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UNRESTRICTED DIFFERENTIAL n-PARAMETER FAMILIES II. RELATION WITH DISCONJUGACY

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0. Definitions and notations

In the first part of our paper, [8], were given the definitions and some topological and transformational properties of the unrestricted n-parameter family (abbreviated UnF) and of the unrestricted differential n-parameter family (UDnF). In what follows we will establish a connection between UnF—s and UDnF—s and the disconjugacy properties of the ordinary differential equations. This connection will permit us to present some aspects of our considerations in [8] from the point of view of the theory of disconjugacy. On the other hand, the results in the theory of disconjugacy lead to the solution of some particular cases of our problem in [8] of construction of an UDnF with given tangent spaces (Theorem 4). In the paragraph 4 of our paper we give a disconjugacy criterion for the nonlinear differential equations which makes a connection between the disconjugacy properties of the equation and its equations in variation (Theorem 5).

Denote by $[\alpha, \beta]$ a finite, closed interval of the real axis $\mathbf{R}$. Let $C[\alpha, \beta]$ and $C^n[\alpha, \beta]$ be the linear spaces of the real valued continuous functions, and respectively, the functions with continuous n^{th} derivatives defined on $[\alpha, \beta]$, each endowed with the usual norm.

For the notions of UnF-s and UDnF-s see Definition 1 and 2 in [8]. The UDnF which is linear with respect to the parameters is called an *unrestricted Chebyshev space* (UCSp). It is in fact an n-dimensional subspace of the space $C^n[\alpha, \beta]$ in which any Hermite-type interpolation problem has a solution. A basis of an UCSp is called an *unrestricted Chebyshev system* (UCS).

Consider the differential equation

(1) $$y^{(n)} = G(x, y, y', \ldots, y^{(n-1)}),$$

where G is a real valued function defined on the direct product $[\alpha, \beta] \times \mathbf{R}^n$.

Definition 1. *The differential equation* (1) *will be said to be disconjugated* (on the interval $[\alpha, \beta]$), *if for any natural number* m, $1 \leq m \leq n$, *any distinct points* $x_1, \ldots, x_m$ *in* $[\alpha, \beta]$, *any natural numbers* $k_1, \ldots, k_m$ *with the property* $\sum_{i=1}^{m} k_i = n$, *there exists a single solution* y *of* (1) *having the properties*

$$y^{(j)}(x_i) = y_i^j,\ j = 0, 1, \ldots, k_i - 1, i = 1, \ldots, m,$$

for any given real numbers y_i^j.

An important particular case of the differential equation (1) is the linear differential equation

(2) $$y^{(n)} = a_1(x)\ y^{(n-1)} + a_2(x)y^{(n-2)} + \ldots + a_n(x)y,$$

where a_i are elements of the space $C[\alpha, \beta]$.

It is obvious that (2) is a disconjugated differential equation if and only if the space of its solutions is an UCSp. Moreover, any UCSp is the solution space for a linear differential equation of the form (2) with a_i in $C[\alpha, \beta]$, $i = 1, \ldots, n$. In the first part of our paper (paragraphs 1 and 2), we will prove that this property is more general: an UnF (respectively an UDnF) is an integral variety of a (disconjugated) differential equation (1) with G a continuous function (respetively, with G having continuous partial derivatives with respect to its last n variables).

Denote by C_n^0 the direct product space of n exemplars of the space $C[\alpha, \beta]$ and by C_n^n the direct product space of n exemplars of the space $C^n[\alpha, \beta]$. By the coefficients $a_i \in C[\alpha, \beta]$ $i = 1, \ldots, n$ a correspondence is defined between the set $\mathfrak{D}$ of all the linear differential equations of the form (2) and the set C_n^0. In what follows we will identify these sets and will consider the set $\mathfrak{D}$ of differential equations of form (2) endowed with the topology of C_n^0. The set $\mathfrak{M}$ of all the disconjugated differential equations of form (2) is open in C_n^0 (see [2] or [6]), and therefore it is a locally convex topological space.

In the paper [6] a topology is introduced in the set $\mathcal{M}$ of all UCSp of the dimension n in $C^n[\alpha, \beta]$ with a factorization of the direct product space C_n^n by the equivalence relation $\sim$ introduced as follows: the elements $f = (f_1, \ldots, f_n)$ and $g = (g_1, \ldots, g_n)$ are equivalent, $f \sim g$, if the linear spaces $L(f_1, \ldots, f_n)$ and $L(g_1, \ldots, g_n)$ coincide. $\mathcal{M}$ will be considered a subset of the factor space $C_n^n/\sim$. The canonical projection in this factorization is denoted by p. In the Lemma 4 of [7] was proved that the sets $\mathcal{M}$ and $\mathfrak{M}$ are homeomorphic.

1. Auxiliary results

In order to avoid an interruption in the following proofs, we will prove here some auxiliary results.

Lemma 1. *Suppose that*

(i) $f\colon [\alpha, \beta] \times \mathbf{R}^n \to \mathbf{R}^n$ *is a continuous mapping;*

(ii) *for any fixed* x_0 *in* $[\alpha, \beta]$ *the mapping*

$$a \mapsto f(x_0, a)$$

is a homeomorphic mapping of $\mathbf{R}^n$ *onto itself.*

Then the equality

$$f(x, a) = b$$

determines an implicit function $a = g(x, b)$, *which has similar properties* (i) *and* (ii).

Proof. Denote by Ψ the mapping defined by

$$\Psi: (x, a) \mapsto (x, f(x,a)).$$

Then Ψ will be a homeomorphism of $[\alpha, \beta] \times \mathbf{R}^n$ onto itself. The mapping Ψ is one to one, according the condition (ii) on f. It is also continuous by (i). To prove the continuity of Ψ^{-1} we will prove in fact that Ψ is an open mapping. For this, let U be an open set in $[\alpha, \beta] \times \mathbf{R}^n$, and let be (x_0, a_0) in U. Then the point (x_0, a_0) has a neighbourhood U_0 of form $U_0 = V \times W$, which is in U, V is open in $[\alpha, \beta]$ and W is open in $\mathbf{R}^n$. We have

$$\Psi(U_0) = V \times f(V, W).$$

The set $f(V, W) \subset \mathbf{R}^n$ is open. Really,

$$f(V, W) = \bigcup_{x \in V} f(x, W)$$

and the openess follows from the condition (ii). This proves that $\Psi(U_0)$ is open, and completes the proof of continuity of Ψ^{-1}.

Denote
$$g(x, b) = \mathrm{pr}_2 \circ \Psi^{-1}(x, b),$$
where pr_2 denotes the projection onto $\mathbf{R}^n$ in the direct product $[\alpha, \beta] \times \mathbf{R}^n$. Then we have
$$f(x, g(x, b)) = f(x, \mathrm{pr}_2 \circ \Psi^{-1}(x, b)).$$
But $\Psi^{-1}(x, b) = (x, a)$, where $b = f(x, a)$, and therefore
$$f(x, g(x, b)) = f(x, a) = b, \tag{3}$$
that is, $g(x, b)$ is the required implicit function. It is obviously continuous on $[\alpha, \beta] \times \mathbf{R}^n$. Let x_0 fixed in $[\alpha, \beta]$. Then from (ii) and (3) it follows that
$$b \mapsto g(x_0, b)$$
is a homeomorphic mapping of $\mathbf{R}^n$ onto itself, which completes the proof of the lemma.

Lemma 2. *Suppose that*

(i) $f: [\alpha, \beta] \times \mathbf{R}^n \to \mathbf{R}^n$ *is a continuous mapping;*

(ii) *all the partial derivatives of the coordinate functions of f with respect to their last n variables are continuous on $[\alpha, \beta] \times \mathbf{R}^n$;*

(iii) *for any fixed x_0 in $[\alpha, \beta]$ the mapping*
$$a \mapsto f(x_0, a)$$
is a diffeomorphism of $\mathbf{R}^n$ onto itself.

Then the relation
$$f(x, a) = b$$
determines an implicit function $a = g(x, b)$, which maps the set $[\alpha, \beta] \times \mathbf{R}^n$ onto $\mathbf{R}^n$ and has the properties (i), (ii) *and* (iii).

Proof. The condition (i) for g is obviously fulfilled according Lemma 1. Denote $f = (f^1, \ldots, f^n)$ and $g = (g^1, \ldots, g^n)$ and let be
$$\eta^i(x, a, b) = f^i(x, a) - b^i, \; i = 1, \ldots, n.$$
Applying these notations, from the implicit function theorem we have for fixed x
$$\left.\frac{\partial g^i}{\partial b^j}\right|_b = -\frac{1}{J(a)} \frac{D(\eta^1, \ldots, \eta^n)}{D(a^1, \ldots, a^{i-1}, b^j, a^{i+1}, \ldots, a^n)}$$
$$i, j = 1, \ldots, n$$

where g is the implicit function determined as in the above lemma and $J(a)$ is the Jacobian of the mapping
$$a \mapsto f(x, a).$$
According to Lemma 1, g is continuous of x and b, and therefore if we put in the above formula $a = g(x, b)$, we have the partial derivatives of the components of g functions of x and b. Because all these functions are continuous and $J(g(x, b)) \neq 0$, the continuity of the partial derivatives of g on the direct product $[\alpha, \beta] \times \mathbf{R}^n$ follows. This proves the condition (ii) for g.

The condition (iii) for g is an immediate consequence of the Lemma 1, (ii) and of the implicit function theorem.

2. Relation with the disconjugacy theory

In this paragraph we will prove theorems of technical character, which make a connection between UnF-s (UDnF-s) and the disconjugated differential equations. We prove first

THEOREM 1. *If the function $Y(x, a)$ generates an* UnF *(respectively, a* UDnF*), then it forms an integral variety for a disconjugated differential equation of the form* (1), *where G is continuous (respectively, it is continuous and has continuous first order partial derivatives with respect to its last n variables) on the direct product $[\alpha, \beta] \times \mathbf{R}^n$.*

Proof. Let $(y, y', \ldots, y^{(n-1)})$ denotes a point in $\mathbf{R}^n$. Then for x_0 fixed there exists a vector a in $\mathbf{R}^n$ such that
$$Y^{(j)}(x_0, a) = y^{(j)}, \; j = 0, 1, \ldots, n-1, \tag{4}$$
where we have denoted by $Y^{(j)}$ the j^{th} partial derivative of Y with respect to x. Denote
$$Y^{(n)}(x_0, a) = y^{(n)}. \tag{5}$$
The functions in (4) for fixed x are the coordinate functions of a homeomorphism of $\mathbf{R}^n$ onto itself (see Theorem 1 in [8]), and they are continuous functions of x. Then according to Lemma 1 there exists a continuous function $g: [\alpha, \beta] \times \mathbf{R}^n \to \mathbf{R}^n$, such that
$$a = g(x_0, y, y', \ldots, y^{(n-1)}).$$
By a substitution in (5) we obtain
$$\begin{aligned} y^{(n)} &= Y^{(n)}(x_0, g(x_0, y, y', \ldots, y^{(n-1)})) = \\ &= G(x_0, y, y', \ldots, y^{(n-1)}). \end{aligned} \tag{6}$$

Considering now x variable, $y^{(i)}$, $i = 0, 1, \ldots, n$ may be considered to be functions of x, and according (4) and (5) they are in fact the 0, 1, ..., n order derivatives of the function y. The disconjugacy of the obtained differential equation follows immediatelly from the definition of the UnF-s. This completes the proof of the first part of the theorem.

In the case of $Y(x, a)$ an UDnF, we observe that according the point (ii) in the Lemma 2, the relations (4) determine a function g which has continuous partial derivatives on $[\alpha, \beta] \times \mathbf{R}^n$ with respect to its last n variables. From this and from the definition of the UDnF-s it follows that the function G in (6) has continuous partial derivatives with respect to its last n variables on the set $[\alpha, \beta] \times \mathbf{R}^n$. This completes the proof of the theorem.

The notion of the equation in variation for a differential equation, used in the followings is that in the book [9], Theorem 15.

THEOREM 2. *$Y(x, a)$ generates an UDnF if and only if it is an integral variety of a differential equation of form (1), where G is continuous and has continuous partial derivatives with respect to the last n variables on $[\alpha, \beta] \times \mathbf{R}^n$, and the equations in variation of this differential equation are also disconjugated.*

Proof. For proving the „only if" part of the above theorem, it suffices, according Theorem 1, to prove that the equations in variation of the differential equation of form (1) determined by this theorem, has a fundamental system of solutions which forms an UCS. We have obviously the identity

$$Y^{(n)}(x, a) = G(x, Y(x, a), Y'(x, a), \ldots, Y^{(n-1)}(x, a)),$$

from which by derivation with respect to a^i, we obtain

$$\frac{\partial Y^{(n)}(x, a)}{\partial a^i} = \sum_{j=0}^{n-1} \frac{\partial G}{\partial y^{(j)}} \frac{\partial Y^{(j)}(x, a)}{\partial a^i}, \quad i = 1, \ldots, n. \tag{7}$$

Because of the continuity of the partial derivatives we have

$$\frac{\partial Y^{(j)}(x, a)}{\partial a^i} = \left(\frac{\partial Y(x, a)}{\partial a^i}\right)^{(j)}, \quad j = 0, 1, \ldots, n.$$

From the definition of the UDnF the functions $\dfrac{\partial Y(x, a)}{\partial a^i}$, $i = 1, \ldots, n$ form an UCS, and then from (7) our assertion follows.

For proving the sufficience of the condition, suppose that $Y(x, a)$ is the integral variety of the differential equation (1) which is parametrized with the Cauchy type initial value conditions in the point α, that is, the function $Y(x, a)$ for a fixed a is the solution y of the differential equation (1) for which

$$y^{(j-1)}(\alpha) = a^j, \quad j = 1, \ldots, n. \tag{8}$$

Then from the Theorem 15 in [9], it follows that $Y(x, a)$ is continuous on $[\alpha, \beta] \times \mathbf{R}^n$ and has partial derivatives with respect to a^i, $i = 1, \ldots, n$, which are also continuous on this set. From the disconjugacy of the differential equation (1) it follows that $Y(x, a)$ is an UnF. From the same Theorem 15 in [9] it follows also that for the fixed parameter a the functions

$$\frac{\partial Y(x, a)}{\partial a^i}, \quad i = 1, \ldots, n \tag{9}$$

are solutions of the equation in variation of the differential equation (1) in the point a, and they are satisfying the initial value conditions

$$\frac{\partial Y^{(j-1)}(\alpha, a)}{\partial a^i} = \delta_i^j, \quad i, j = 1, \ldots, n. \tag{10}$$

It follows then that (9) is a fundamental system of solutions for the equation in variation, and because of the disconjugacy of this equation in variation, this system is an UCS. From this follows that Y generates an UDnF, which proves the theorem.

3. Construction of UDnF-s

From the considerations in the papers [6] and [7] it follows that the set $p^{-1}(\mathfrak{M})$ of all the n-tuples $(f_1, \ldots, f_n)$ in C_n^n for which the functions $f_1, \ldots, f_n$ form an UCS, is open in the space C_n^n. From this and from the locally convex structure of $p^{-1}(\mathfrak{M})$, it follows that if $f_1, \ldots, f_n$ form an UCS, then in a sufficiently small neighbourhood of $(f_1, \ldots, f_n)$ in C_n^n, all the elements $(g_1, \ldots, g_n)$ have the property that they form UCS-s, and moreover, $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ are composable in the sense of Definition 5 in [8]. Therefore from Theorem 5 in [8], we conclude that it may be constructed an UDnF with the tangent spaces $L(f_1, \ldots, f_n)$ and $L(g_1, \ldots, g_n)$. But the condition of composability has an implicit formulation and we do not have simple methods to decide that two UCSp have or not composable bases.

In this paragraph we will give a construction similar to that in the Theorem 3 in [5] (or Theorem 5 in [8]), in which we will use some results in the nonlinear disconjugacy theory of the nature of those obtained by A. LASOTA and Z. OPIAL [3], [4] and by G. A. BESSMERTNYH and A. YU. LEVIN [1].

We begin with a general disconjugacy theorem which follows directly from our theorems 1 and 2 and from Theorem 2 of A. LASOTA and Z. OPIAL in [4].

Suppose that the nonnegative functions $P_i(x)$, $i = 1, \ldots, n$ have the following property: there exists a positive number $\varepsilon > 0$ such that each linear differential equation of the form (2) which has the coefficients satisfying the inequalities

$$|a_i(x)| \leq P_i(x) + \varepsilon, \quad x \in [\alpha, \beta], \quad i = 1, \ldots, n,$$

is disconjugated. A set of functions with this property will be called during this paragraph an *admissible system of functions.*

THEOREM 3. *Let us given the differential equation* (1) *and let* $P_i(x)$, $i = 1, \ldots, n$ *form an admissible system of functions.*

(i) *If G is a continuous function on the set* $[\alpha, \beta] \times \mathbf{R}^n$ *satisfying the Lipschitz condition*

$$|G(x, u_0, u_1, \ldots, u_{n-1}) - G(x, v_0, v_1, \ldots, v_{n-1})| \leq \sum_{i=0}^{n-1} P_{n-i}(x)\, |u_i - v_i|,$$

then the integral variety $Y(x, a)$ *of* (1) *parametrized by the Cauchy type conditions in a point of* $[\alpha, \beta]$, *forms an* UnF.

(ii) *If G has continuous partial derivatives on the set* $[\alpha, \beta] \times \mathbf{R}^n$, *with respect to its last n variables, which satisfy the inequalities*

$$\left|\frac{\partial G(x, u_0, u_1, \ldots, u_{n-1})}{\partial u_i}\right| \leq P_{n-i}(x), \quad i = 0, 1, \ldots, n-1,$$

then the integral variety $Y(x, a)$ *of the above differential equation, parametrized by the Cauchy type conditions in a point of* $[\alpha, \beta]$, *form an* UDnF.

We shall give in what follows a constructive proof of the following theorem:

THEOREM 4. *Let* (e_1) *and* (e_2) *denote two differential equations of the form* (2) *with the coefficients* a_i, *respectively* b_i, $i = 1, \ldots, n$, *satisfying the inequalities*

$$|a_i(x)| < P_i(x), \quad |b_i(x)| < P_i(x), \quad i = 1, \ldots, n,$$

where $P_i(x)$, $i = 1, \ldots, n$ *form an admissible system of functions. Suppose also that* (e_2) *has a nonvanishing solution with nonvanishing derivatives of orders* $1, \ldots, n-1$. *Then there exists a disconjugated differential equation* (1), *the integral variety of which is an* UDnF, *and which has the linear differential equations* (e_1) *and* (e_2) *as equations in variation in two distinct points.*

Proof. Let y_0 be a solution of (e_2) which is nonvanishing and has nonvanishing derivatives of orders $1, \ldots, n-1$. We may suppose that $|y_0^{(i)}(x)| > 1$, $i = 0, 1, \ldots, n-1$. Denote $\varepsilon_i = \operatorname{sgn} y^{(i)}(\alpha)$, $i = 0, 1, \ldots, n-1$. Then we have

(11) $$\varepsilon_i y^{(i)}(x) > 1, \quad i = 0, 1, \ldots, n-1.$$

Let be the function G in the right hand side of (1) defined by the equality

$$G(x, u_0, u_1, \ldots, u_{n-1}) = \sum_{i=0}^{n-1} F_i(x, u_i),$$

where

(12) $$F_i(x, u_i) = \int_0^{u_i} f_i(x, u)\, du$$

is a primitive function with respect to u_i of

(13) $$f_i(x, u_i) = \begin{cases} a_{n-i}(x), & \varepsilon_i u_i \leq 0, \\ u_i b_{n-i}(x) + (1 - u_i) a_{n-i}(x), & 0 < \varepsilon_i u_i < 1, \\ b_{n-i}(x), & \varepsilon_i u_i \geq 1, \end{cases}$$

$$i = 0, 1, \ldots, n-1.$$

We prove that y_0 is a solution of the differential equation

(14) $$y^{(n)} = \sum_{i=0}^{n-1} F_i(x, y^{(i)}).$$

Because of (11) we have according (12)

$$f_i(x, y_0^{(i)}(x)) = b_{n-1}(x), \quad i = 0, 1, \ldots, n-1.$$

For $\varepsilon_i u_i \geq 1$ we have by (13)

$$F_i(x, u_i) = b_{n-i}(x) u_i, \quad i = 0, 1, \ldots, n-1,$$

and then

$$F_i(x, y_0^{(i)}(x)) = b_{n-i}(x) y_0^{(i)}(x), \quad i = 0, 1, \ldots, n-1,$$

and therefore (14) for $y = y_0$ becomes the differential equation (e_2) in which is substituted y_0, which proves our assertion.

From the above construction it follows that

$$\left|\frac{\partial G(x, u_0, u_1, \ldots, u_{n-1})}{\partial u_i}\right| = \left|\frac{\partial F_i(x, u_i)}{\partial u_i}\right| = |f_i(x, u_i)| < P_{n-i}(x),$$

$$i = 0, 1, \ldots, n-1,$$

and according to Theorem 3, the differential equation (14) will be disconjugated and its integral variety forms an UDnF.

Suppose that $a = (y_0(\alpha), y_0'(\alpha), \ldots, y_0^{(n-1)}(\alpha))$ and that the integral variety $Y(x, a)$ of (14) is parametrized by the Cauchy type conditions in the point α. Then in a neighbourhood of the point a all the solutions of (14) will be also solutions of (e_2) but this means that the equation of variation in a of (14) will be (e_2).

From the special form of (14) it also follows that $y = 0$ is its solution. The equation of variation of (14) in the point $a = (0, \ldots, 0)$ will be (e_1), according our construction. This completes the proof of the theorem.

Corollary. *If (e_1) is a differential equation of the form (2) with the coefficients a_i satisfying the inequalities $|a_i(x)| < P_i(x)$, $i = 1, \ldots, n$, where $P_i(x)$, $i = 1, \ldots, n$ form an admissible system of solutions, if (e_2) is a differential equation with constant coefficients b_i, $|b_i| < P_i(x)$, $i = 1, \ldots, n$, the characteristic equation of which has a real, nonzero root, then there exists a disconjugate differential equation with the integral variety an UDnF, which has as equations in variation in two distinct points the equations (e_1), and (e_2).*

Proof. In this case (e_2) has a solution satisfying the conditions in Theorem 4.

4. A disconjugacy criterion

Consider the differential equation (1) with G having partial derivatives with respect to the last n variables, continuous on the direct product $[\alpha, \beta] \times \mathbf{R}^n$. If $Y(x, a)$ is the integral variety of (1) parametrized by the initial value conditions (8) in the point α, then in any point $a \in \mathbf{R}$ it is defined the equation in variation of (1) and the functions (9) form a fundamental system of solutions for this equation in variation, and satisfy (10).

Suppose now that $Y(x, a)$ is given and suppose that the set

$$\mathfrak{M}(G) = \left\{\left(\frac{\partial G(x, Y, \ldots, Y^{(n-1)})}{\partial y^{(n-1)}}, \ldots, \frac{\partial G(x, Y, \ldots, Y^{(n-1)})}{\partial y}\right), \ a \in \mathbf{R}^n\right\}$$

has the properties:

(i) $\overline{\mathfrak{M}(G)} \subset \mathfrak{M}$, where $\mathfrak{M}$ denotes the set of all disconjugated differential equations of form (2);

(ii) $\overline{\mathfrak{M}(G)}$ is compact.

We will define the mapping

$$\varphi : C_n^0 \to C_n^n$$

as follows: if $a = (a_1, \ldots, a_n)$ is an element of C_n^0, then let $(y_1, \ldots, y_n) = \varphi(a)$ be the fundamental system of solutions of the differential equation (2) (with these coefficients a_i) which satisfies the initial value conditions

$$y_i^{(j-1)}(\alpha) = \delta_i^j, \quad i, j = 1, \ldots, n.$$

The mapping φ will be an imbedding of C_n^0 in C_n^n. We have also with our above notations

$$\varphi\left(\frac{\partial G(x, Y, \ldots, Y^{(n-1)})}{\partial y^{(n-1)}}, \ldots, \frac{\partial G(x, Y, \ldots, Y^{(n-1)})}{\partial y}\right) = \left(\frac{\partial Y}{\partial a^1}, \ldots, \frac{\partial Y}{\partial a^n}\right). \tag{15}$$

Let be now m a natural number $1 \leq m \leq n$, let $x_1, \ldots, x_m$ be distinct points in $[\alpha, \beta]$, and let be $k_1, \ldots, k_m$ natural numbers with the property that $\sum_{i=1}^{m} k_i = n$. Let J be the functional on C_n^n defined by the relation

$$J : (y_1, \ldots, y_n) \mapsto$$
$$\mapsto \det \left\| y_i(x_1), \ldots, y_i^{(k_1-1)}(x_1), \ldots, y_i(x_m), \ldots, y_i^{(k_m-1)}(x_m) \right\|, \tag{16}$$

where the determinant in the right hand side is represented by the i^{th} column.

Because the set $\overline{\mathfrak{M}(G)}$ is in $\mathfrak{M}$ it follows that for any element $a \in \overline{\mathfrak{M}(G)}$, $\varphi(a)$ is an n-tuple of functions $y_1, \ldots, y_n$ which form an UCS. But this means that the determinant in the right hand side of (16) is different from zero, i.e., for any a in $\overline{\mathfrak{M}(G)}$ we have $J(\varphi(a)) \neq 0$. From the continuity of φ and of J and from the compactness of $\overline{\mathfrak{M}(G)}$ it follows then that there exist the positive numbers δ and M with the property that

$$0 < \delta < |J(\varphi(\overline{\mathfrak{M}(G)}))| < M. \tag{17}$$

Consider now the mapping

$$\Phi : \mathbf{R}^n \to \mathbf{R}^n \tag{18}$$

which has the coordinate functions defined by the equalities

$$\Phi^r(.) = Y^{(j)}(x_i, .), \quad i = 1, \ldots, m, \quad j = 0, 1, \ldots, k_i - 1, \tag{19}$$

where $r = 1 + k_1 + \ldots + k_{i-1} + j$ and $k_0 = 0$.

From (15) and (16) it follows that for a given parameter a, the expression

$$J\left|\varphi\left(\frac{\partial G(x, Y, \ldots, Y^{(n-1)})}{\partial y^{(n-1)}}, \ldots, \frac{\partial G(x, Y, \ldots, Y^{(n-1)})}{\partial y}\right)\right|$$

will be the value of the Jacobian of the mapping (18)—(19) in the point a of $\mathbf{R}^n$. From the relation (17) we have

$$0 < \delta < |J(\varphi(\mathfrak{M}(G)))| < M,$$

and from this relation and the global implicit function theorem in [5] (for a different proof of this theorem see also [10]), it follows that the mapping Φ is a diffeomorphism of $\mathbf{R}^n$ onto itself. According the Theorem 2 in [8], it follows that $Y(x, a)$ forms an UDnF. We have proved such the following

THEOREM 5. *Suppose that the differential equation* (1) *with* G *having partial derivatives with respect to its last* n *variables which are continuous on the set* $[\alpha, \beta] \times \mathbf{R}^n$, *has the property that the set* $\overline{\mathfrak{M}(G)}$, *where* $\mathfrak{M}(G)$ *denotes the set of all the equations in variation of* (1), *is compact and is contained in* $\mathfrak{M}$, *the set of all disconjugated linear differential equations of the form* (2). *Then the differential equation* (1) *is disconjugated and its integral variety* $Y(x, a)$ *obtained by the parametrization by the initial value conditions* (8), *is an* UDnF.

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ОБ АСИМПТОТИЧЕСКИХ СВОЙСТВАХ СОБСТВЕННЫХ ФУНКЦИЙ ОПЕРАТОРА ЛАПЛАСА

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Обозначим через $\mathbf{D}$ единичный квадрат на плоскости $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, а через $\mathbf{B}$ его границу. Рассмотрим краевую задачу

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda u = 0, \ (x, y) \in \mathbf{D} \tag{1}$$

$$u = 0 \qquad (x, y) \in \mathbf{B}.$$

Собственные числа этой задачи вычисляются непосредственно

$$\lambda = \pi^2(n^2 + m^2), \tag{2}$$

где $n \geqslant 1$, $m \geqslant 1$ целые числа. Кратность собственного числа равна количеству его представлений в виде (2). Обозначим через $N(T)$ количество собственных значений задачи, не превосходящих растущей границы T

$$N(T) = \sum_{\lambda_n \leqslant T} 1, \tag{3}$$

где каждое собственное значение λ_n засчитывается столько раз, какова его кратность. Легко видеть, что

$$N(T) = \frac{N_1(T) - 1}{4} - \left[\frac{\sqrt{T}}{\pi}\right] = \frac{N_1(T)}{4} - \frac{\sqrt{T}}{\pi} + O(1), \tag{4}$$

где $N_1(T)$ количество целых точек, лежащих внутри и на границе круга

$$x^2 + y^2 \leqslant \left(\frac{\sqrt{T}}{\pi}\right)^2.$$