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THE BEST CONSTANT IN AN INTEGRAL INEQUALITY

by

ALEXANDRU LUPAŞ

Cluj

Let $L_2'[a, b]$ be the space of all real functions which are defined and absolutely continuous on $[a, b]$, $-\infty < a < b < +\infty$, whose derivatives are square integrable (in the sense of Lebesgue) on $[a, b]$. If $f, g : [a, b] \to \mathbf{R}$ are Lebesgue integrable and $h \in L_2'[a, b]$, let us denote

$$I(f) = \frac{1}{b-a}\int_a^b f(t)\, dt$$

$$T(f, g) = I(fg) - I(f)I(g) \tag{1}$$

$$\|h'\|_2 = \left[\frac{1}{b-a}\int_a^b |h'(t)|^2 \cdot dt\right]^{1/2}.$$

A. M. OSTROWSKI [3] has proved that if $f, g \in L_2'[a, b]$, then there exists a constant C, $0 \leqslant C \leqslant \frac{1}{8}(b-a)^2$, such that

$$|T(f, g)| \leqslant C\|f'\|_2\|g'\|_2. \tag{2}$$

Our purpose is to find

$$\max_{f,g \in L_2'[a,b]} \frac{|T(f,g)|}{\|f'\|_2 \cdot \|g'\|_2}, \quad \|f'\|_2 \cdot \|g'\|_2 \neq 0,$$

that is the best constant in (2).

THEOREM 1. *If $f, g \in L_2'[a, b]$, then*

$$|T(f, g)| \leqslant \frac{(b-a)^2}{\pi^2}\|f'\|_2 \cdot \|g'\|_2 \tag{3}$$

where $T(f, g)$ *is defined by* (1). *The constant* $\frac{1}{\pi^2}$ *cannot be improved: there is at least a pair* (f_0, g_0), *of elements from* $L_2'[a, b]$, *for which the equality holds. For instance*

$$(4) \qquad \begin{aligned} f_0(x) &= A + B\sin\left(\frac{\pi}{2}\,\frac{a+b-2x}{b-a}\right) \\ g_0(x) &= C + D\sin\left(\frac{\pi}{2}\,\frac{a+b-2x}{b-a}\right) \end{aligned}, \quad x \in [a, b].$$

Proof. Firstly we observe that

$$|T(f, f)|^2 = \frac{1}{(b-a)^2}\left|\int_a^b [f(t) - I(f)]\left[f(t) - f\left(\frac{a+b}{2}\right)\right] dt\right|^2 \leqslant$$

$$\leqslant \frac{1}{(b-a)^2}\int_a^b |f(t) - I(f)|^2\,dt \cdot \int_a^b \left|f(t) - f\left(\frac{a+b}{2}\right)\right|^2 dt.$$

On the other hand

$$\frac{1}{b-a}\int_a^b |f(t) - I(f)|^2\,dt = T(f, f) \geqslant 0.$$

Therefore, for f non-constant on $[a, b]$, we have

$$(5) \qquad T(f, f) \leqslant \frac{1}{b-a}\int_a^b \left|f(t) - f\left(\frac{a+b}{2}\right)\right|^2 dt.$$

According to an inequality due to J. B. DIAZ and F. T. METCALF [1] $\left(\text{see Theorem 1 with } \xi = \frac{a+b}{2}\right)$

$$\int_a^b \left|f(t) - f\left(\frac{a+b}{2}\right)\right|^2 dt \leqslant \frac{(b-a)^2}{\pi^2}\int_a^b |f'(t)|^2\,dt.$$

Therefore (5) becomes

$$(6) \qquad T(f, f) \leqslant \frac{(b-a)^2}{\pi^2}(\|f'\|_2)^2.$$

Likewise for $g \in L_2'[a, b]$

$$T(g, g) \leqslant \frac{(b-a)^2}{\pi^2}(\|g'\|_2)^2.$$

Now it is known that (see [3], p. 361)

$$(7) \qquad |T(f, g)|^2 \leqslant T(f, f)\,T(g, g).$$

Combining (6)—(7) we obtain

$$|T(f, g)| \leqslant \frac{(b-a)^2}{\pi^2}\|f'\|_2 \cdot \|g'\|_2.$$

which proves (3). If $f_0, g_0 : [a, b] \to \mathbf{R}$ are defined as in (4), then the equalities

$$I(f_0 g_0) = AC + \frac{BD}{2}, \; I(f_0) = A, \; I(g_0) = C$$

$$\|f_0'\|_2 = \frac{\pi|B|\sqrt{2}}{2(b-a)}, \; \|g_0'\|_2 = \frac{\pi|D|\sqrt{2}}{2(b-a)},$$

imply

$$|T(f_0, g_0)| = \frac{(b-a)^2}{\pi^2}\|f_0'\|_2 \cdot \|g_0'\|_2 = \frac{|BD|}{2}.$$

Therefore (3) cannot be improved.

Another inequality derived by A. M. OSTROWSKI (see [3], p. 370) is the following

$$|T(f, g)| \leqslant \frac{b-a}{4\sqrt{2}}\|f'\|_2 \cdot \operatorname*{Osc}_{[a,b]}(g)$$

where

$$\operatorname*{Osc}_{[a,b]}(g) = \sup_{t\in[a,b]} g(t) - \inf_{t\in[a,b]} g(t).$$

The following proposition gives a sharper estimate.

THEOREM 2. *If* $f \in L_2'[a, b]$ *and* $g : [a, b] \to \mathbf{R}$ *is a measurable, bounded function, then*

$$(8) \qquad |T(f, g)| \leqslant \frac{b-a}{2\pi}\|f'\|_2 \cdot \operatorname*{Osc}_{[a,b]}(g).$$

Proof. By means of Grüss inequality ([2], p. 70)

$$T(g, g) \leqslant \frac{1}{4}\left[\operatorname*{Osc}_{[a,b]}(g)\right]^2$$

and from (6)—(7) we conclude with (8).

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VALEURS UNITAIRES DE POLYNÔMES SUR UN CORPS QUADRATIQUE IMAGINAIRE

par

MAURICE MIGNOTTE

Saint-Denis

Préliminaires[1]

On a bien sûr les inégalités

$$u_i \leqq d \text{ pour } i = 1, \ldots, n$$
$$f \leqq (n-1)\, d$$
$$f \leqq v.$$

Notre but est d'obtenir une majoration la meilleure possible de f.

On peut d'abord montrer un résultat général simple.

Lemme 1: *On a toujours* $v \leqq k$ (et donc $f \leqq k$), *de plus* $v = 0$ *si* $u_1 > k$.

Si $v = 0$, il n'y a rien à démontrer. Sinon soit x tel que $P(x) = e_i$, $i \geqq 2$, alors

$$P(x) = (x - x_1) \ldots (x - x_{u_1}) Q(x) + 1 = e_i$$

donc les $x - x_i$ sont dans D. Ainsi

— il y a au plus k choix de x, donc $v \leqq k$,

— on a $u_1 \leqq k$.

Ceci achève la démonstration du lemme.

Dans la suite on se limitera au cas où $v \geqq 1$. Quitte à faire une translation $x \to x + t$ on supposera que $P(0) \in U'$, ainsi les $x_i \in D$.

Soit x tel que $P(x) \in U'$, alors $(x - x_1) \ldots (x - x_{u_1}) Q(x) \in D$, ce qui amène à considérer les notions suivantes.

[1]) Voir notations en fin d'article.