4

in the neighbourhood there exists the inverse function (with respect to y) to the function $\gamma(x, y)$.

Equation (5) is then locally equivalent to that of the form

$$\psi(x) = h(x, \psi[f(x)]). \tag{11}$$

In view of the local equivalence $z = \gamma(x, y) \equiv y = h(x, z)$ and by (10) we get directly the inequality

$$|h(x, z_1) - h(x, z_2)| \leq \vartheta\, |z_1 - z_2|, \tag{12}$$

where $0 < \vartheta = \frac{1}{\theta} < 1$.

In a neighbourhood of zero there are fulfilled hypotheses of the theorem 3.4 [2], and whence we conclude that there exists a continuous solution of equation (11) fulfilling (7). This fact completes the proof.

Similarly one can prove the following

THEOREM 2. *Suppose that $f(x) \in U^p$, $g\left(\frac{1}{x}\right) \in U^{-p}$, $p > 1$, and that the functions $f(x)$, $g\left(\frac{1}{x}\right)$ are continuous in a neighbourhood of zero and $0 < f(x) < x$ there. Further, let the function $x^p g\left(\frac{1}{x}\right)$ fulfil a Lipschitz condition in a neighbourhood of zero. Then for every $\alpha < 0$ there exists exactly one function $\varphi \in U^\alpha$ satisfying equation (1) in a neighbourhood of zero. This function $\varphi(x)$ is continuous in the neighbourhood and fulfils condition*

$$\lim_{x \to 0+} x^{-\alpha}\varphi(x) = \eta,$$

where $\eta = \eta(\alpha)$ denotes the positive number such that

$$\eta^{p-1} = \lim_{x \to 0+} \frac{f(x)^\alpha}{g\left(\frac{1}{x}\right) x^{p(1+\alpha)}}.$$

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SOME RELATIONS BETWEEN INTERVAL FUNCTIONS (I)

by

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Introduction

Let $\mathbf{I} = \{[a_1, a_2] | a_1 \leqslant a_2\} = \{A | a_1 \leqslant a_2\}$ the set of real intervals. If α is a real number, *i.e.* $\alpha \in \mathbf{R}$, then we consider $\alpha = [\alpha, \alpha] \in \mathbf{I}$, therefore $\mathbf{I} \subset \mathbf{R}$.

For an interval $A = [a_1, a_2]$ we denote by $\bar{A}$ the width $a_2 - a_1$.

If $\mathbf{H} \subseteq \mathbf{I}^n$ (n is a natural number), then the map $f: \mathbf{H} \to \mathbf{I}$ is called interval function, if for

$$(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbf{H} \cap \mathbf{R}^n$$

we have

$$f(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbf{R}.$$

Let $\circ \in \{+, -, \cdot\}$ and $\mathbf{H} = \mathbf{I}^2$. The map $f: (A, B) \mapsto A \circ B$ is an interval function. If $\mathbf{H} = \{(A, B) | 0 \notin B\}$, then $f: (A, B) \mapsto A/B$ is an interval function. If $\mathbf{H} = \mathbf{I}^3$, then the maps

$$f: (A, B, C) \mapsto A(B + C),\ g: (A, B, C) \mapsto AB + AC$$

are also interval functions.

If n is a natural number, then we denote

$$nA = [na_1, na_2],\ A^2 = \{uv | u, v \in A\},\quad sqA = \{u^2 | u \in A\}.$$

For $\mathbf{H} = \mathbf{I}$, the maps

$$f: A \mapsto nA,\ g: A \mapsto A^2,\ h: A \mapsto sqA$$

are interval functions.

The map

$$f: [a_1, a_2] \mapsto [a_1 - k, a_2 + k],$$

where k is a positive number, is not an interval function.

If $A, B \in \mathbf{I}$, then we have one of the following relations between the intervals A and B:

$$A = B \text{ i.e. } a_1 = b_1 \,\&\, a_2 = b_2,$$
$$A \leqslant B \text{ i.e. } a_2 \leqslant b_1,$$
$$A \subseteq B \text{ i.e. } b_1 \leqslant a_1 \leqslant a_2 \leqslant b_2,$$
$$A =\!\mid B \text{ i.e. } a_1 \leqslant b_1 \leqslant a_2 \leqslant b_2,$$
$$A \rho B \text{ i.e. } \overline{A} \leqslant \overline{B}$$
$$A \geqslant B \text{ i.e. } B \leqslant A,$$
$$A \supseteq B \text{ i.e. } B \subseteq A$$

and

$$A \mid\!= B \text{ i.e. } B =\!\mid A.$$

The relations $\leqslant$ and $\subseteq$ are transitive. For the relation $=\!\mid$ we have

$$(A =\!\mid B \,\&\, B =\!\mid C) \Rightarrow (A =\!\mid C \vee A \leqslant C).$$

The purpose of the first section of the present paper is the study of the relations between some interval functions, as

$$2AB \text{ and } A^2 + B^2$$

or

$$A^2 - B^2 \text{ and } (A + B)(A - B).$$

We prove that

(1) $$2AB(=\!\mid \cup \leqslant \cup \subseteq) A^2 + B^2$$

and

(2) $$A^2 - B^2 \subseteq (A + B)(A - B).$$

The comparison of the subdistributive law

$$A(B + C) \subseteq AB + AC$$

and the relation (2) is given in the second section. The subdistributive law is verified for many-valued quantities [5] and in a more general case of

the so-called distributive system, [1]. The relations (2) are not valid for manyvalued quantities. The study of distinction between the subdistributive law and the relation (2) is given in the second section of our paper.

The study of the so-called quasioperations with intervals is the purpose of the third section. The quasioperations are generate by the solutions of some interval equations. We give in this introduction the following simple examples of quasioperations.

Let $B \ominus A$ be the solution of the interval equation $A + X = B$ and $B \ominus' A$ the solution of the interval equation $B - X = A$. We have

$$B \ominus A = \begin{cases} \text{interval}, & \text{if } \overline{A} < \overline{B} \\ \varnothing, & \text{if } \overline{A} > \overline{B} \\ \text{real number} & \text{if } \overline{A} = \overline{B} \end{cases}$$

and

$$B \ominus' A = \begin{cases} \text{interval}, & \text{if } \overline{A} > \overline{B} \\ \varnothing, & \text{if } \overline{A} < \overline{B} \\ \text{real number}, & \text{if } \overline{A} = \overline{B}. \end{cases}$$

We have the relations

$$B \ominus A \subseteq B - A, \quad B \ominus' A \subseteq B - A.$$

In the fourth section is given the study of the interval relations in the point of view given in the papers [2, 3]. For the product of relations we have the definition:

if $u, v \in \mathbf{I}^2$, then

$$uv = \{(A, B) \mid \exists C : (A, C) \in u \,\&\, (C, B) \in v\}.$$

For the above interval relations we have for example the formulae:

$$(\leqslant \cdot =\!\mid) = (=\!\mid \cdot \leqslant) = (\leqslant)$$
$$(\subseteq \cdot =\!\mid) = (=\!\mid \cup \subseteq \cup \leqslant)$$
$$(=\!\mid)^2 = (=\!\mid \cup \leqslant)$$
$$(=\!\mid \cdot \mid\!=) = (\mid\!= \cdot =\!\mid) = (=\!\mid \cup \mid\!= \cup \leqslant \cup \subseteq)$$

etc.

If f is an interval function and

(3) $$f(X + Y) \subseteq f(X) + f(Y),$$

then the subdistributive law gives the solution

$$f(X) = AX$$

(where A is an arbitrary interval) of (3). If $A(X+Y) \doteq AX + AY$, then we can give some interval function $g(x)$ for which

$$AX + g(X)$$

is also solution of (3). If $\overline{X} = \overline{Y}$, then (3) is the classical Cauchy functional equation. Some considerations on the solutions of (3) are given in the fifth section.

In the sixth section is given a study of some complex intervals and complex many-valued quantities.

For the definition of operations in the arithmetic of intervals, one can for example see [4].

SECTION 1

The relations $2AB\,(=\mid \cup \leqslant \cup \subseteq) A^2 + B^2$ and $A^2 - B^2 \subseteq (A + B)(A - B)$

§1. The comparison of the intervals $2AB$ and $A^2 + B^2$

1. 1 Let A be a fixed positive interval. If B is a positive interval, then

$$2AB = [2a_1b_1,\ 2a_2b_2],\quad A^2 + B^2 = [a_1^2 + b_1^2,\ a_2^2 + b_2^2].$$

We can have one of the following relations between the endpoints of the intervals $2AB$ and $A^2 + B^2$:

(4) $$2a_1b_1 \leqslant a_1^2 + b_1^2 \leqslant 2a_2b_2 \leqslant a_2^2 + b_2^2$$

or

(5) $$2a_1b_1 \leqslant 2a_2b_2 \leqslant a_1^2 + b_1^2 \leqslant a_2^2 + b_2^2.$$

If A and B are positive intervals, then

$$(4) \Leftrightarrow \left(b_2 \geqslant b_1 \,\&\, b_2 \geqslant \frac{a_1^2 + b_1^2}{2a_2}\right) \Leftrightarrow \Bigg(\bigg(\Big(0 \leqslant b_1 \leqslant a_2 - \sqrt{a_2^2 - a_1^2} \vee b_1 \geqslant a_2 + \sqrt{a_2^2 - a_1^2}\Big) \,\&\, \left(b_2 \geqslant \frac{a_1^2 + b_1^2}{2a_2}\right)\bigg) \vee \Big(a_2 - \sqrt{a_2^2 - a_1^2} \leqslant b_1 \leqslant a_2 + \sqrt{a_2^2 - a_1^2}\Big) \,\&\, (b_2 \geqslant b_1)\Bigg).$$

Thus (4) follows for the domain bounded by the parabola

$$b_2 = \frac{a_1^2 + b_1^2}{2a_2}$$

and the bisector $b_2 = b_1$ (in the second octant, fig. 1), domain denoted by D_1.

If A and B are positive intervals, then

(5) $$\Leftrightarrow b_1 \leqslant b_2 \leqslant \frac{a_1^2 + b_1^2}{2a_2} \Leftrightarrow \left(b_1 \leqslant b_2 \leqslant \frac{a_1^2 + b_1^2}{2a_2} \,\&\, \Big(0 \leqslant b_1 \leqslant a_2 - \sqrt{a_2^2 - a_1^2} \vee b_1 \geqslant a_2 + \sqrt{a_2^2 - a_1^2}\Big)\right).$$

It follows that (5) is valid in the domain D_2 of second octant (Fig. 1).

On the other hand, if A is a positive interval, then

$$(4) \Leftrightarrow 2AB =\mid A^2 + B^2 \Leftrightarrow (b_1,\ b_2) \in D_1 \ (B \text{ is a positive interval})$$

and

$$(5) \Leftrightarrow 2AB \leqslant A^2 + B^2 \Leftrightarrow (b_1,\ b_2) \in D_2 \ (B \text{ is a positive interval}).$$

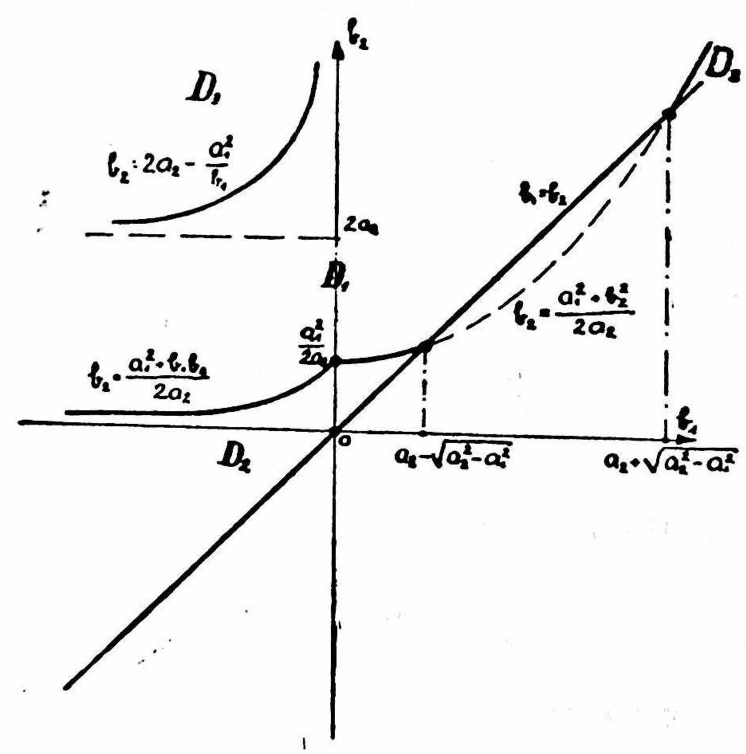

Fig. 1

If B is a mixed interval, then (since A is a positive interval) it follows:

$$2AB = [2a_2b_1,\ 2a_2b_2],\quad A^2 + B^2 = [a_1^2 + b_1b_2,\ a_2^2 + \max(b_1^2,\ b_2^2)].$$

We have one of the following cases:

(6) $$2a_2b_1 \leqslant a_1^2 + b_1b_2 \leqslant 2a_2b_2 \leqslant a_2^2 + \max(b_1^2,\ b_2^2)$$

(7) $$2a_2b_2 \leqslant a_1^2 + b_1b_2$$

and

(8) $$a_1^2 + b_1b_2 \leqslant 2a_2b_1 \leqslant 2a_2b_2 \leqslant a_2^2 + \max(b_1^2,\ b_2^2).$$

If A is a positive interval, B is a mixed interval, then

$$(6) \Leftrightarrow \frac{a_1^2 + b_1b_2}{2a_2} \leqslant b_2 \leqslant 2a_2 - \frac{a_1^2}{b_1} \Leftrightarrow 2AB =\!\!| A^2 + B^2$$

$$(7) \Leftrightarrow b_2 \leqslant \frac{a_1^2 + b_1b_2}{2a_2} \Leftrightarrow 2AB \leqslant A^2 + B^2$$

and

$$(8) \Leftrightarrow b_2 \geqslant 2a_2 - \frac{a_1^2}{b^1} \Leftrightarrow 2AB \subseteq A^2 + B^2.$$

The domains (Fig. 1 second quadrant)

$$D_1 = \{(b_1,\ b_2) | 2AB =\!\!| A^2 + B^2\},$$

$$D_2 = \{(b_1,\ b_2) | 2AB \leqslant A^2 + B^2\}$$

$$D_3 = \{(b_1,\ b_2) | 2AB \subseteq A^2 + B^2\}$$

are delimited by the hyperbolas:

$$b_2 = \frac{a_1^2}{2a_2 - b_1} \text{ and } b_2 = 2a_2 - \frac{a_1^2}{b_1}$$

and by the straight lines

$$b_2 = 0,\ b_1 = 0.$$

If A is a positive and B is a negative interval, then

$$2AB = [2a_2b_1,\ 2a_1b_2],\quad A^2 + B^2 = [a_1^2 + b_2^2,\ a_2^2 + b_1^2]$$

hence

$$2AB \leqslant A^2 + B^2 \Leftrightarrow b_1 \leqslant b_2 \leqslant 0 \Leftrightarrow (b_1,\ b_2) \in D_2, \text{ (third quadrant, Fig.1)}$$

From the above considerations follows

THEOREM 1. *If A is a positive interval, then*

$$2AB\ (=\!\!|\ \cup \leqslant \cup \subseteq)\ A^2 + B^2.$$

We have (fig. 2) (for fixed positive interval A):

$$2AB =\!\!| A^2 + B^2 \Leftrightarrow (b_1, b_2) \in D_1,$$

$$2AB \leqslant A^2 + B^2 \Leftrightarrow (b_1,\ b_2) \in D_2$$

and

$$2AB \subseteq A^2 + B^2 \Leftrightarrow (b_1,\ b_2) \in D_3$$

(see fig. 1).

1. 2. Let A be a mixed fixed interval.

If B is a positive interval, then

$$2AB = [2a_1b_2,\ 2a_2b_2],\quad A^2 + B^2 = [a_1a_2 + b_1^2,\ \max(a_1^2, a_2^2) + b_2^2].$$

From

$$2a_1b_2 \leqslant a_1a_2 + b_1^2 \Leftrightarrow b_2 \geqslant \frac{a_2}{2} + \frac{b_1^2}{2a_1},$$

$$2a_2b_2 \leqslant a_1a_2 + b_1^2 \Leftrightarrow b_1 \leqslant b_2 \leqslant \frac{a_1}{2} + \frac{b_1^2}{2a_2},$$

$$2a_1b_2 \leqslant a_1b_2 + b_1^2$$

and

$$2a_2b_2 \leqslant \max(a_1^2, a_2^2) + b_2^2$$

it follows that (in the case: A is a mixed interval and B is a positive interval):

$$2AB =\mid A^2 + B^2 \Leftrightarrow 2a_1b_2 \leqslant a_1b_2 + b_1^2 \leqslant 2a_2b_2 \leqslant \max (a_1^2, a_2^2) + b_2^2 \Leftrightarrow$$

$$\Leftrightarrow b_2 \geqslant \frac{a_2}{2} + \frac{b_1^2}{2a_1} \;\&\; b_2 \geqslant \frac{a_1}{2} + \frac{b_1^2}{2a_2} \Leftrightarrow (b_1, b_2) \in D_1$$

(fig. 2 second octant),

$$2AB \leqslant A^2 + B^2 \Leftrightarrow 2a_2b_2 \leqslant a_1a_2 + b_1^2 \Leftrightarrow b_1 \leqslant b_2 \leqslant \frac{a_1}{2} + \frac{b_1^2}{2a_2} \Leftrightarrow (b_1, b_2) \in D_2$$

(fig. 2, second octant) and

$$2AB \subseteq A^2 + B^2 \Leftrightarrow a_1a_2 + b^2 \leqslant 2a_1b_2 \leqslant 2a_2b_2 \leqslant \max (a_1^2, a_2^2) + b_2^2 \Leftrightarrow$$

$$\Leftrightarrow b_2 \geqslant \frac{a_2}{2} + \frac{b_1}{2a_1} \Leftrightarrow (b_1, b_2) \in D_3$$ (fig. 2 second octant).

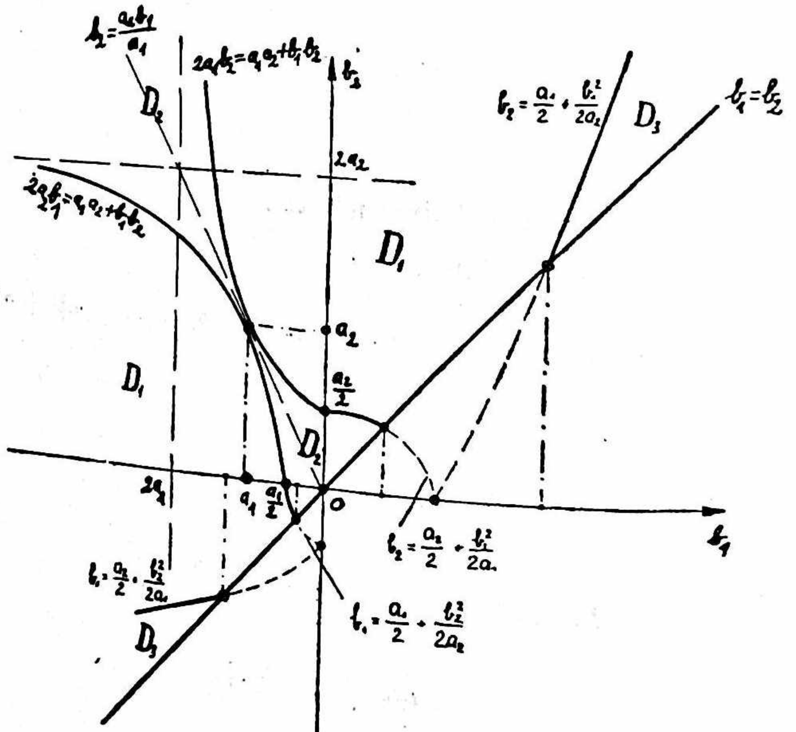

Fig. 2

Let A be a mixed interval and B also a mixed interval. In this case the values of the intervals $2AB$ and $A^2 + B^2$ are given by the table:

Case	$2AB$	$A^2 + B^2$
$a_1 + a_2 \geqslant 0,\ 0 \leqslant b_2 < \frac{a_1b_2}{a_2}$	$[2a_2b_1, 2a_1b_1]$	$[a_1a_2 + b_1b_2, a_2^2 + b_1^2]$
$a_1 + a_2 \geqslant 0,\ \frac{a_1b_1}{a_2} < b_2 < -b_1$	$[2a_2b_1, 2a_2b_2]$	$[a_1a_2 + b_1b_2, a_2^2 + b_1^2]$
$a_1 + a_2 \geqslant 0,\ -b_1 < b_2 < \frac{a_2b_1}{a_1}$	$[2a_2b_1, 2a_2b_2]$	$[a_1a_2 + b_1b_2, a_2^2 + b_2^2]$
$a_1 + a_2 \geqslant 0,\ b_2 \geqslant \frac{a_2b_1}{a_1}$	$[2a_1b_2, 2a_2b_2]$	$[a_1a_2 + b_1b_2, a_2^2 + b_2^2]$
$a_1 + a_2 < 0,\ 0 \leqslant b_2 < \frac{a_2b_1}{a_1}$	$[2a_2b_1, 2a_1b_1]$	$[a_1a_2 + b_1b_2, a_1^2 + b_1^2]$
$a_1 + a_2 < 0,\ \frac{a_2b_1}{a_1} < b_2 < -b_1$	$[2a_1b_2, 2a_1b_1]$	$[a_1a_2 + b_1b_2, a_1^2 + b_1^2]$
$a_1 + a_2 < 0,\ -b_1 < b_2 < \frac{a_1b_1}{a_2}$	$[2a_1b_2, 2a_1b_1]$	$[a_1a_2 + b_1b_2, a_1^2 + b_2^2]$
$a_1 + a_2 < 0,\ b_2 \geqslant \frac{a_1b_1}{a_2}$	$[2a_1b_2, 2a_2b_2]$	$[a_1a_2 + b_1b_2, a_1^2 + b_2^2]$

The hyperbolas

$$2a_2b_1 = a_1a_2 + b_1b_2 \quad \text{and} \quad 2a_1b_2 = a_1a_2 + b_1b_2$$

are tangent in the point (a_1, a_2) to the straight line

$$b_2 = \frac{a_2b_1}{a_1}$$

and are the bounds of the domains D_1 and D_2 (second quadrant, Fig. 2) in which

$$2AB =\mid A^2 + B^2 \text{ respectively } 2AB \leqslant A^2 + B^2.$$

Let A be a mixed interval and B a negative interval. In this case

$$2AB = [2a_2b_1,\ 2a_1b_1]$$

and

$$A^2 + B^2 = [a_1a_2 + b_2^2,\ \max(a_1^2,\ a_2^2) + b_1^2].$$

We have

$$2AB \mathrel{=\!|} A^2 + B^2 \Leftrightarrow 2a_2b_1 \leqslant a_1a_2 + b_2^2 \leqslant 2a_1b_1 \leqslant \max\,(a_1^2,\ a_2^2) + b_1^2 \Leftrightarrow$$
$$\Leftrightarrow b_1 \leqslant \frac{a_1}{2} + \frac{b_2^2}{2a_2}\ \&\ b_1 \leqslant \frac{a_2}{2} + \frac{b_2^2}{2a_1}\ (D_1 \text{ in the third quadrant Fig. 2}),$$
$$2AB \leqslant A^2 + B^2 \Leftrightarrow 2a_1b_1 \leqslant a_1a_2 + b_2^2 \Leftrightarrow b_1 \geqslant \frac{a_2}{2} + \frac{b_2^2}{2a_1}\ (D_2 \text{ in the third quadrant Fig. 2})$$

and

$$2AB \subseteq A^2 + B^2 \Leftrightarrow a_1a_2 + b_2^2 \leqslant 2a_2b_1 \leqslant 2a_1b_1 \leqslant \max\,(a_1^2,\ a_2^2) + b_1^2 \Leftrightarrow$$
$$\Leftrightarrow \frac{a_1}{2} + \frac{b_2^2}{2a_2} \leqslant b_1 \leqslant b_2\ (D_3 \text{ in the third quadrant, Fig. 2}).$$

Hence we have the following result:

THEOREM 2. *If A is a mixed interval and B is an arbitrary interval then*

$$2AB(\mathrel{=\!|} \cup \leqslant \cup \subseteq) A^2 + B^2$$

and we have the domains D_1, D_2 and D_3 in which $2AB \mathrel{=\!|} A^2 + B^2$, $2AB \leqslant A^2 + B^2$ respectively $2AB \subseteq A^2 + B^2$.

1.3. Let A be a negative fixed interval. If B is a positive interval, then $2AB = [2a_1b_2,\ 2a_2b_2]$, $A^2 + B^2 = [a_2^2 + b_1^2,\ a_1^2 + b_2^2]$, hence

$$2AB \leqslant A^2 + B^2\quad (D_2 \text{ in the second octant, Fig. 3}).$$

If B is a mixed interval, then

$$2AB = [2a_1b_2,\ 2a_1b_1],\ A^2 + B^2 = [a_2^2 + b_1b_2,\ a_1^2 + \max\,(b_1^2,\ b_2^2)],$$

hence

$$2AB \mathrel{=\!|} A^2 + B^2 \Leftrightarrow b_1 \leqslant 2a_1 - \frac{a_2^2}{b_2}\ \&\ b_2 \geqslant 2a_1 - \frac{a_2^2}{b_1} \Leftrightarrow$$
$$\Leftrightarrow (b_1,\ b_2) \in D_1\quad (\text{fig. 3, second quadrant})$$
$$2AB \leqslant A^2 + B^2 \Leftrightarrow 2a_1b_1 \leqslant a_2^2 + b_1b_2 \Leftrightarrow (b_1,\ b_2) \in D_2\ (\text{fig. 3, second quadrant}),$$

and

$$2AB \subseteq A^2 + B^2 \Leftrightarrow b_1 \leqslant 2a_1 - \frac{a_2^2}{b_2} \Leftrightarrow (b_1,\ b_2) \in D_3\ (\text{fig. 3, second quadrant}),$$

If B is also a negative interval, then

$$2AB = [2a_2b_2,\ 2a_1b_1],\quad A^2 + B^2 = [a_2^2 + b_2^2, a_1^2 + b_1^2]$$

and in this case

$$2AB \mathrel{=\!|} A^2 + B^2 \Leftrightarrow b_1 \leqslant \frac{a_2^2 + b_2^2}{2a_1} \Leftrightarrow (b_1,\ b_2) \in D_1\ (\text{fig. 3, third quadrant})$$

and

$$2AB \leqslant A^2 + B^2 \Leftrightarrow \frac{a_2^2 + b_2^2}{2a_1} \leqslant b_1 \leqslant b_2 \Leftrightarrow (b_1,\ b_2) \in D_2\ (\text{fig. 3, third quadrant}).$$

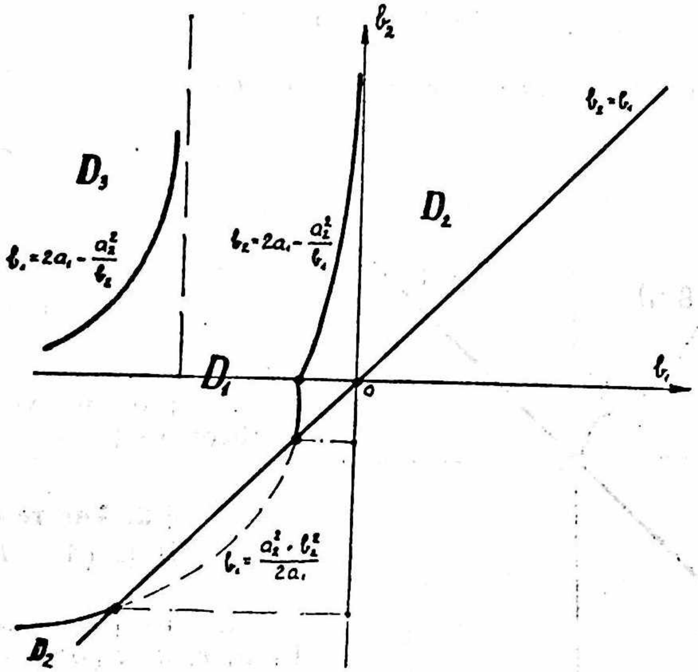

Fig. 3

In this way we have

THEOREM 3. *If A is a negative interval, then*

$$2AB\,(=\!| \ \cup \leqslant \cup \subseteq)\ A^2 + B^2$$

and the domains in which $2AB =\!| A^2 + B^2$, $2AB \leqslant A^2 + B^2$ *respectively*, $2AB \subseteq A^2 + B^2$ *(the domains D_1, D_2, D_3) are represented in Fig. 3.*

From the above theorems we have:

THEOREM 4. *If A and B are arbitrary intervals, then*

$$2AB\,(=\!| \ \cup \leqslant \cup \subseteq)\,A^2 + B^2.$$

Observation. *Let $\Gamma = \{B(t) \equiv (\alpha(t),\ \beta(t)) \mid t \in (0,\ +\infty)\}$ a conti-nuous line so that $B(0) = (0,0)$, $\alpha(t) \neq \beta(t)$ (for $t < 0$) and*

$$t_1 \neq t_2 \Rightarrow B(t_1) \neq B(t_2).$$

In this case $B = B(t)$ is a variable interval and we have

THEOREM 5. *If A is an interval with $\overline{A} > 0$ and t_3 is a real number for which*

$$2AB(t_3) \subseteq A^2 + B^2$$

then $\alpha(t_3)\beta(t_3) < 0$ and there are the real numbers t_1, t_2 for which

$$0 < t_2 < t_1 < t_3$$

and

$$2AB(t_2) \leqslant A^2(t_2) + B^2,$$

$$2AB(t_1) =\!| A^2(t_1) + B^2$$

(see Fig. 4).

The proof is given by the above theorems (1—3).

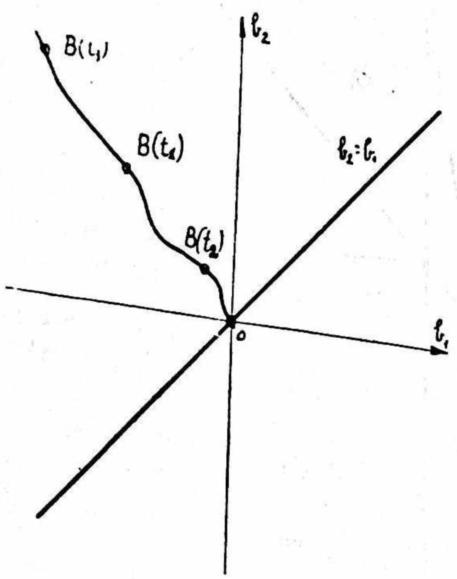

Fig. 4

§ 2. The relation $A^2 - B^2 \subseteq (A + B)(A - B)$

In this paragraph we denote by u, v, w, z positive numbers for which

$$0 < u < v < w < z.$$

If

$$A \in \{[u,\ v],\ [-v,\ -u]\} \quad \text{and} \quad B \in \{[w,\ z],\ [-z,\ -w]\}^{*},$$

then

$$A^2 - B^2 = [u^2 - z^2,\ v^2 - w^2] \ \text{and} \ (A + B)(A - B) = \\ = [-(v + z)(z - u),\ -(u + w)(w - v)]$$

and

$$-(v + z)(z - u) < -(z^2 - u^2) < -(w^2 - v^2) < -(u + w)(w - v).$$

If $A \in \{[-u,\ v],\ [-v,\ u]\}$ and $B \in \{[w,\ z],\ [-z,\ -w]\}$,

then

$$A^2 - B^2 = [-uv - z^2,\ v^2 - w^2] \ \text{and} \ (A + B)(A - B) = \\ = [-(u + z)(v + z),\ (w - u)(v - w)]$$

and

$$-(u + z)(v + z) < -(uv + z^2) < -(w^2 - v^2) < (w - v)(v + z)$$

If $A \in \{[u,\ v],\ [-v,\ -u]\}$ and $B \in \{[-w,\ z],\ [-z,\ w]\}$,

then

$$A^2 - B^2 = [u^2 - z^2,\ v^2 + wz] \ \text{and} \ (A + B)(A - B) = \\ = [-(z - u)(v + z),\ (v + w)(v + z)]$$

and

$$-(z - u)(v + z) < -(z^2 - u^2) < v^2 + wz < (v + w)(v + z).$$

If $A = \{[-u,\ v],\ [-v,\ u]\}$ and $B = \{[-w,\ z],\ [-z,\ w]\}$,

then

$$A^2 - B^2 = [-uv - z^2,\ v^2 + wz] \ \text{and} \ (A + B)(A - B) = \\ = [-(u + z)(v + z),\ (v + w)(v + z)]$$

---

[*]) i.e. A is one of the intervals $[u,\ v]$ or $[-v,\ -u]$ and B is one of the intervals $[w,\ z]$ or $[-z,\ -w]$.

and

$$-(u + z)(v + z) < -(uv + z^2) < v^2 + wz < (v + w)(v + z).$$

In this way we have

THEOREM 6. *If* max $(|a_1|, |a_2|) \leqslant$ min $(|b_1|, |b_2|)$, *then*

$$A^2 - B^2 \subseteq (A + B)(A - B).$$

Let

$$A \in \{[u, w], [-w, -u]\} \text{ and } B \in \{[v, z], [-z, -v]\}.$$

In this case

$$A^2 - B^2 = [u^2 - z^2, w^2 - v^2] \text{ and } (A + B)(A - B) =$$
$$= [-(z - u)(w + z), (w + z)(w - v)]$$

and

$$-(z - u)(w + z) < -(z^2 - u^2) < w^2 - v^2 < (w + z)(w - v).$$

If

$$A \in \{[-u, w], [-w, u]\} \text{ and } B \in \{[v, z], (-z, -v]\}$$

then

$$A^2 - B^2 = [-uw - z^2, w^2 - v^2] \text{ and } (A + B)(A - B) =$$
$$= [-(w + z)(z - u), (w + z)(w - v)]$$

and

$$-(w + z)(z + u) < -(uv + z^2) < w^2 - v^2 < (w + z)(w - v).$$

If

$$A \in \{[u, w], [-w, -u]\} \text{ and } B \in \{[-v, z], [-z, v]\}$$

then

$$A^2 - B^2 = [u^2 - z^2, w^2 + vz] \text{ and } (A + B)(A - B) =$$
$$= [-(w + z)(z - u), (w + z)(w + v)]$$

and

$$-(w + z)(z - u) < -(z^2 - u^2) < w^2 + vz < (w + z)(w + v).$$

If

$$A \in \{[-u, w], [-w, u]\} \text{ and } B \in \{[-v, z], [-z, v]\}$$

then

$$A^2 - B^2 = [-uw - z^2, w^2 + vz] \text{ and } (A + B)(A - B) =$$
$$= [-(u + z)(w + z), (v + w)(z + w)]$$

and

$$-(u + v)(w + z) < -(uw + z^2) < w^2 + vz < (v + w)(z + w).$$

Ih this way we have

THEOREM 7. *If*

$$\min(|a_1|, |a_2|) \leqslant \min(|b_1|, |b_2|) \leqslant \max(|a_1|, |a_2|) \leqslant \max(|b_1|, |b_2|),$$

then

$$A^2 - B^2 \subseteq (A + B)(A - B).$$

Similarly we have the proofs of the following results:

THEOREM 8. *If*

$$\min(|a_1|, |a_2|) \leqslant, \min(|b_1|, |b_2|) \leqslant \max(|b_1|, |b_2|) \leqslant \max(|a_1|, |a_2|),$$

then

$$A^2 - B^2 \subseteq (A + B)(A - B).$$

THEOREM 9. *If*

$$\min(|b_1|, |b_2|) \leqslant \min(|a_1|, |a_2|) \leqslant \max(|a_1|, |a_2|) \leqslant \max(|b_1|, |b_2|)$$

then

$$A^2 - B^2 \subseteq (A + B)(A - B).$$

THEOREM 10. *If*

$$\min(|b_1|, |b_2|) \leqslant \min(|a_1|, |a_2|) \leqslant \max(|b_1|, |b_2|) \leqslant \max(|a_1|, |a_2|)$$

then

$$A^2 - B^2 \subseteq (A + B)(A - B).$$

THEOREM 11. *If*

$$\max(|b_1|, |b_2|) \leqslant \min(|a_1|, |a_2|)$$

then

$$A^2 - B^2 \subseteq (A + B)(A - B).$$

The above conditions represent a full system for the choice of the interval-pairs (A, B), hence follows

THEOREM 12. *If A and B are intervals, then*

$$A^2 - B^2 \subseteq (A + B)(A - B). \tag{2}$$

SECTION 2

THE COMPARISON OF THE SUBDISTRIBUTIVE LAW AND OF THE RELATION (2)

If A, B, C are intervals, then it follows the subdistributive law

$$A(B + C) \subseteq AB + AC. \tag{9}$$

The subdistributive law is valid for the many valued quantities [5] and in particular for intervals.

For the so-called distributive systems we have also the subdistributive law [1]. Let $(M, +, \cdot)$ a nonemty set M and $+, \cdot$ two binary operations for which

$$a, b, c \in M \Rightarrow a.(b + c) = ab + ac.$$

Let $\mathfrak{M}$ be the family of the subsets of M, $\mathcal{N} \subseteq \mathfrak{M}$ and for

$$A, B \in \mathfrak{M}$$

(i.e. $A, B \subseteq \mathfrak{M}$) and

$$\circ \in \{+, \cdot\},$$

let

$$A \circ B = \{r \circ s \mid r \in A,\ s \in B\}.$$

The system $(\mathfrak{M}, +, \cdot)$ is called associated to the system $(M, +, \cdot)$. In this case we have

THEOREM 13. *If $(M, +, \cdot)$ is a distributive system and $(\mathfrak{M}, +, \cdot)$ is the associated system and $\mathcal{N} \subseteq \mathfrak{M}$ with the closure property*

$$\circ \in \{+, \cdot\} \ \&\ A, B \in \mathcal{N} \Rightarrow A \circ B \in \mathcal{N}$$

then for A, B, $C \in \mathcal{N}$ we have the subdistributive law:

$$A(B + C) \subseteq AB + AC.$$

In the special case $M = \mathbf{R}$, $\mathcal{N} = \mathfrak{M} = \mathfrak{L}(\mathbf{R})$ we have the subdistributive law for the many valued quantities. If $\mathcal{N}$ is the set of real intervals, then we have the subdistributive law for intervals.

For many valued quantities the relation (2) is not valid.

We give the following example. Let

$$A = \{1, 5\} \text{ and } B = \{-2, 6\}.$$

In this case

$$A^2 = \{1, 5, 25\},\ B^2 = \{-12, 4, 36\},$$

$$A^2 - B^2 = \{-35, -31, -11, -3, 1, 13, 17, 21, 37\}$$

$$A + B = \{-1, 3, 7, 11\},\ A - B = \{-5, -1, 3, 7\}$$

and

$$(A + B)(A - B) = \{-55, -35, -15, -11, -7, -3, 1, 5, 9, 21, 33, 49, 77\}.$$

In this way

$$A^2 - B^2 \not\subseteq (A + B)(A - B).$$

If U is a bounded set of real numbers, then let

$$\mathfrak{J}\,(U) = [u_1, u_2]$$

the smalest closed interval for which $U \subseteq [u_1, u_2]$.

For many valued quantities we have the following result:

THEOREM 14. *If U and V are bounded sets of real numbers, then*

$$\mathfrak{J}\,(U^2 - V^2) \subseteq \mathfrak{J}\,((U + V)(U - V)). \tag{10}$$

Proof. If U and V are closed intervals, then

$$U^2 - V^2 = \mathfrak{J}\,(U^2 - V^2) \subseteq (U + V)(U - V) = \mathfrak{J}\,((U + V)(U - V)).$$

If $U^2 - V^2$ or $(U + V)(U - V)$ is not an interval, then we suppose that exists a real number w for which

$$w \in \mathfrak{J}\,(U^2 - V^2) \;\&\; w \notin \mathfrak{J}\,((U + V)(U - V)).$$

If we prove that this hypotesis is false, then the theorem is proved.

$\mathfrak{J}\,((U + V)(U - V))$ is an interval and in this way from

$$w \in \mathfrak{J}\,(U^2 - V^2) \quad \text{and} \quad w \notin \mathfrak{J}\,((U + V)(U - V))$$

it follows that there is an interval which is contained in

$$\mathfrak{J}\,(U^2 - V^2)$$

and is not contained in

$$\mathfrak{J}\,((U + V)(U - V)).$$

This configuration gives the proof that the above hypotesis is not valid.

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COMPLÉMENTS AU TRAITÉ DE MITRINOVIC (I). QUELQUES INÉGALITÉS INTÉGRALES

par

BORISLAV CRSTICI et GHEORGHE TUDOR

à Timişoara à Timişoara

1. Soit u une fonction réelle telle que $|u(x)| \leqslant 1$ pour tout x de son ensemble de définition D. Si pour un $x_1 \in D$ on a $u(x_1) \geqslant 0$ alors nous avons

$$1 \geqq u(x_1) \geqq u^2(x_1) \geqq u^3(x_1) \geqq \ldots . \geqq u^{n-1}(x_1) \geqq 0$$

Considérons la somme

$$F_n(x) \stackrel{\mathrm{df}}{=} \sum_{k=1}^{n} u^{k-1}(x_1) \frac{\sin kx}{k} \tag{1}$$

Si $u(x_1) = 0$, nous convenons de prendre $u^0(x_1) = 1$.

On peut écrire la somme (1) de la manière suivante

$$F_n(x) = \sum_{k=1}^{n-1} [u^{k-1}(x_1) - u^k(x_1)] \sum_{j=1}^{k} \frac{\sin jx}{j} + u^{n-1}(x_1) \sum_{j=1}^{n} \frac{\sin jx}{j} \tag{2}$$

d'où on déduit compte tenant de l'inégalité de Fejér-Jackson

$$F_n(x) \stackrel{\mathrm{df}}{=} \sum_{k=1}^{n} u^{k-1}(x_1) \frac{\sin kx}{k} > 0, \;\; x \in \,]\,0, \pi\,[\tag{3}$$

Si pour un $x_2 \in D$ on a $u(x_2) < 0$ alors nous avons

$$1 \geqq |u(x_2)| \geqq u^2(x_2) \geqq |u^3(x_2)| \geqq \ldots > 0$$

Soit la somme

$$G_n(x) \overset{\text{df}}{=} \sum_{k=1}^{n} u^{k-1}(x_2) \frac{\sin kx}{k} \tag{4}$$

Nois écrivons cette somme sous la forme

$$\begin{aligned} G_n(x) &= \sum_{k=1}^{n-1} (-1)^{k-1} [u^{k-1}(x_2) + u^k(x_2)] \sum_{j=1}^{k} (-1)^{j-1} \frac{\sin jx}{j} + \\ &+ (-1)^{n-1} u^{n-1}(x_2) \sum_{j=1}^{n} (-1)^{j-1} \frac{\sin jx}{j} \end{aligned} \tag{5}$$

En tenant compte de l'inégalité

$$\sum_{j=1}^{n} (-1)^{j-1} \frac{\sin jx}{j} > 0, \quad x \in]0, \pi[$$

nous déduisons de (4) l'inégalité

$$G_n(x) \overset{\text{df}}{=} \sum_{k=1}^{n} u^{k-1}(x_2) \frac{\sin kx}{k} > 0, \quad x \in]0, \pi[\tag{6}$$

Nous pouvons donc énoncer le

THÉORÈME 1. *Soit la fonction* $u : D \to [-1, 1]$, *D ensemble quelconque. On a l'inégalité*

$$H_n(x ; y) \overset{\text{df}}{=} \sum_{k=1}^{n} u^{k-1}(y) \frac{\sin kx}{k} > 0, \ y \in D, \ x \in]0, \pi[\tag{7}$$

Cas particulier. Si $D =]0, \pi[$ alors nous avons

$$H_n(x, y) \overset{\text{df}}{=} \sum_{k=1}^{n} u^{k-1}(y) \frac{\sin kx}{k} > 0, \ (x, y) \in]0, \pi] \times]0, \pi[\tag{7'}$$

et plus particulièrement

$$H_n(x, x) \overset{\text{df}}{=} \sum_{k=1}^{n} u^{k-1}(x) \frac{\sin kx}{k} > 0, \ x \in]0, \pi[\tag{7''}$$

En s'appuyant sur l'inégalité de W. H. Young

$$1 + \sum_{k=1}^{n} \frac{\cos kx}{k} > 0, \quad x \in]0, \pi[\tag{8}$$

et sur l'inégalité

$$1 + \sum_{k=1}^{n} (-1)^k \frac{\cos kx}{k} > 0, \quad x \in]0, \pi[\tag{9}$$

nous déduisons de la même manière le

THÉORÈME 2. *Soit la fonction* $u : D \to [-1, 1]$, *D ensemble quelconque. On a l'inégalité*

$$K_n(x ; y) \overset{\text{df}}{=} 1 + \sum_{k=1}^{n} u^k(y) \frac{\cos kx}{k} > 0, \quad y \in D, \ x \in]0, \pi[\tag{10}$$

Cas particulier. Si $D =]0, \pi[$, alors nous avons

$$K_n(x ; y) \overset{\text{df}}{=} 1 + \Sigma\, u^k(y) \frac{\cos kx}{k} > 0, \ (x, y) \in]0, \pi[\times]0, \pi[\tag{10'}$$

et plus particulièrement

$$K_n(x, x) \overset{\text{df}}{=} 1 + \sum_{k=1}^{n} u^k(x) \frac{\cos kx}{k} > 0, \quad x \in]0, \pi[\tag{10''}$$

Dans la suite nous allons utiliser les notations

$$h_n(x) = H_n(x ; x), \quad b(x) = K_n(x, x)$$

2. Les théorèmes donnés nous permettent d'établir deux classes d'inégalités intégrales intéressantes.

Nous avons successivement

$$u(x) h_n(x) = \sum_{k=1}^{n} u^k(x) \frac{\sin kx}{k} \tag{11}$$

$$[u(x) h_n(x)]' = u'(x) \sum_{k=1}^{n} u^{k-1}(x) \sin kx + u(x) \sum_{k=1}^{n} u^{k-1}(x) \cos kx$$

$$\begin{aligned} [u(x) h_n(x)]' &= u'(x) \frac{u^{n+1}(x) \sin nx - u^n(x) \sin (n+1)x + \sin x}{u^2(x) - 2u(x) \cos x + 1} + \\ &+ u(x) \frac{u^{n+1}(x) \cos nx - u^n(x) \cos (n+1)x + \cos x - u(x)}{u^2(x) - 2u(x) \cos x + 1} \end{aligned}$$

En intégrant (11) de 0 à $x < \pi$ et en tenant compte que $h_n(x) > 0$ pour $x \in]0, \pi[$, nous obtenons l'inégalité intégrale

$$\int_0^x [u'(t)S_n(t, u(t)) + u(t)C_n(t, u(t))]dt > 0 \quad (< 0) \tag{12}$$

valable pour tout $x \in]0, \pi[$ tel que $u(x) > 0$ $(u(x) < 0)$ où $S_n(x, u(x))$ et $C_n(x, u(x))$ sont les deux fractions dans le membre droit de la formule (11), facteurs auprès de $u'(x)$ respectivement de $u(x)$.

D'une manière analogue nous obtenons l'inégalité intégrale

$$\int_0^x [u'(t)C_n(t, u(t)) - u(t)S_n(t, u(t))]\, dt + 1 + \sum_{k=1}^{n} \frac{u^k(0)}{k} > 0, \quad x \in]0, \pi[\tag{13}$$

3. Voici maintenant quelques exemples d'inégalités intégrales que nous pouvons tirer de l'inégalité (12) et de l'inégalité (13). Prenons dans (12)

$$u(x) = \cos x + \alpha \sin x \tag{14}$$

avec α vérifiant la condition

$$-\operatorname{ctg} t < \alpha \leqslant \operatorname{tg} \frac{t}{2} \tag{15}$$

quel que soit $t \in]0, x[$

En effectuant tous les calculs dans (12) pour ce cas particulier nous aboutissons à l'inégalité

$$\int_0^x (\cos t + \alpha \sin t)^n \frac{\sin (n+1)t}{\sin t}\, dt > x \tag{16}$$

pour $x \in]0, \pi[$ et α vérifiant la condition (15).

L'inégalité (13) nous donne dans le cas particulier de la fonction (14) et avec α vérifiant les conditions

$$-\operatorname{ctg} \frac{t}{2} \leqslant \alpha \leqslant \operatorname{tg} \frac{t}{2}, \quad \forall\, t \in]0, x[, \ \forall\, x \in]0, \pi[\tag{17}$$

l'inégalité intégrale

$$\begin{aligned} 0 < \int_0^x \frac{1 - (\cos t + \alpha \sin t)^n \cos (n+1)t}{\sin t}\, dt < 1 + \sum_{k=1}^{n} \frac{1}{k} + \\ + \ln \left(\sec \frac{x}{2}\right)^2 < 2 + \ln n + \ln \left(\sec \frac{x}{2}\right)^2 < \ln \left(e\sqrt{n} \sec \frac{x}{2}\right)^2 \end{aligned} \tag{18}$$

pour $x \in]0, \pi[$ et α vérifiant (17).

Nous opinons que les cas particuliers qu'on peut obtenir de (16) et (18) pour $\alpha = 0$ sont très intéressants. Nous avons

$$\int_0^x \cos^n t \frac{\sin (n+1)t}{\sin t}\, dt > x, \quad x \in \left]0, \frac{\pi}{2}\right[\tag{19}$$

$$0 < \int_0^x \frac{1 - \cos^n t \cos (n+1)t}{\sin t}\, dt < \ln \left(e\sqrt{n} \sec \frac{x}{2}\right)^2 \quad x \in]0, \pi[. \tag{20}$$

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