=1}^{j-1} \frac{a_u A_j}{A_i} \cdot x_u - x_j - \sum_{d=j+1}^{i-1} \frac{a_d A_j}{A_i} \cdot x_d + x_i - \sum_{t=i+1}^{n} \frac{a_t A_j}{A_i} \cdot x_t = 0$$

therefore, we obtain again an element of the set $\mathcal{H}_{1,2}$. Thus we are led to the following result:

THEOREM 4. *The set* $\mathcal{H}_{1,2}$ *is a normal subgroup of the group* $\mathfrak{G} = (\mathcal{H}, \cdot)$.

6. We extend these considerations on the (k, l) product in the case when $M = \mathbf{C}$, where $\mathbf{C}$ is the set of all complex numbers. We suppose that

$$c_p = a_p^{(1)} + i a_p^{(2)}; \quad d = b^{(1)} + i b^{(2)} \quad (p = 1, \ldots, n)$$

where the numbers k and l are fixed and belong to the set $\{1, \ldots, n\}$. By definition $H(c_1, \ldots, c_n, d)$ is a complex relation and it can be considered as a complex hyperplane. Let us denote by $\mathfrak{K}$ the set of all complex hyperplanes of the set $\mathbf{C}^n$ with $c_k \neq 0$, $c_l \neq 0$. Then by definition of the (k, l) product we have that

$$H(c_1, \ldots, c_n, d) \cdot H(C_1, \ldots, C_n, D) = H(\ldots, c_u C_l - c_k C_u, \ldots, c_l C_l, \ldots, c_d C_l - c_k C_d, \ldots, - c_k C_k, \ldots, c_t C_l - c_k C_t, \ldots, dC_l - c_k D),$$

where

$$u = 1, \ldots, l-1; \quad d = l+1, \ldots, k-1; \quad t = k+1, \ldots, n.$$

We use the following symbols:

$$K \equiv x_l + i x_k = 0; \quad \overline{K} \equiv x_l - i x_k = 0.$$

It is easily to verify that we have following decomposition of the real unit hyperplane: $U = K \cdot \overline{K}$. Similarly, we can deduce the decomposition

$$H(\ldots, x_u X_l + i x_k X_u, \ldots, x_l X_l, \ldots, x_d X_l + i x_k X_d, \ldots, i x_k X_k, \ldots, x_t X_l + i x_k X_t, \ldots, b X_l + i x_k B) =$$
$$= H(\ldots, x_u, \ldots, x_l, \ldots, x_d, \ldots, x_k, \ldots, x_t, \ldots, b) \,.\, K \cdot$$
$$\cdot H(\ldots, X_u, \ldots, X_l, \ldots, X_d, \ldots, X_k, \ldots, X_t, \ldots, B).$$

If we suppose that H belongs to $\mathcal{H}$, we define the m-th radical of the H by

$$\sqrt[m]{H} = \{h \in \mathcal{H} \mid h^m = h \cdot h \cdot \ldots \cdot h = H\},$$

where „·" means the (k, l) product defined above. Hence for $m = 2$ we have

$$\sqrt{H(a_1, \ldots, a_n, b)} = H(A_1, \ldots, A_n, B) \quad \text{if} \quad H(a_1, \ldots, a_n, b) =$$
$$= H^2(A_1, \ldots, A_n, B) = H[\ldots, (A_l - A_k) A_u, \ldots, A_l^2, \ldots, (A_l - A_k) A_d, \ldots, - A_k^2, \ldots, (A_l - A_k) A_t, \ldots, (A_l - A_k) B]$$

what is equivalent with the system

$$\begin{cases} (A_l - A_k) A_p = a_p & (p = 1, \ldots, l-1, l+1, \ldots, k-1, k+1, \ldots, n) \\ A_l^2 = a_l & \\ -A_k^2 = a_k & \\ (A_l - A_k) B_k = b. & \end{cases}$$

Without loss of generality we may assume that $a_l > 0$. We shall tell two cases:

a) $a_k < 0$. In this case, if $a_k^2 \neq a_l^2$ we obtain two elements $h_1 \in \mathcal{H}$ and $h_2 \in \mathcal{H}$ such that $h_1^2 = h_2^2 = H$, where

$$h_1 = H\left(\ldots, \frac{a_u}{\sqrt{a_l} - \sqrt{-a_k}}, \ldots, \sqrt{a_l}, \ldots, \frac{a_d}{\sqrt{a_l} - \sqrt{-a_k}}, \ldots, \sqrt{-a_k}, \ldots, \frac{a_t}{\sqrt{a_l} - \sqrt{-a_k}}, \ldots, \frac{b}{\sqrt{a_l} - \sqrt{-a_k}}\right)$$

and

$$h_2 = H\left(\ldots, \frac{a_u}{\sqrt{a_l} + \sqrt{-a_k}}, \ldots, \sqrt{a_l}, \ldots, \frac{a_d}{\sqrt{a_l} + \sqrt{-a_k}}, \ldots, -\sqrt{-a_k}, \ldots, \frac{a_t}{\sqrt{a_l} + \sqrt{-a_k}}, \ldots, \frac{b}{\sqrt{a_l} + \sqrt{-a_k}}\right).$$

If $a_k^2 = a_l^2$, then we obtain a single element $h \in \mathcal{H}$ such that $h^2 = H$, where

$$h = H\left(\ldots, \frac{a_u}{\sqrt{a_l} + \sqrt{-a_k}}, \ldots, \sqrt{a_l}, \ldots, \frac{a_d}{\sqrt{a_l} + \sqrt{-a_k}}, \ldots, -\sqrt{-a_k}, \ldots, \frac{a_t}{\sqrt{a_l} + \sqrt{-a_k}}, \ldots, \frac{b}{\sqrt{a_l} + \sqrt{-a_k}}\right).$$

b) $a_k > 0$. If $a_k \neq - a_l$ then we obtain two elements h_1' and h_2' such that $h_1'^2 = h_2'^2 = H$ where $h_1' \in \mathcal{H}$, $h_2' \in \mathcal{H}$ and $h_1' \equiv \sum_{s=1}^{n} A_s' x_s + B' = 0$

$$A_p' = \frac{a_p \sqrt{a_l}}{a_l + a_k} + i \frac{a_p \sqrt{a_l}}{a_l + a_k} \quad (p = 1, \ldots, l-1, l+1, \ldots, k-1, k+1, \ldots, n)$$

$$B' = \frac{b\sqrt{a_l}}{a_l + a_k} + i \frac{b\sqrt{a_k}}{a_l + a_k}; \quad A_l' = \sqrt{a_l}; \quad A_k' = i\sqrt{a_k}; \quad h_2' \equiv \sum_{s=1}^{n} A_s'' x_s + B'' = 0$$

$$A_s'' = \frac{a_p \sqrt{a_l}}{a_l + a_k} - i \frac{a_p \sqrt{a_k}}{a_l + a_k} \quad (p = 1, \ldots, l-1, l+1, \ldots, k-1, k+1, \ldots, n)$$

$$B'' = \frac{b\sqrt{a_l}}{a_l + a_k} - i \frac{b\sqrt{a_k}}{a_l + a_k}; \quad A_l'' = \sqrt{a_l}; \quad A_k'' = - i\sqrt{a_k}.$$

From the above decomposition we can establish that

$$h_1' = H\left(\dots,\frac{a_u\sqrt{a_l}}{a_l+a_k},\dots,\sqrt{a_l},\dots,\frac{a_d\sqrt{a_l}}{a_l+a_k},\dots,\sqrt{a_k},\dots,\right.$$
$$\left.,\frac{a_t\sqrt{a_l}}{a_l+a_k},\dots,\frac{b\sqrt{a_l}}{a_l+a_k}\right).$$
$$\cdot K\cdot H\left(\dots,\frac{a_u}{a_l+a_k},\dots,1,\dots,\frac{a_d}{a_l+a_k},\dots,1,\dots,\frac{a_t}{a_l+a_k},\dots,\frac{b}{a_l+a_k}\right)$$
$$h_2' = H\left(\dots,\frac{a_u\sqrt{a_l}}{a_l+a_k},\dots,\sqrt{a_l},\dots,\frac{a_d\sqrt{a_l}}{a_l+a_k},\dots,\right.$$
$$\left.,-\sqrt{a_k},\dots,\frac{a_t\sqrt{a_l}}{a_l+a_k},\dots,\frac{b\sqrt{a_l}}{a_l+a_k}\right).$$
$$\cdot K\cdot H\left(\dots,\frac{a_u}{a_l+a_k},\dots,1,\dots,\frac{a_d}{a_l+a_k},\dots,1,\dots,\frac{a_t}{a_l+a_k},\dots,\frac{b}{a_l+a_k}\right)$$

which are the reprezentations of the complex radicals with the aid of the real elements and the complex unit.

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A NOTE ON A SPECIAL CLASS OF DEMAND FUNCTIONS

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1. Introduction*

By demand functions we mean a system of n functions $q_i(m, p_1, \dots, p_n)$, $i = 1, \dots, n$, where $q_i \geq 0$, $m \geq 0$, $p_i > 0$, satisfying the following conditions known as Slutsky's conditions:

(1) $$\sum_{i=1}^{n} p_i\, q_i = m$$

(2) $$k_{ij} \overset{df}{=} \frac{\partial q_i}{\partial p_j} + \frac{\partial q_i}{\partial m} q_j = \frac{\partial q_j}{\partial p_i} + \frac{\partial q_j}{\partial m} q_i = k_{ji} \text{ for all } i, j$$

(3) $$\sum_{j=1}^{n} p_j\, k_{ij} = 0.$$

The natural way to derive demand functions is to start with a given direct or indirect utility function. In the first case it is practically possible to do this if the utility function (which has to be concave) is quadratic. In the second case demand functions are obtained from the well known Roy formula.

One may use also another approach to derive demand functions starting directly from Slutsky's conditions. So did McFadden in an unpubli-

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