Now (5.1) and (5.2) imply respectively that $A^T y + c \in C^*$ and $A^T y - c$ are both *CONS*. Hence, it is impossible for sup $(-c, x)$ subject to $Ax = b$, $x \in C$ to be *AUBD*, which would necessarily follow if (I.C) were (*IAC*). Hence, (II.C*) is *CONS* and (I.C) is not (*IAC*).

Corollary 3.1. (I.C) $IAC \Rightarrow I_{(-r)}$ is $H^{-}AC$.

Proof Assume (I.C) is *IAC*. Then by Lemma 2, $I_{(r)}$ is *HSINC*. If, to the contrary, $I_{(-r)}$ were $H^{-}SINC$, then it would follow that $\{u_k\} \in AF_0(1) \Rightarrow$ $\Rightarrow (c, u_k) \to 0$, which by Lemma 3 is a contradiction. Hence, $I_{(-r)}$ is $H^{-}AC$.

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UNRESTRICTED DIFFERENTIAL n-PARAMETER FAMILIES I. CHARACTERIZATION AND TRANSFORMATION THEOREMS

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0. Introduction and definitions

Denote by $[\alpha, \beta]$ a finite closed interval of the real axis $\mathbf{R}$. Let $C[\alpha, \beta]$ and $C^n[\alpha, \beta]$ be the linear spaces of real and continuous functions, and respective the real functions with continuous, n^{th} derivatives, with the usual norms.

In a previous paper [1] we have introduced the notion of the differential n-parameter families. The differential n-parameter families are n-parameter families in the sense of L. TORNHEIM [2], which considered as manifolds in the space $C[\alpha, \beta]$, are differential manifolds and have as tangent spaces n-dimensional Chebyshev subspaces of $C[\alpha, \beta]$.

The case when $C[\alpha, \beta]$ is changed in $C^n[\alpha, \beta]$, and the Chebyshev spaces in unrestricted Chebyshev spaces is particularly important, because it has a strong connection with the disconjugate n^{th} order differential equations. The aim of our paper is the transpozition of the results in [1] to this particular case with releaving of the special problems, and the applications of the obtained results to many point boundary value problems for nonlinear differential equations. We will deal with this last problem in the second part of our paper.

Definition 1. *Suppose that $Y(x, a)$ is a real valued function defined on the direct product $[\alpha, \beta] \times \mathbf{R}^n$, having partial derivatives with respect to x of orders $1, 2, \ldots, n$, denoted by $Y^{(j)}(a, x)$, $j = 1, 2, \ldots, n$, which are continuous functions on $[\alpha, \beta] \times \mathbf{R}^n$. It will be said that Y forms (or is) an unrestricted n-parameter family (abbreviated, UnF), if for any natural*

m, $1 \leqslant m \leqslant n$, *any distinct points* $x_1, x_2, \ldots, x_m$ *in* $[\alpha, \beta]$ *and any natural numbers* $k_1, k_2, \ldots, k_m$ *with the property* $\sum_{i=1}^{m} k_i = n$, *there exists a single point* a *in* $\mathbf{R}^n$ *such that*

$$Y^{(j)}(x_i, a) = y_i^j,\ j = 0, \ldots, k_i - 1,\ i = 1, 2, \ldots, m$$

for any fixed system of reals y_i^j, $j = 0, 1, \ldots, k_i - 1$, $i = 1, 2, \ldots, m$.

If Y is satisfying Definition 1 and is linear (additive and homogeneous) with respect to a, then the corresponding *UnF* will be called an unrestricted Chebyshev space. As it is easy to see, in this case Y is an n-dimensional subspace of $C^n[\alpha, \beta]$.

The space $\mathbf{R}^n$ is called the space of parameters. The coordinates of an element a in $\mathbf{R}^n$ will be denoted by $a^1, a^2, \ldots, a^n$ and are called the parameters of the corresponding function $Y(x, a)$ in $C^n[\alpha, \beta]$.

In what follows we will refer sometimes to the function Y satisfying Definition 1. as to an *UnF*.

Definition 2. *Suppose that* Y *forms an UnF in the sense of Definition* 1, *and that* $\frac{\partial Y^{(j)}(a, x)}{\partial a^i}$, $j = 0, 1, \ldots, n$, $i = 1, 2, \ldots, n$ *exist and are continuous functions on the direct product* $[\alpha, \beta] \times \mathbf{R}^n$. *If the functions* $\frac{\partial Y(x, a)}{\partial a^i}$, $i = 1, 2, \ldots, n$ *generate an unrestricted Chebyshev space for any* $a \in \mathbf{R}^n$ i.e., *they form a so called unrestricted Chebyshev system*), *then the corresponding UnF is called unrestricted differential n-parameter family (abbreviated, UDnF).*

If Y satisfies the conditions in the Definition 2, then it may be considered a differential manifold in $C^n[\alpha, \beta]$, which has in any point a tangent space $L(a)$ which is an n-dimensional unrestricted Chebyshev subspace of the space $C^n[\alpha, \beta]$.

1. The topological characterisation of UnF-s and UDnF-s.

In this paragraph we will give simple topological characterizations of *UnF*-s and *UDnF*-s. The idea in the following characterization theorems is to interchange the role of the variable x and the parameter a, i.e., to consider the function $Y(x, a)$ as a family of functions from R^n to R^n depending on the parameter x in $[\alpha\ \beta]$.

THEOREM 1. *The real valued function* $Y(x, a)$ *defined on the direct product* $[\alpha, \beta] \times \mathbf{R}^n$, *having partial derivatives of orders* $1, 2, \ldots, n$, *with respect to* x *in* $[\alpha, \beta]$, *denoted by* $Y^{(j)}(x, a)$ $j = 1, 2, \ldots, n$ *which are continuous on* $[\alpha, \beta] \times \mathbf{R}^n$, *forms an UnF, if and only if for any natural number* m, $1 \leqslant m \leqslant n$, *any distinct points* $x_1, x_2, \ldots, x_m$ *in* $[\alpha, \beta]$ *and any natural numbers* $k_1, k_2, \ldots, k_m$ *which satisfy* $\sum_{i=1}^{m} k_i = n$, *the mapping with the coordinate functions*

$$f^{i,j}(a) = Y^{(j)}(x_i, a),\ j = 0, 1, \ldots, k_i - 1,\ i = 1, 2, \ldots, m, \tag{1}$$

is a homeomorphism of $\mathbf{R}^n$ *onto itself.*

The proof of the theorem is analogous with the proof of Theorem 1 in [1].

THEOREM 2. *The real valued function* $Y(x, a)$, *defined on the direct product* $[\alpha, \beta] \times \mathbf{R}^n$, *having partial derivatives of orders* $1, 2, \ldots, n$ *with respect to* x *in* $[\alpha\ \beta]$, *denoted by* $Y^{(j)}(x, a)$, $j = 1, 2, \ldots, n$, *and partial derivatives of form* $\frac{\partial Y^{(j)}(x, a)}{\partial a^i}$, $j = 0, 1, \ldots, n$, $i = 1, 2, \ldots, n$ *which are all continuous on* $[\alpha, \beta] \times \mathbf{R}^n$, *forms an UDnF, if and only if for any natural number* m, $1 \leqslant m \leqslant n$, *any distinct points* $x_1, x_2, \ldots, x_m$ *in* $[\alpha, \beta]$, *and any natural numbers* $k_1, k_2, \ldots, k_m$, *which satisfy* $\sum_{i=1}^{m} k_i = n$, *the mapping with the coordinate functions* (1) *is a diffeomorphism of* $\mathbf{R}^n$ *onto itself.*

Proof. From the Definition 2. it follows that if $Y(x, a)$ is an *UDnF*, then the Jacobian determinant of the mapping (1) is everywhere different from zero, i.e., the mapping (1) is a local diffeomorphism in any point a, and according our Theorem 1, it is also a global diffeomorphism. Conversely, the condition of non vanishing of the Jacobian determinant of the mapping (1) is equivalent with the final part of the Definition 2, and the proof is complete.

2. A transformation theorem

Applying the results of the paragraph 1, we will prove here the following theorem:

THEOREM 3. *Suppose that* $Y(x, a)$ *is an UnF (respectively, an UDnF) and let be:*

(i) ψ *a homeomorphism (respectively, a diffeomorphism) of the space* $\mathbf{R}^n$ *onto itself;*

(ii) χ *a diffeomorphism of class* C^n *of the interval* $[\alpha, \beta]$ *onto itself;*

(iii) $\varkappa$ *a diffeomorphism of class* C^n *(respectively, of class* C^{n+1}*) of the space* $\mathbf{R}$ *onto itself;*

(iv) ψ *a fixed element of the space* $C^n[\alpha, \beta]$;

then the function

$$Z(x, a) = \varkappa(Y(\chi(x), \Psi(a))) + \psi(x)$$

is also an UnF (respectively, an UDnF).

Proof. (i) Let Ψ be as in the theorem. If m; $x_1, x_2, \ldots, x_m$; $k_1, k_2, \ldots, k_m$ satisfy the conditions in the Definition 1, then the mapping with the coordinate functions

$$Y^{(j)}(x_i, \Psi(a)),\ j = 0, 1, \ldots, k_i - 1,\ i = 1, 2, \ldots, m$$

will be a homeomorphism of $\mathbf{R}^n$ onto itself, because it is the composition of the homeomorphism Ψ and the homeomorphism with the coordinate functions

$$Y^{(j)}(x_i, b),\ j = 0, 1, \ldots, k_i - 1,\ i = 1, 2, \ldots, m,$$

which are both onto. But then, according to the Theorem 1, the function

$$Y(x, \Psi(a))$$

will be an *UnF*.

In the case when Y is an *UDnF* and Ψ is a diffeomorphism, we have

$$\left\| \frac{\partial Y^{(j)}(x_i, \Psi(a))}{\partial a^k} \right\|_{j=0,1,\ldots,k_i-1,\ i=1,2,\ldots,m\ k=1,2,\ldots,n} =$$

$$= \left\| \frac{\partial Y^{(j)}(x_i), \Psi(a))}{\partial b^k} \right\|_{j=0,\ldots,k_i-1,\ i=1,\ldots,m\ k=1,\ldots,n} \left\| \frac{\partial \Psi^l(a)}{\partial a^k} \right\|_{k,\,l=1,2,\ldots,n},$$

where we have denoted $b = \Psi(a)$, $b = (b^1, b^2, \ldots, b^n)$ and $\Psi = (\Psi^1, \Psi^2, \ldots, \Psi^n)$. The matrices in the right hand side are nonsingular according to the Theorem 2 and the definition of the mapping Ψ. Then their product is nonsingular, and therefore from the Theorem 2 it follows that

$$Y(x, \Psi(a))$$

is an *UDnF*.

(ii) Consider the function

$$Z(x, a) = Y(\chi(x), a),$$

where $\chi(x)$ is as in the theorem. Let m; $x_1, x_2, \ldots, x_m$; $k_1, k_2, \ldots, k_m$ be as in the Definition 1. Differentiating j times the function Z with respect to x, we obtain

$$Z^{(j)}(x_i, a) = Y'(\chi(x_i), a)\, \chi^{(j)}(x_i) + \ldots + Y^{(j)}(\chi(x_i), a)(\chi'(x_i))^j \tag{2}$$

$$j = 0, 1, \ldots, k_i - 1,\ i = 1, 2, \ldots, m,$$

where $Y^{(k)}$ denotes the k^{th} partial derivative of the function Y with respect to its first variable. From this relation it follows that the mapping with the coordinate functions

$$Z^{(j)}(x_i, a),\quad j = 0, 1, \ldots, k_i - 1,\ i = 1, 2, \ldots, m$$

is the composition of the mapping with the coordinate functions

$$Y^{(j)}(\chi(x_i), a),\ j = 0, 1, \ldots, k_i - 1,\ i = 1, 2, \ldots, m,$$

and the linear mapping with the matrix

$$M = \begin{Vmatrix} M_1 & 0 & \ldots & 0 \\ 0 & M_2 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & M_m \end{Vmatrix},$$

where

$$M_i = \begin{Vmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & \chi'(x_i) & 0 & \ldots & 0 \\ 0 & \chi''(x_i) & (\chi'(x_i))^2 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \chi^{(k_i-1)}(x_i) & \ldots & \ldots & (\chi'(x_i))^{k_i-1} \end{Vmatrix},$$

$$i = 1, 2, \ldots, m.$$

We have then

$$\det M = (\chi'(x_1))^{\frac{k_1(k_1-1)}{2}} (\chi'(x_2))^{\frac{k_2(k_2-1)}{2}} \ldots (\chi'(x_m))^{\frac{k_m(k_m-1)}{2}},$$

and because of the condition (ii) in the theorem $\det M \neq 0$, i.e., the linear mapping is nonsingular. This means that the composite mapping is a homeomorphism onto, and therefore, by the Theorem 1, $Z(x, a)$ is an *UnF*.

If we suppose that $Y(x, b)$ is an $UDnF$, then differentiating (2) with respect to a^l, we obtain

$$\frac{\partial Z^{(j)}(x_i, a)}{\partial a^l} = \frac{\partial Y'(\chi(x_i), a)}{\partial a^l}\chi^{(j)}(x_i) + \dots + \frac{\partial Y^{(j)}(\chi(x_i)\, a)}{\partial a}(\chi'(x_i))^j$$

and therefore we have

$$\left\| \frac{\partial Z^{(j)}(x_i, a)}{\partial a^l} \right\|_{j=0,1,\dots,k_i-1,\ i=1,2,\dots,m,\ l=1,2,\dots,n} =$$

$$= M \cdot \left\| \frac{\partial Y^{(j)}(\chi(x_i), a)}{\partial a^l} \right\|_{j=0,1,\dots,k_i-1,\ i=1,2,\dots,m,\ l=1,2,\dots,n}.$$

From the condition about Y, and the nonsingulartiy of M, it follows that the matrix in the left hand side of this equality is nonsingular. This proves that the corresponding mapping is a local diffeomorphism and according to the first part of our proof, it is also a global homeomorphism, i.e.,

$$Z(x, a) = Y(\chi(x), a)$$

is an $UDnF$ by Theorem 2.

(iii) Let us consider the function

$$Z(x, a) = \varkappa(Y(x, a)), \tag{3}$$

where Y and $\varkappa$ are as in the theorem. The functions Z is continuous and has continuous partial derivatives on the direct product $[\alpha, \beta] \times \mathbf{R}^n$ of orders $1, 2, \dots, n$ with respect to x according the conditions about Y and $\varkappa$. Similarly, if Y is an $UDnF$ and $\varkappa$ is of class C^{n+1}, then $\frac{\partial Z^{(j)}(x, a)}{\partial a^i}$, $j = 0, 1, \dots, n$, $i = 1, 2, \dots, n$ exist and are continuous on the direct product $[\alpha, \beta] \times \mathbf{R}^n$. Suppose that m; $x_1, x_2, \dots, x_m$; $k_1, k_2, \dots, k_m$ are as in Definition 1. By derivation j times with respect to x in (3), we obtain:

$$Z^{(j)}(x_i, a) = \varkappa^{(j)}(Y(x_i, a))\,(Y'(x_i, a))^j + \dots + \varkappa'(Y(x_i, a))\,Y^{(j)}(x_i, a) \tag{4}$$

$$j = 0, 1, \dots, k_i - 1,\ i = 1, 2, \dots, m.$$

The values $Y^{(j)}(x_i, a)$ for $j = 0, 1, \dots, k_i - 1$, $i = 1, 2, \dots, m$ can be uniquely determined as functions of $Z^{(j)}(x_i, a)$, $j = 0, 1, \dots, k_i - 1$, $i = 1, 2, \dots, m$, from the system (4). Really, for a given index i we have:

$$Z(x_i, a) = \varkappa(Y(x_i, a)),$$

$$Z'(a_i, a) = \varkappa'(Y(x_i, a))\,Y'(x_i, a),$$

$$Z''(x_i, a) = \varkappa''(Y(x_i, a))(Y'(x_i, a))^2 + \varkappa'(Y(x_i, a))Y''(x_i, a),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

and it is easy to see that according the definition of $\varkappa$, from these relations may be recursively determined $Y(x_i, a)$, $Y'(x, a)$, $Y''(x_i, a)$, etc. This means that the conditions in Definition 1 for the function Z may be transposed via (4) in conditions of similar nature for Y. Because (4) determines uniquely the values $Y^{(j)}(x_i, a)$, and Y is an UnF, it follows that there exists a single a for which the conditions as those in Definition 1 are fulfilled. This proves that Z forms an UnF.

In the case when Y is an $UDnF$ and $\varkappa$ is of class C^{n+1}, differentiating with respect to a we have

$$\frac{\partial Z^{(j)}(x, a)}{\partial a^l} = \left(\varkappa'(Y(x_j, a))\left(\frac{\partial Y(x, a)}{\partial a^l}\right)\right)^{(j)},$$

where j denotes the order of derivation with respect to x. We have then

$$\frac{\partial Z^{(j)}(x_i, a)}{\partial a^l} = \varkappa^{(j+1)}(Y(x_i, a))\frac{\partial Y(x_i, a)}{\partial a^l}(Y'(x_i, a))^j + \dots + \varkappa'(Y(x_i, a))\frac{\partial Y^{(j)}(x_i, a)}{\partial a^l}$$

$$j = 0, 1, \dots, k_i - 1, \quad i = 1, 2, \dots, m, \quad l = 1, 2, \dots, n,$$

and from these relations it follows that

$$\left\| \frac{\partial Z^{(j)}(x_i, a)}{\partial a^l} \right\|_{j=0,1,\dots,k_i-1,\ i=1,2,\dots,m,\ l=1,2,\dots,n} = \tag{5}$$

$$= N \left\| \frac{\partial Y^{(j)}(x_i, a)}{\partial a^l} \right\|_{j=0,1,\dots,k_i-1,\ i=1,2,\dots,m,\ l=1,2,\dots,n},$$

where

$$N = \begin{Vmatrix} N_1 & 0 & \dots & 0 \\ 0 & N_2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & N_m \end{Vmatrix},$$

and

$$N_i = \begin{Vmatrix} \varkappa'(Y) & 0 & \dots & 0 \\ \varkappa''(Y)Y' & \varkappa'(Y) & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \varkappa^{(k_i)}(Y)(Y')^{k_i-1} & & \dots & \varkappa'(Y) \end{Vmatrix}$$

$$i = 1, 2, \dots, m,$$

the argument of Y and Y' being (x_i, a). From the form of the matrix N it follows that

$$\det N = (\varkappa'(Y(x_1, a)))^{k_1} (\varkappa'(Y(x_2, a)))^{k_2} \dots (\varkappa'(Y(x_m, a)))^{k_m}$$

and because $\varkappa'(x) \neq 0$ for any x in $\mathbf{R}$, we have that $\det N \neq 0$. From this and from the condition about Y it follows that the matrix in the left hand side of the relation (5) is nonsingular, which together with the first part of our proof, proves that Z is an $UDnF$, according Theorem 2.

(iv) The proof of the fact that if Y is an UnF (or an $UDnF$) and ψ is a fixed element in $C^n[\alpha, \beta]$, then $Y + \psi$ is also an UnF (respectively, an $UDnF$) is an immediate consequence of the Definitions 1 and 2. This completes the proof of the Theorem 3.

This theorem permits to introduce an equivalence relation in the set of UnF—s and the set of $UDnF$—s. Two UnF—s (or $UDnF$—s) are equivalent, if there exist Ψ, χ and $\varkappa$ which satisfy the conditions in Theorem 3, such that one of them is the transformation by these functions of the other.

3. Unispatial UDnF—s

Definition 4. *If the UDnF formed by the function $Y(x, a)$ has the property that its tangent space $L(a)$ in any point $a \in \mathbf{R}^n$ is the same unrestricted Chebyshev space L, then it will be called unispatial UDnF with the tangent space L.*

THEOREM 4. *The function $Y(x, a)$ forms an unispatial UDnF with the tangent space L if and only if it may be represented in the form*

$$Y(x, a) = \sum_{i=1}^{n} f^i(a)\, \varphi_i(x) + \psi(x),$$

where φ_i, $i = 1, 2, \dots, n$ form a basis of the space L, f^i, $i = 1, 2, \dots, n$ are the coordinate functions of a diffeomorphism of $\mathbf{R}^n$ onto itself, and ψ is an element of the space $C^n[\alpha, \beta]$.

The proof of the theorem is similar to the proof of Theorem 2 in [1].

As it was observed at the end of the paragraph 2, applying Theorem 3 an equivalence relation may be introduced in the set of all $UDnF$—s. The property to be unispatial of an $UDnF$ is not preserved by a transformation given in Theorem 3. By a direct method it is easy to show that the tangent spaces of an $UDnF$ which is obtained from an unispatial $UDnF$ by the transformations in the Theorem 3, will be of form $\varphi \cdot L$, where L is a constant space, and $\varphi(x, a)$ is a nonvanishing function on $[\alpha, \beta] \times \mathbf{R}^n$.

Really, applying the notations in Theorem 3, we have

$$\frac{\partial Z}{\partial a} = \varkappa'(Y(\chi(x), \Psi(a))) \sum_{j=1}^{n} \frac{\partial Y(\chi(x), \Psi(a))}{\partial b^i} \frac{\partial \Psi^j(a)}{\partial a^i},$$

where $b = (b^1, b^2, \dots, b^n)$ is the second argument of Y, and $\Psi = (\Psi^1, \Psi^2, \dots, \Psi^n)$. From this equality it follows then our affirmation.

4. An UDnF having given tangent spaces

We introduce first the notion of composable unrestricted Chebyshev systems:

Definition 5. *Let $\varphi_1, \varphi_2, \dots, \varphi_n$ and $\psi_1, \psi_2, \dots, \psi_n$ be unrestricted Chebyshev systems in $C^n[\alpha, \beta]$* (i.e., they span unrestricted Chebyshev spaces). *They will be said to be composable if:*

(i) *every sequence of n functions, $\eta_1, \eta_2, \dots, \eta_n$, where η_i is φ_i* or ψ_i (generally depending on i), *form an unrestricted Chebyshev system,*

(ii) *if m; $x_1, x_2, \dots, x_m$; $k_1, k_2, \dots, k_m$ is a given system of numbers satisfying the conditions of Definition 1, then the determinants*

$$\det \left\| \eta_k^{(j)}(x_i) \right\|_{j=0,1,\dots,k_i-1,\ i=1,2,\dots,m,\ k=1,2,\dots,n} \tag{6}$$

are of the same sign independently of the manner of choosing φ_i or ψ_i for η_i, $i = 1, 2, \dots, n$.

We observe first, that from the condition (i) and (ii) it follows that if we have for example the ordering $x_1 < x_2 < \dots < x_m$, then all the determinants of form (6) have the same sign, independently of values of m, x_i and k_i, with the single condition of permanence of the ordering of knots x_i.

Consider the systems $e^{\alpha_1 x}, e^{\alpha_2 x}, \ldots, e^{\alpha_n x}$ and $e^{\beta_1 x}, e^{\beta_2 x}, \ldots, e^{\beta_n x}$, where $\alpha_1 \leqslant \beta_1 < \alpha_2 \leqslant \beta_2 < \ldots < \alpha_n \leqslant \beta_n$ are real numbers. These systems form composable unrestricted Chebyshev systems. The non vanishing of the determinants of from (6) for $\eta_i = e^{\gamma_i x}$, where γ_i is α_i or β_i, for simple knots x_i (i.e., for $m = n$) follows from the fact that $e^{\gamma_1 x}, e^{\gamma_2 x}, \ldots, e^{\gamma_n x}$ form an unrestricted Chebyshev system for any pairwise different numbers γ_i, $i = 1, 2, \ldots, n$. All the corresponding determinants of form (6) have the same sign by a similar consideration as in [1], & 4. But this means that all the conditions of Definition 5 are fulfilled.

By a complet analogy with the Theorem 3 in the paper [1], we have

THEOREM 5. *If $\varphi_1, \varphi_2, \ldots, \varphi_n$ and $\psi_1, \psi_2, \ldots, \psi_n$ are composable unrestricted Chebyshev systems in the space $C^n[\alpha, \beta]$, then the function*

$$Y(x, a) = \sum_{i=1}^{n} \int^{a^i} f(x, a^i)\, da^i,$$

where

$$f(x, a^i) = \begin{cases} \psi_i(x), & a^i \leqslant 0 \\ a^i \varphi_i(x) + (1 - a^i)\psi_i(x), & 0 < a^i \leqslant 1, \\ \varphi_i(x), & a^i < 1, \end{cases}$$

$$i = 1, 2, \ldots, n,$$

is an UDnF, having in $a_0 = (0, 0, \ldots, 0)$ and $a_1 = (1, 1, \ldots, 1)$ as tangent spaces the space $L(\psi_1, \psi_2, \ldots, \psi_n)$ and respectively the space $L(\varphi_1, \varphi_2, \ldots, \varphi_n)$.

Consider the *UDnF* constructed in Theorem 5 for $n = 2$ and the composable unrestricted Chebyshev systems $e^{\alpha_1 x}$, $e^{\alpha_2 x}$ and $e^{\beta_1 x}$, $e^{\beta_2 x}$, where $\alpha_1 \leqslant \beta_1 < \alpha_2 \leqslant \beta_2$. The tangent spaces of this *UDnF* in the point $a_0 = (0, 0)$ and $a_1 = (1, 1)$ will be spanned by $e^{\beta_1 x}$, $e^{\beta_2 x}$, respectively, by $e^{\alpha_1 x}$, $e^{\alpha_2 x}$. In what follows we will prove that if the differences $\beta_1 - \alpha_1$, $\beta_2 - \alpha_1$, $\beta_1 - \alpha_2$, $\beta_2 - \alpha_2$ are pairwise different, then this *UDnF* cnnot be obtained from an unispatial *UDnF* by a transformation given in Theorem 3.

At the end of the paragraph 3 it was observed that if an *UDnF* is obtained by a transformation as that in the Theorem 3 from an unispatial *UDnF*, then it has its tangent spaces of the form $\varphi \cdot L$, where L is a fixed n-dimensional unrestricted Chebyshev space, and $\varphi(x, a)$ is a nonvanishing function on the direct product $[\alpha, \beta] \times \mathbf{R}^n$.

Suppose that the *UDnF* considered above can be obtained from an unispatial *UDnF* by a transformation. Then it follows that there exist the nonvanishing functions $\rho(x)$ und $\sigma(x)$ such that

$$\rho(x) L(e^{\alpha_1 x}, e^{\alpha_2 x}) = \sigma(x) L(e^{\beta_1 x}, e^{\beta_2 x})$$

i.e., we have

$$\rho(x) e^{\alpha_1 x} = (c_1 e^{\beta_1 x} + c_2 e^{\beta_2 x})\sigma(x),$$

$$\rho(x) e^{\alpha_2 x} = (k_1 e^{\beta_1 x} + k_2 e^{\beta_2 x})\sigma(x), \tag{7}$$

from which it follows that

$$c_1 e^{(\beta_1 - \alpha_1)x} + c_2 e^{(\beta_2 - \alpha_1)x} = k_1 e^{(\beta_1 - \alpha_2)x} + k_2 e^{(\beta_2 - \alpha_2)x}. \tag{8}$$

According (7), not all c_1, c_2, k_1, k_2 are zero, and therefore from (8) it follows that the functions $e^{(\beta_i - \alpha_j)x}$, $i, j = 1, 2$, are linearly dependent, which is a contradiction.

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