

THE HOMOTOPICAL CHARACTERIZATION OF THE SET OF UNRESTRICTED CHEBYSHEV SPACES

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0. The purpose of this paper is to establish the contractibility to a point of the set $\mathfrak{M}$ of the n -dimensional unrestricted Chebyshev subspaces in the linear normed space $C^n[0, 1]$ (for definitions see 1. and 2.). The introduction of a topology in the set $\mathfrak{M}$ and the investigations about the topological properties of this set is motivated by some problems in the theory of nonlinear approximation of the nature of that in the paper [6].

In a recent paper [7], it was established the pathwise connectendess of the set $\mathfrak{M}$, and also the pathwise connectendess of the set $\mathcal{C}$ of all the n -dimensional Chebyshev subspaces in the linear normed space $C[0, 1]$. The methods in [7] are based on some integral representations which were given by S. KARLIN and W. STUDDEN in [3] (Chapter XI, Theorem 1.2.) for the unrestricted Chebyshev systems and by M. A. RUTMAN in [9] for the Chebyshev-Markov sequences of functions in the space $C[0, 1]$.

In this paper we follow another way which is based on the employing of the theorem of DE LA VALLÉE POUSSIN [11] in the qualitative theory of linear differential equations. It will be proved that all the homotopy groups of $\mathfrak{M}$ are trivial and that the introduced topology in $\mathfrak{M}$ is equivalent with a locally convex one, and from these, applying the method of N. H. KUIPER [4] it will follow the homotopical triviality of the set $\mathfrak{M}$.*

Because this method makes use of the theory of ordinary differential equations, it is inaplicable in a similar investigation about the set $\mathcal{C}$ of all the n -dimensional Chebyshev subspaces of the space $C[0, 1]$. So the problem of the complete homotopical characterization of $\mathcal{C}$ remains open.

We intend to make all proofs as independent as possible, except the last step in the proof of our theorem, which in fact would be a mutatis

* As it was observed in this paper, this method represent an explicit exposition of an implicit result of J. MILNOR in [5].

mutandis transposition of the proof of the Theorem 2 in the paper [4]. For the sake of completeness and the continuity in the exposition, some implicit, immediate consequences of some classical theorems in the theory of the ordinary differential equations are explicitly stated and proved in the sequel.

1. Denote by $C^v[0, 1]$ the linear normed space of the real valued functions having continuous derivatives of order v on the interval $[0, 1]$ (in the endpoints 0 and 1 the one sided derivatives are considered), in which the norm is introduced by

$$\|f\| = \max_{0 \leq i \leq v} \max_{x \in [0, 1]} |f^{(i)}(x)|,$$

where the symbol (i) denotes the i^{th} derivative.

Definition (i) The set of n elements $f_1, f_2, \dots, f_n$ of the space $C^n[0, 1]$ is said to form an *unrestricted Chebyshev system* (abbreviated U.C.S.) if for any natural number m , $1 \leq m \leq n$, for any distinct points $x_1, x_2, \dots, x_m$ in the interval $[0, 1]$, and for any natural numbers k_i , $i = 1, 2, \dots, m$ having the property $\sum_{i=1}^m k_i = n$, the determinant

$$\det \|f_j^{(k_i)}(x_i)\| \quad k = 0, 1, \dots, k_i - 1, i = 1, 2, \dots, m, j = 1, 2, \dots, n$$

is different from zero.

(ii) The subspace L of the dimension n in $C^n[0, 1]$ is said to be an *unrestricted Chebyshev space* (U.C.SP.), if it has a basis which is an U.C.S..

If a basis of a subspace of dimension n in the space $C^n[0, 1]$ is an U.C.S., then by simple algebraic considerations it follows that all its bases have this property.

2. Denote by $\mathfrak{M}$ the set of all n -dimensional U.C.SP. in the space $C^n[0, 1]$. In order to introduce a topology in $\mathfrak{M}$, we will introduce a topology in the set of all subspaces of the space $C^n[0, 1]$ of dimension n at most, and consider the set $\mathfrak{M}$ topologized by the induced topology.

Consider the linear normed space

$$C_n^n = C^n[0, 1] \times \dots \times C^n[0, 1]$$

where the expression in the right of this equality denotes the direct product of n exemplars of the space $C^n[0, 1]$, and the norm in C_n^n is one of the usual norms in a direct product of linear normed spaces. The elements in C_n^n are the n -tuples of form $f = (f_1, f_2, \dots, f_n)$, where f_i , $i = 1, 2, \dots, n$ are elements of the space $C^n[0, 1]$. We introduce the equivalence relation $\sim$ in C_n^n considering $(f_1, f_2, \dots, f_n) = f \sim g = (g_1, g_2, \dots, g_n)$, if $L(f) = L(g)$,

where $L(h) = L(h_1, h_2, \dots, h_n)$ denotes the linear space spanned by the functions $h_1, h_2, \dots, h_n$. Consider the topological space

$$\mathfrak{L} = C_n^n / \sim$$

and denote the canonical projection of this factorization by p . The mapping p maps an element in C_n^n in its equivalence class in $\mathfrak{L}$, and the topology in $\mathfrak{L}$ is the strongest topology which may be introduced in $\mathfrak{L}$ with the condition to p should be continuous in this topology. The elements of the set $\mathfrak{L}$ may be identified with the subspaces of dimension n at most in $C^n[0, 1]$. Accordingly, we will consider that $\mathfrak{M}$ is in $\mathfrak{L}$ and that $\mathfrak{M}$ is topologized by the induced topology.

As it was mentioned in the introduction, in the paper [7] it was proved the pathwise connectendess of the set $\mathfrak{M}$ in the above introduced topology. In the present paper we will prove the following theorem:

THEOREM. The topological space $\mathfrak{M}$ of all the n -dimensional unrestricted Chebyshev subspaces in the linear normed space $C^n[0, 1]$ is contractible to a point.

We remind that the contractibility to a point of a topological space means the homotopical equivalence of the identity mapping of this space with any constant mapping of this space onto one of its points.

In order to avoid an interruption in the following exposition, we will prove at this point the following lemma:

Lemma 1. The canonical projection

$$p: C_n^n \rightarrow \mathfrak{L}$$

of the factorization defined above, restricted to the subset L^n in C_n^n of all the n -tuples $(f_1, f_2, \dots, f_n)$ for which $f_1, f_2, \dots, f_n$ are linearly independent elements in $C^n[0, 1]$, is an open mapping.

Proof. Suppose that the set U is an open subset of L^n . Then we have

$$p^{-1}(p(U)) = \{g \in L^n: g = Af, f \in U, A \in GL_n\},$$

where GL_n denotes the set of all real, nonsingular $n \times n$ matrices. The above equality may be written also in the following equivalent form:

$$p^{-1}(p(U)) = \bigcup_{A \in GL_n} A(U),$$

where $A(U) = \{g \in L^n: g = Af, f \in U\}$. Because A is a nonsingular $n \times n$ matrix, the mapping

$$f \mapsto Af$$

is a homeomorphism of L^n onto itself and therefore $A(U)$ is an open subset in L^n . But then $\bigcup_{A \in GL_n} A(U)$ is also open. From the definition of the factor topology then it follows that $p(U)$ is open in the set $\mathcal{E}^n = p(L^n)$, that is, $p|L^n$ is an open mapping.

3. Consider the differential equation

$$(1) \quad y^{(n)} + a_1(x) y^{(n-1)} + \dots + a_{n-1}(x) y' + a_n(x) y = 0,$$

where the coefficients $a_i(x)$, $i = 1, 2, \dots, n$ are real valued continuous functions for x in $[0, 1]$, i.e., they are elements of the linear normed space $C[0, 1] = C^0[0, 1]$. The n -tuple of functions $a = (a_1, a_2, \dots, a_n)$, which will be considered by us to be an element of the linear normed space

$$C_n^0 = C[0, 1] \times \dots \times C[0, 1]$$

(in which the norm is one of the standard norms of the direct product of normed spaces), determines the differential equation (1), and equally the linear differential operator

$$(2) \quad \frac{d^n}{dx^n} + a_1(x) \frac{d^{n-1}}{dx^{n-1}} + \dots + a_{n-1}(x) \frac{d}{dx} + a_n(x) E,$$

where E denotes the identical operator of the space $C^n[0, 1]$.

In this way a one to one correspondence is established between the set of all differential operators of form (2), and the set C_n^0 introduced above. In what follows we will identify these sets, and will consider that the set of all differential operators of the form (2) is topologized by the topology of the linear normed space C_n^0 . We adopt the convention that in the sequel the symbol C_n^0 will denote the set of all differential operators of form (2). Accordingly, the elements of form $(a_1, a_2, \dots, a_n)$ in C_n^0 will sometimes be called "differential operators".

Let D denote the set in C_n^0 of all n -tuples $(f_1, f_2, \dots, f_n)$ having the property that the functions $f_1, f_2, \dots, f_n$ form a fundamental system of solutions for a differential equation of the form (1). The set D is then formed by the n -tuples for which

$$(3) \quad W(f_1, f_2, \dots, f_n; x) \neq 0, \quad x \in [0, 1].$$

where $W(f_1, f_2, \dots, f_n; x)$ denotes the value of the Wronskian of the functions $f_1, f_2, \dots, f_n$ in the point x .

We define the mapping

$$r: D \rightarrow C_n^0,$$

by setting

$$r: (f_1, f_2, \dots, f_n) \mapsto (a_1, a_2, \dots, a_n),$$

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where

$$(4) \quad a_{n-i}(x) = \frac{(-1)^{n-i}}{W(f_1, f_2, \dots, f_n; x)} \begin{vmatrix} f_1(x) & f_2(x) & \dots & f_n(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(i-1)}(x) & f_2^{(i-1)}(x) & \dots & f_n^{(i-1)}(x) \\ f_1^{(i+1)}(x) & f_2^{(i+1)}(x) & \dots & f_n^{(i+1)}(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n)}(x) & f_2^{(n)}(x) & \dots & f_n^{(n)}(x) \end{vmatrix}$$

$$i = 0, 1, \dots, n-1.$$

Lemma 2. The mapping r is continuous and open.

Proof. The continuity of r follows from the fact that $W(f_1, f_2, \dots, f_n; x)$ is a non vanishing, continuous function on the direct product $D \times [0, 1]$, and from the fact that the second determinant in (4) is also continuous on this set.

In order to prove the openness of the mapping r , suppose that U is an open set of D and that $r(U)$ is not open in C_n^0 . Suppose that $(a_1, a_2, \dots, a_n)$ is in $r(U) \setminus \text{int } r(U)$ and let be $\{(a_1^v, a_2^v, \dots, a_n^v)\}$, $v = 1, 2, \dots$ a sequence in $C_n^0 \setminus r(U)$ which tends to $(a_1, a_2, \dots, a_n)$ as $v \rightarrow \infty$. Consider a point $(f_1, f_2, \dots, f_n)$ in $r^{-1}(a_1, a_2, \dots, a_n) \cap U$ and let be $(f_{1,v}, f_{2,v}, \dots, f_{n,v})$ the point in $r^{-1}(a_1^v, a_2^v, \dots, a_n^v)$ having the property that $f_{i,v}^{j-1}(0) = f_i^{j-1}(0)$, $i, j = 1, 2, \dots, n$. Then it follows that $(f_{1,v}, f_{2,v}, \dots, f_{n,v})$ tends to $(f_1, f_2, \dots, f_n)$ as $v \rightarrow \infty$ (see for ex. P. HARTMAN [2], 2.3., Theorem II.). But then, according to the openness of the set U , it follows that $(f_{1,v}, f_{2,v}, \dots, f_{n,v})$ is in U for sufficiently great v , and therefore $(a_1^v, a_2^v, \dots, a_n^v) = r(f_{1,v}, f_{2,v}, \dots, f_{n,v})$ is in the set $r(U)$, which is a contradiction.

Lemma 3. Consider the subset D_0 in D of all the n -tuples $(f_1, f_2, \dots, f_n)$, which has the property

$$f_i^{(j-1)}(0) = \delta_i^j, \quad i, j = 1, 2, \dots, n.$$

Then $s = r|D_0$ is a homeomorphic mapping of D_0 onto C_n^0 .

Proof. The mapping s is obviously onto. It is one to one according to the unicity of solutions which satisfy the initial value condition in the lemma of a differential equation of the form (1). The continuity of s follows from the continuity of r (Lemma 2.). The continuity of the inverse of s follows from the theory of differential equations ([2], 2.3, Theorem II.).

4. We will say that the differential operator (2) has the interpolatory property, if the space of solutions of the corresponding differential equation (1) is an U.C.SP., or, equivalently, if any many point boundary value problem for this differential equation has a solution for the points in $[0, 1]$.

There exists a one to one correspondence between the set C_n^0 of all the differential operators of the form (2) and the set of all n -dimensional subspaces in $C^n[0, 1]$, which have bases $h_1, h_2, \dots, h_n$, with the property (3). This correspondence may be realized by attaching to the differential operator (2) the corresponding differential equation (1) and considering the n -dimensional linear space of solutions of this equation. In this way is defined a mapping

$$\chi: C_n^0 \rightarrow \mathfrak{L}$$

Because the condition (3) is equivalent with the condition in the point (i) of our Definition for the particular case $m = 1$, it follows that there exists a subset $\mathfrak{M}$ of the set C_n^0 such that there exists a one to one correspondence by χ between the sets $\mathfrak{M}$ and $\mathfrak{M}$. According to the above definition of the mapping χ , it follows that $\mathfrak{M} = \chi^{-1}(\mathfrak{M})$ will be the set of all differential operators of the form (2) which have the interpolatory property. In what follows will be proved the topological equivalence of the sets $\mathfrak{M}$ and $\mathfrak{M}$.

Lemma 4. *If χ is the mapping defined above, we have*

$$\chi = \phi r^{-1},$$

χ is an imbedding of C_n^0 into $\mathfrak{L}$, and $\chi|_{\mathfrak{M}}$ is a homeomorphic mapping between $\mathfrak{M}$ and $\mathfrak{M}$.

Proof. We will prove that

$$\phi r^{-1}: C_n^0 \rightarrow \mathfrak{L}^n$$

is a homeomorphic imbedding of the set C_n^0 into $\mathfrak{L}^n$ (for the definition of the set $\mathfrak{L}^n$ see the Lemma 1.). Observe that ϕr^{-1} is well defined. Really, if $(a_1, a_2, \dots, a_n)$ is in C_n^0 , then the set $r^{-1}(a_1, a_2, \dots, a_n)$ contains all the n -tuples $(f_1, f_2, \dots, f_n)$ in $D \subset L^n$, for which $f_1, f_2, \dots, f_n$ form a fundamental system of solutions of the differential equation (1). All these n -tuples are transformed by ϕ in the same element L in $\mathfrak{L}^n$, which is the n -dimensional subspace in $C^n[0, 1]$ of the solutions of differential equation (1). This mapping is obviously one to one. The mappings ϕr^{-1} and $r\phi^{-1} = (\phi r^{-1})^{-1}$ are open maps. Really, if V is an open set of C_n^0 , then $r^{-1}(V)$ is open by the continuity of r (Lemma 2.). Then $\phi(r^{-1}(V))$ is open by the openness of $\phi|_{L^n}$. Indeed, in Lemma 1. it was proved that $\phi|_{L^n}$ is open mapping. But $r^{-1}(V)$ is an open subset in D . According to the condition (3) about the elements in D , it follows that D is open in L^n and then $r^{-1}(V)$ is open also in L^n , i.e., $\phi|_D$ is open mapping. Similarly, if U is an open subset of $\phi r^{-1}(C_n^0)$, then $\phi^{-1}(U)$ is open by the continuity of ϕ (ϕ being the canonical projection in a factorization), and $r(\phi^{-1}(U))$ is open by the openness of r (Lemma 2.). The last part in the lemma follows from the definition of the set $\mathfrak{M}$.

From the Lemma 4. it follows that our Theorem stated in the point 2. above, has the following equivalent formulation:

THEOREM. *The set $\mathfrak{M}$ of all differential operators of the form (2) with a_i in $C[0, 1]$, $i = 1, 2, \dots, n$, which have the interpolatory property on $[0, 1]$ (or, for which any many point boundary value problem for the points in $[0, 1]$ has a solution), is contractible to a point.*

We will prove in what follows this form of our theorem.

Lemma 5. *The set $\mathfrak{M}$ is open in the space C_n^0 .*

Proof. We will prove first that the differential operator (2) has the interpolatory property on the interval $[0, 1]$ if and only if there exists a fundamental system of solutions $f_1, f_2, \dots, f_n$ of the corresponding differential equation (1) such that

$$(5) \quad f_1(x) = W(f_1; x) > 0, \quad W(f_1, f_2; x) > 0, \dots, W(f_1, f_2, \dots, f_n; x) > 0,$$

for x in $[0, 1]$.

Suppose that there exists a fundamental system of solutions of the differential equation (1) such that the inequalities (5) hold. Suppose that the functions $\bar{f}_1, \bar{f}_2, \dots, \bar{f}_n$ are extensions of the functions in this system, which are of class C^n on the interval $(\alpha, \beta) \supset [0, 1]$. Because the Wronskians $W(\bar{f}_1, \bar{f}_2, \dots, \bar{f}_i; x)$, $i = 1, 2, \dots, n$ are continuous functions on (α, β) , according to (5) there exists a positive number ε , such that $[-\varepsilon, 1 + \varepsilon] \subset (\alpha, \beta)$ and the conditions (5) hold with f_i changed in $\bar{f}_i$, $i = 1, 2, \dots, n$ for x in $[-\varepsilon, 1 + \varepsilon]$. But then, from Theorem II of G. PÓLYA in [8], it follows that the differential operator corresponding to the differential equation with the system of functions $\bar{f}_1, \bar{f}_2, \dots, \bar{f}_n$ as fundamental system of solutions, has the interpolatory property on the interval $(-\varepsilon, 1 + \varepsilon)$ and such also on the interval $[0, 1]$, but here it coincides with (2), and the sufficiency of the condition (5) is proved.

Suppose now that (2) has the interpolatory property on $[0, 1]$, and extend the coefficients of (2) as constant functions to $(-\infty, 0]$ and $[1, +\infty)$. From a result of O. ARAMA [1] it follows then that the maximal interval in which the differential operator (2) (with extended coefficients) has the interpolatory property and which begins at 0 in the positive direction, is of the form $[0, \delta)$. But $[0, 1] \subset [0, \delta)$, and therefore $\delta > 1$. Let be now $\beta \in (1, \delta)$. Then the differential operator (2) has the interpolatory property on the interval $[0, \beta]$. The maximal interval beginning with β in negative direction in which (2) has the interpolatory property is of the form $(\alpha, \beta]$. But $[0, \beta] \subset (\alpha, \beta]$ and therefore $\alpha < 0$. In conclusion the differential operator (2) has the interpolatory property on the interval $(\alpha, \beta) \supset [0, 1]$, and then from the Theorem IV in [8], it follows that there exists a fundamental system $f_1, f_2, \dots, f_n$ of solutions of the differential equation (1) corresponding to the differential operator (2) (with extended coefficients) for which the condition (5) is fulfilled for x in (α, β) and therefore also for x in $[0, 1]$. This proves the necessity of the condition (5).

Let us given a differential operator (2) in $\mathcal{M}$ and suppose that $f_1, f_2, \dots, f_n$ is a fundamental system of solutions of the corresponding differential equation (1), having the property (5). Let be r the mapping defined in the point 3. Then $(f_1, f_2, \dots, f_n)$ is in $r^{-1}(a_1, a_2, \dots, a_n)$. If the point $(g_1, g_2, \dots, g_n)$ of C_n^0 is sufficiently near to $(f_1, f_2, \dots, f_n)$, then because of continuity of the Wronskians in (5) we have

$$g_1(x) = W(g_1; x) > 0, W(g_1, g_2; x) > 0, \dots, W(g_1, g_2, \dots, g_n; x) > 0,$$

for x in $[0, 1]$. This means that $r^{-1}(\mathcal{M})$ is open in C_n^0 and also in D . Because of the openness of the mapping r , proved in the Lemma 2., it follows then that $\mathcal{M} = r(r^{-1}(\mathcal{M}))$ is open, which completes the proof of the lemma.

6. Consider the differential equation (1), where the coefficients a_i , $i = 1, 2, \dots, n$ are elements of $C[0, 1]$. The theorem of DE LA VALLÉE POUSSIN [11] asserts that the solutions of the equation (1) form an U.C.S.P. on any interval of the form $[\alpha, \alpha + \beta]$ in $[0, 1]$ for $0 < \beta < h_0$, where h_0 is the positive solution of the n order algebraic equation

$$\sum_{i=1}^n \|a_i\| \frac{h^i}{i!} - 1 = 0,$$

i.e., to any differential equation of form (1), except the equation $y^{(n)} = 0$, corresponds a positive real number h_0 , which will be called the de la Vallée Poussin number (VP-number) of the equation (1), or of the differential operator (2), and this differential operator will have the interpolatory property in any closed interval in $[0, 1]$ of smaller length of its VP-number. In what follows we will attach to the differential equation $y^{(n)} = 0$ the de la Vallée Poussin number equal with $+\infty$. The solutions of this differential equation are the polynomials of order $n - 1$ at most, and they obviously form an U.C.S.P. on all the real line.

Denote by $[\mathbb{R}^+]$ the compactified positive half line.

Lemma 6. Consider the mapping

$$\varphi: C_n^0 \rightarrow [\mathbb{R}^+],$$

which maps a differential operator $a = (a_1, a_2, \dots, a_n)$ in C_n^0 into its VP-number. The mapping φ is continuous.

Proof. We prove first the continuity of the function φ in the set $C_n^0 \setminus \{0\}$, $0 = (0, 0, \dots, 0)$. Denote by $a = (a_1, a_2, \dots, a_n)$ and $b = (b_1, b_2, \dots, b_n)$ two elements of this set, and let be h_a , respective h_b their VP-numbers. Then we have

$$(6) \quad 0 = \sum_{i=1}^n \|a_i\| \frac{h_a^i}{i!} - \sum_{i=1}^n \|b_i\| \frac{h_b^i}{i!} = \sum_{i=1}^n (\|a_i\| - \|b_i\|) \frac{h_a^i}{i!} - \sum_{i=1}^n \|b_i\| \frac{h_b^i - h_a^i}{i!}.$$

If we suppose that a is fixed and let $b \rightarrow a$, then $b_i \rightarrow a_i$ and the first term in the right hand side of the above equality tends to zero. Because it was supposed that $a \neq 0$, there exists a positive number ε and an index i_0 such that $\|a_{i_0}\| > \varepsilon$, and therefore for b sufficiently near to a , we have also $\|b_{i_0}\| > \frac{\varepsilon}{2} > 0$. Because $\|b_i\| \geq 0$, $i = 1, 2, \dots, n$, for a given b all the terms in the last sum of (6) have the same sign. But then it follows that $h_b^i - h_a^i \rightarrow 0$ as $b \rightarrow a$ and therefore $h_b \rightarrow h_a$, and the continuity of φ follows.

To prove the continuity in the point $+\infty$, we will prove that if A is any given positive number, there exists a neighbourhood U of the point θ such that if $a \in U$, then $\varphi(a) > A$. Suppose that $\varepsilon = e^{-A}$ and let be

$$U = \{a = (a_1, a_2, \dots, a_n) \in C_n^0: \|a_i\| < \varepsilon\}.$$

Denote $h_a = \varphi(a)$. Then we have for any $a \in U$, $a \neq \theta$

$$1 = \sum_{i=1}^n \|a_i\| \frac{h_a^i}{i!} < \varepsilon \sum_{i=1}^n \frac{h_a^i}{i!} < \varepsilon e^{h_a} = e^{h_a - A},$$

and therefore $h_a - A > 0$, which completes the proof of our lemma.

Lemma 7. Denote by $\mathcal{V}$ the set in $\mathcal{M}$ of all the differential operators which have their VP-numbers greater then 1. Then $\mathcal{V}$ is a convex set in the linear structure of the space C_n^0 .

Proof. Suppose that the differential operators $a = (a_1, a_2, \dots, a_n)$ and $b = (b_1, b_2, \dots, b_n)$ are in $\mathcal{V}$ and let h_a , respective h_b denote their VP-numbers. Consider the differential operator $c = \lambda a + (1 - \lambda) b$, where $0 \leq \lambda \leq 1$, $c = (c_1, c_2, \dots, c_n)$, and let be h_c the VP-number of c . Then we have

$$\|c_i\| \leq \lambda \|a_i\| + (1 - \lambda) \|b_i\|$$

and therefore for any $h > 0$ the inequality

$$(7) \quad \sum_{i=1}^n \|c_i\| \frac{h^i}{i!} \leq \lambda \sum_{i=1}^n \|a_i\| \frac{h^i}{i!} + (1 - \lambda) \sum_{i=1}^n \|b_i\| \frac{h^i}{i!}$$

holds. Suppose now that $1 < h_a \leq h_b$. Then

$$\sum_{i=1}^n \|b_i\| \frac{h_a^i}{i!} \leq 1,$$

and therefore from the inequality (7) it follows that

$$\sum_{i=1}^n \|c_i\| \frac{h_a^i}{i!} \leq 1.$$

This last inequality implies $h_c \geq h_a > 1$, which proves the lemma.

7. Let the elements $f_1, f_2, \dots, f_n$ in $C^n[0, 1]$ form an U.C.S., and let δ be a real number, $0 < \delta \leq 1$. Consider the mapping $x = \delta t$, $t \in [0, 1]$ of the interval $[0, 1]$ onto the interval $[0, \delta]$. From our definition in 1. it follows immediately that the functions.

$$(8) \quad g_i(t) = f_i(\delta t), \quad t \in [0, 1], \quad i = 1, 2, \dots, n$$

form an U.C.S. Similarly, if we have $f = (f_1, f_2, \dots, f_n) \in D$, then $g = (g_1, g_2, \dots, g_n) \in D$.

Lemma 8. Let be $f = (f_1, f_2, \dots, f_n)$ and $g = (g_1, g_2, \dots, g_n)$ two elements in D which are related by the formula (8). Denote by h_a , respectively by h_b the VP-numbers of the differential operators $r(f_1, f_2, \dots, f_n) = (a_1, a_2, \dots, a_n) = a$ and $r(g_1, g_2, \dots, g_n) = (b_1, b_2, \dots, b_n) = b$. Then

$$h_b \geq \delta^{-1} h_a.$$

Proof. In the notations adopted in the lemma, we have for the coefficients of the differential operator corresponding by r to $g = (g_1, g_2, \dots, g_n)$

$$b_{n-j}(t) = \frac{(-1)^{n-j}}{W(g_1, g_2, \dots, g_n; t)} \begin{vmatrix} g_1(t) & g_2(t) & \dots & g_n(t) \\ \vdots & \vdots & \ddots & \vdots \\ g_1^{(j-1)}(t) & g_2^{(j-1)}(t) & \dots & g_n^{(j-1)}(t) \\ g_1^{(j+1)}(t) & g_2^{(j+1)}(t) & \dots & g_n^{(j+1)}(t) \\ \vdots & \vdots & \ddots & \vdots \\ g_1^{(n)}(t) & g_2^{(n)}(t) & \dots & g_n^{(n)}(t) \end{vmatrix} =$$

$$= \frac{(-1)^{n-j} \delta^{\frac{n(n-1)}{2} + n-j}}{\delta^{\frac{n(n-1)}{2}} W(f_1, \dots, f_n; \delta t)} \begin{vmatrix} f_1(\delta t) & f_2(\delta t) & \dots & f_n(\delta t) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(j-1)}(\delta t) & f_2^{(j-1)}(\delta t) & \dots & f_n^{(j-1)}(\delta t) \\ f_1^{(j+1)}(\delta t) & f_2^{(j+1)}(\delta t) & \dots & f_n^{(j+1)}(\delta t) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n)}(\delta t) & f_2^{(n)}(\delta t) & \dots & f_n^{(n)}(\delta t) \end{vmatrix},$$

$$j = 0, 1, \dots, n-1.$$

From this formula it follows that

$$b_{n-j}(t) = \delta^{n-j} a_{n-j}(\delta t),$$

where a_i , $i = 1, 2, \dots, n$ denote the coefficients of the differential equation corresponding by r to the element $f = (f_1, f_2, \dots, f_n)$. From this relation it follows that

$$\|b_{n-j}\| \leq \delta^{n-j} \|a_{n-j}\|, \quad j = 0, 1, \dots, n-1,$$

and hence

$$\sum_{i=1}^n \|b_i\| \frac{h^i}{i!} \leq \sum_{i=1}^n \|a_i\| \frac{(\delta h)^i}{i!}$$

for any $h > 0$. If $h = \frac{h_a}{\delta}$, then we have

$$\sum_{i=1}^n \|b_i\| \frac{\left(\frac{h_a}{\delta}\right)^i}{i!} \leq \sum_{i=1}^n \|a_i\| \frac{h_a^i}{i!} = 1,$$

and from this inequality it follows that

$$\frac{h_a}{\delta} \leq h_b.$$

8. Lemma 9. All the homotopy groups of the topological space $\mathfrak{M}$ are trivial.

Proof. Let be S a topological sphere of any dimension, and let be

$$\Phi: S \rightarrow \mathfrak{M}$$

a continuous mapping of S in $\mathfrak{M}$. Then $\Phi(S)$ is a compact subset of $\mathfrak{M}$. For any a in $\Phi(S)$ we have for the corresponding VP-number the inequality $\varphi(a) > 0$. Because the continuity of φ (Lemma 6.) and the compactness of $\Phi(S)$, it follows that there exists a positive number ρ such that $\varphi(a) > \rho > 0$ for any differential operator in the set $\Phi(S)$. Suppose $\rho < 1$.

Consider now the linear mapping

$$x = (1 - \lambda + \lambda\rho)t, \quad t \in [0, 1],$$

of the interval $[0, 1]$ onto the interval $[0, 1 - \lambda + \lambda\rho]$, where λ is supposed fixed in $[0, 1]$. (We have obviously $0 < 1 - \lambda + \lambda\rho \leq 1$.) Let be a in $\mathfrak{M}$ and suppose that $f = (f_1, f_2, \dots, f_n) \in r^{-1}(a)$. Introduce the functions

$$g_i(t) = f_i((1 - \lambda + \lambda\rho)t), \quad t \in [0, 1], \quad i = 1, 2, \dots, n.$$

As it was observed at the beginning of the point 7., $(g_1, g_2, \dots, g_n)$ is in $r^{-1}(\mathfrak{M})$. Denote $b = r(g_1, g_2, \dots, g_n)$. With this procedure it was defined a function $b = F(a, \lambda)$ from the set $\mathfrak{M} \times [0, 1]$ to $\mathfrak{M}$. It is easy to see that

the function F is well defined. For this is sufficient to observe that the element b in $\mathcal{M}$ defined in the above correspondence, is independent under a changing of the fundamental system of solutions $f_1, f_2, \dots, f_n$ of the differential equation (1) corresponding to the differential operator a . In what follows it will be proved that the above defined function

$$F: \mathcal{M} \times [0, 1] \rightarrow \mathcal{M}$$

is continuous.

Suppose that U is an open subset of $\mathcal{M}$ and let be (a', λ') a fixed element of $F^{-1}(U)$. Denote by $f_1, f_2, \dots, f_n$ the fundamental system of solutions of the differential equation corresponding to a' , which satisfies the initial value conditions

$$(9) \quad f_i^{(j-1)}(0) = \delta_i^j, \quad i, j = 1, 2, \dots, n.$$

From the definition of the function F it follows that the differential equation which corresponds to the differential operator $F(a', \lambda')$ has a fundamental system of solutions of the form

$$h_i(t) = f_i((1 - \lambda' + \lambda'\rho)t), \quad t \in [0, 1], \quad i = 1, 2, \dots, n,$$

i.e., $(h_1, h_2, \dots, h_n) \in r^{-1}(F(a', \lambda')) \subset r^{-1}(U)$. Because the set $r^{-1}(U)$ is open in $r^{-1}(\mathcal{M})$ and this second set is open in D (this follows from the Lemma 2 and the Lemma 5.), and therefore also in C_n^n according the condition (3) which characterizes the set D , there exists a number $\varepsilon > 0$ such that for any $l = (l_1, l_2, \dots, l_n)$ in C_n^n for which $\|h_i - l_i\| < \varepsilon$, $i = 1, 2, \dots, n$, we have $l \in r^{-1}(U)$.

Let be $\delta_1 > 0$ so chosen that

$$|f_i^{(j)}(x') - f_i^{(j)}(x'')| < \frac{\varepsilon}{3}, \quad i = 1, 2, \dots, n, \quad j = 0, 1, \dots, n,$$

if $x', x'' \in [0, 1]$ and $|x' - x''| < \delta_1$. Introduce the notation

$$M = \max_{1 \leq i \leq n} \|f_i\|,$$

and suppose that $\delta_2 > 0$ is so chosen that

$$|(1 - \lambda' + \rho\lambda')^j - (1 - \lambda'' + \rho\lambda'')^j| < \frac{\varepsilon}{3M}, \quad j = 1, 2, \dots, n,$$

for any $\lambda'' \in [0, 1]$ for which $|\lambda' - \lambda''| < \delta_2$. Denote $\delta = \min(\delta_1, \delta_2)$.

From the Lemma 3. it follows that the mapping $s|B$, where $B = D_0 \cap r^{-1}(\mathcal{M})$, is a homeomorphic mapping of B onto $\mathcal{M}$. Consider the open set

$$G = \{g = (g_1, g_2, \dots, g_n) \in B : \|f_i - g_i\| < \frac{\varepsilon}{3}, \quad i = 1, 2, \dots, n\}.$$

Then because $s|B$ is a homeomorphism, $U_0 = s(G)$ is an open subset of $\mathcal{M}$. Consider also the open set

$$V_0 = \{\lambda'' \in [0, 1] : |\lambda' - \lambda''| < \delta\}$$

The open set $U_0 \times V_0$ in $\mathcal{M} \times [0, 1]$ contains the element (a', λ') and is contained in $F^{-1}(U)$. Really, in order to prove the second affirmation let be $(a'', \lambda'') \in U_0 \times V_0$ and let $(g_1, g_2, \dots, g_n) = s^{-1}(a'')$. Consider the functions

$$l_i(t) = g_i((1 - \lambda'' + \lambda''\rho)t), \quad t \in [0, 1], \quad i = 1, 2, \dots, n.$$

We have for t in $[0, 1]$:

$$\begin{aligned} |h_i^{(j)}(t) - l_i^{(j)}(t)| &= \\ &= |(1 - \lambda' + \lambda'\rho)^j f_i^{(j)}((1 - \lambda' + \lambda'\rho)t) - \\ &\quad - (1 - \lambda'' + \lambda''\rho)^j g_i^{(j)}((1 - \lambda'' + \lambda''\rho)t)| \leq \\ &\leq |(1 - \lambda' + \lambda'\rho)^j - (1 - \lambda'' + \lambda''\rho)^j| |f_i^{(j)}((1 - \lambda' + \lambda'\rho)t)| + \\ &\quad + |f_i^{(j)}((1 - \lambda' + \lambda'\rho)t) - f_i^{(j)}((1 - \lambda'' + \lambda''\rho)t)| (1 - \lambda'' + \lambda''\rho)^j + \\ &\quad + |f_i^{(j)}((1 - \lambda'' + \lambda''\rho)t) - g_i^{(j)}((1 - \lambda'' + \lambda''\rho)t)| (1 - \lambda'' + \lambda''\rho)^j < \\ &< M \frac{\varepsilon}{3M} + \frac{\varepsilon}{3} + |f_i^{(j)}((1 - \lambda' + \lambda'\rho)t) - f_i^{(j)}((1 - \lambda'' + \lambda''\rho)t)|, \\ &j = 0, 1, \dots, n \quad i = 1, 2, \dots, n. \end{aligned}$$

Since $|(1 - \lambda' + \lambda'\rho)t - (1 - \lambda'' + \lambda''\rho)t| = |(\lambda'' - \lambda')(1 - \rho)t| \leq |\lambda' - \lambda''| < \delta \leq \delta_1$, it follows that $|f_i^{(j)}((1 - \lambda' + \lambda'\rho)t) - f_i^{(j)}((1 - \lambda'' + \lambda''\rho)t)| < \frac{\varepsilon}{3}$. The above inequalities then give $|h_i^{(j)}(t) - l_i^{(j)}(t)| < \varepsilon$, $j = 0, 1, \dots, n$, $i = 1, 2, \dots, n$, from which it follows that

$$\|h_i - l_i\| < \varepsilon, \quad i = 1, 2, \dots, n,$$

and therefore, by the choosing of ε , $(l_1, l_2, \dots, l_n) \in r^{-1}(U)$. From the definition of the function F it follows that

$$r(l_1, l_2, \dots, l_n) = F(a'', \lambda'')$$

and then $F(a'', \lambda'') \in U$, i.e., $(a'', \lambda'') \in F^{-1}(U)$. This proves that $U_0 \times V_0 \subset F^{-1}(U)$. It was proved then that each point (a', λ') in $F^{-1}(U)$ has a neighbourhood which is contained in this set, that is, $F^{-1}(U)$ is open, which proves the continuity of the mapping F .

We have $F(a, 0) = a$, and therefore $F(\Phi(S), 0) = \Phi(S)$. Let us denote $\Psi(S) = F(\Phi(S), 1)$. From the continuity of F then it follows that $\Phi \simeq \Psi$ where $\simeq$ denotes the homotopical equivalence.

From the Lemma 8. it follows that $\Psi(S) \subset \mathfrak{Y}$. Indeed, let be a in the set $\Psi(S)$ and determine the VP-number of the differential operator a . According to the definition of the function F , there exists an element $a' \in \Phi(S)$, the corresponding differential equation having a fundamental system of solutions $f_1, f_2, \dots, f_n$ such that

$$h_i(t) = f_i(\rho t), \quad i = 1, 2, \dots, n,$$

and $h_1, h_2, \dots, h_n$ form a fundamental system of solutions for the differential equation corresponding to the element a . For the VP-number of the differential operator a' is valid the inequality $h_{a'} > \rho$. Applying now Lemma 8., we have

$$h_a \geq \rho^{-1} h_{a'} > \rho^{-1} \rho = 1,$$

i.e., $a \in \mathfrak{Y}$. This proves that $\Psi(S) \subset \mathfrak{Y}$ and because $\mathfrak{Y}$ is a convex set in C_n^0 (Lemma 7.), it is contractible to a point ([10], Chapter I, 3.4.), and therefore

$$\Psi \simeq C,$$

where C denotes a constant mapping. But then

$$\Phi \simeq C,$$

which completes the proof of the lemma.

9. *The proof of the theorem.* The topological space $\mathfrak{M}$ is in fact a metric space, being a subset of the linear normed space C_n^0 . According to the Lemma 5, the topology of $\mathfrak{M}$ is a local convex one. Hence $\mathfrak{M}$ is a paracompact space with a locally convex topology, which according to the Lemma 9. has all its homotopy groups trivial. Then an analogous procedure of that in §3. of the paper of KUIPER [4] gives the proof of the theorem.

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