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EQUALLY SPACED POINTS FOR FAMILIES OF COMPACT CONVEX SETS IN MINKOWSKI SPACES

by

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In a recent paper [3] we have considered the problem of the existence and the unicity of a point in the n-dimensional Euclidean space, which is equally spaced from the members of a given family of convex and compact sets. The same problem presents some particularities in the more general case of the Minkowski spaces. For example, as far as we know, there was not approached even the problem if for $n + 1$ given independent points in an n-dimensional Minkowski space there is or not a point equally spaced from these points.

In the particular case of the family of hyperplanes the Chebyshev point of the family (see for definition [5]) is also an equally spaced point, but these two notions are in general different.

A somewhat related question has been investigated by DEKNATEL [1], in considering the locus of the equally spaced points from two closed sets in the plane.

Theorem 1 in our paper concerns the problem of the existence of an equally spaced point for a set of $n + 1$ independent points in an n-dimensional Minkowski space. This problem is related to a conjecture stated by P. TURÁN:

If Σ is a closed ,,sufficiently smooth'' $(n - 1)$-surface in the n-dimensional Euclidean space $\mathbf{R}^n$ and σ is a nondegenerated n-dimensional simplex, then there exists a simplex σ' similar to σ, inscribed in Σ. This conjecture was proved by E. G. STRAUS and his students for the case $n = 2$ and $n = 3$, (unpublished). For our purposes it is essential for the inscribed simplex to be of a given orientation and with faces parallel to given hyperplanes. But this restriction modifies essentially the problem, and the method of E. G. Straus seems to be not applicable.

THEOREM 1. *Let be Σ a strictly convex closed surface of class C^1 in $\mathbf{R}^n$ and let be σ a nondegenerated simplex in $\mathbf{R}^n$ with the vertices $p_1, p_2, \ldots, p_{n+1}$. Then there exists a simplex σ' with vertices $p'_1, p'_2, \ldots, p'_{n+1}$, having the same orientation as σ and parallel faces with the faces of σ, which is inscribed in Σ, in the sense that $p'_i \in \Sigma$, $i = 1, 2, \ldots, n+1$.*

We need the following lemma:

Lemma 1. *Suppose that σ and σ' are two nondegenerated simplexes having the vertices $p_1, p_2, \ldots, p_{n+1}$ and $p'_1, p'_2, \ldots, p'_{n+1}$ respectively and the centroids c and c'. If the segments cp_i and $c'p'_i$ are parallel and of the same orientation for $i = 1, 2, \ldots, n+1$, then σ and σ' have the same orientation and parallel faces.*

Proof. By two translations we may realize that c and c' coincide with the origin of the space $\mathbf{R}^n$. Then we have by the definition of the centroid

$$p_1 + p_2 + \ldots + p_{n+1} = 0, \tag{1}$$

and because of our condition we can write $p'_i = \lambda_i p_i$, $\lambda_i > 0$, $i = 1, 2, \ldots, n+1$. Again by the definition of the centroid

$$\lambda_1 p_1 + \lambda_2 p_2 + \ldots + \lambda_{n+1} p_{n+1} = 0. \tag{2}$$

Now, multiplying (1) with $-\lambda_1$ and adding to (2), we obtain

$$(\lambda_2 - \lambda_1)p_2 + \ldots + (\lambda_{n+1} - \lambda_1)p_{n+1} = 0. \tag{3}$$

From the condition that σ is a nondegenerated simplex, it follows that $p_2, \ldots, p_{n+1}$ are linearly independent and therefore from (3) $\lambda_i - \lambda_1 = 0$, $i = 2, \ldots, n+1$, that means that the simplex σ' is centrally homothetic with the coefficient $\lambda_1 > 0$ with σ. This proves the lemma.

Proof of Theorem 1. We define a mapping f of $\mathrm{Co}(\Sigma)$ the convex hull of Σ into itself in the following way: Suppose $q \in \mathrm{Co}(\Sigma)$ and consider for an i $(1 \leqq i \leqq n+1)$ the ray ρ_i starting from q parallel to cp_i and having the same orientation, and denote by $p_i(q)$ the intersection point of ρ_i with Σ, which is farthest from q. From the convexity of Σ it follows that $p_i(q)$ depends continuously on q. Put

$$f(q) = \frac{1}{n+1}(p_1(q) + p_2(q) + \ldots + p_{n+1}(q)).$$

Then f is a well defined and continuous mapping of $\mathrm{Co}(\Sigma)$ into itself. Then by Brouwer's fixed point theorem there exists a point $c' \in \mathrm{Co}(\Sigma)$ such that $f(c') = c'$. This means that c' will be the centroid of the simplex σ' with the vertices $p_i(c') = p'_i$, $i = 1, 2, \ldots, n+1$. We shall show that c' is inside Σ. Suppose the contrary: $c' \in \Sigma$ and let be H the tangent hyperplane in c' to Σ. At least one of the rays ρ_i, say ρ_{i_0} will have points in the open halfspace defined by H, the closure of which contains $\mathrm{Co}(\Sigma)$. But then the point $p_{i_0}(c')$ will be in the same open halfspace, and because all the other points $p_i(c')$ are in the closure of this halfspace, their centroid cannot be on H. This contradiction proves that c' is inside Σ. From Lemma 1, it follows that σ' has the required property. This completes the proof of the theorem.

Let X^n denote the n-dimensional real Minkowski space with a unit sphere with the surface S, having the elements $x = (x^1, \ldots, x^n)$, $x^i \in \mathbf{R}$, $i = 1, \ldots, n$. In what follows we shall use the therm ,,sphere'' to denote centrally homothetic and translated images of the surface S of the unit sphere in X^n.

The unit sphere in X^n determines a norm $||\cdot||$ in this space. The distance $\rho(x, y)$ between the points x and y will then be defined by

$$\rho(x, y) = ||x - y||.$$

Definition 1. *There will be said that the family $\mathcal{K}$ of sets in X^n has a supporting sphere, if there exists a sphere S in X^n, having common points with each member of $\mathcal{K}$, and the interior of S contains no point of any member of $\mathcal{K}$.*

The centre of the supporting sphere will be said to be an equally spaced point for the family $\mathcal{K}$.

Definition 2. *The family $\mathcal{K}$ of sets in X^n will be said to be independent, if for any $n+1$ pairwise distinct members $K_1, K_2, \ldots, K_{n+1}$ of $\mathcal{K}$, any set of points $p_1, \ldots, p_{n+1}$, where $p_i \in K_i$, $i = 1, 2, \ldots, n+1$, determines a nondegenerated simplex in the space X^n*, i.e. *the vectors $p_2 - p_1, \ldots, p_{n+1} - p_1$, are linearly independent.*

From Theorem 1 follows the

Corollary. *Let be $p_1, p_2, \ldots, p_{n+1}$ independent points in the Minkowski space X^n with a unit sphere with surface S which is of class C^1. Then there exists a centrally homothetic and translated image of S, denoted S' with the property that $p_i \in S'$, $i = 1, 2, \ldots, n+1$. This means that if we denote by p the centre of S', we have*

$$\rho(p, p_1) = \rho(p, p_2) = \ldots = \rho(p, p_{n+1}). \tag{4}$$

If $n = 2$, then from Theorem 2 in [4], it follows that the point p with this property is uniquely determined. For $n \geqq 3$ this may be not true.

Definition 3. *The independent family $\mathcal{K}$ of sets in X^n will be called strictly independent if for any pairwise distinct members $K_1, K_2, \ldots, K_{n+1}$*

and any points $p_i \in K_i$, $i = 1, 2, \dots, n+1$, there exists a single point p with the property (4).

If M' and M'' are compact sets in X^n, we denote

$$\rho(M', M'') = \min \{\rho(p', p'') : p' \in M', \ p'' \in M''\}.$$

In the case $M'' = \{p''\}$ a point $p' \in M'$ for which

$$\rho(M', p'') = \rho(p', p'')$$

is called a nearest point of M' to p''.

THEOREM 2. *Let be $K_1, K_2, \dots, K_{n+1}$ strictly independent, convex and compact sets in the Minkowski space X^n with a strictly convex unit sphere with the surface of class C^1. Then this family of sets admits at least one supporting sphere.*

Proof. Let be $p_i \in K_i$, $i = 1, 2, \dots, n+1$, and denote by $S(p_1, p_2, \dots, p_{n+1})$ the sphere determined by these points having the centre $c(p_1, p_2, \dots, p_{n+1})$. We will prove that

(i) the set $\Omega = \{c(p_1, p_2, \dots, p_{n+1}) : p_i \in K_i, \ i = 1, 2, \dots, n+1\}$ is bounded, and that

(ii) c is a continuous function of $p_1, p_2, \dots, p_{n+1}$.

For proving (i) let us consider a piece of S, which can be represented in the form:

$$x_n = f(x_1, \dots, x_{n-1}), \tag{5}$$

where $(x_1, x_2, \dots, x_{n-1}) \in V$, and V is a neighbourhood of the origin in $\mathbf{R}^{n-1} = \{x \in \mathbf{R}^n : x_n = 0\}$. For a differentiable f it is known that if we denote by H the tangent hyperplane in the point $x' = (x_1', x_2', \dots, x_{n-1}', f(x_1', x_2', \dots, x_{n-1}'))$ to f, and if $x'' = (x_1'', x_2'', \dots, x_{n-1}'', f(x_1'', x_2'', \dots, x_{n-1}''))$ is an other point on f, then if we denote by $h(x'')$ the distance from x'' to H, then $\dfrac{h(x'')}{\rho(x', x'')} \to 0$, when $\rho(x', x'') \to 0$, where $\rho(x', x'')$ is the distance from x' to x''. Consider now a homothety with the centre $0 \in \mathbf{R}^n$ and with the coefficient r. This homothety transforms our surface in a surface of the form

$$x_n = rf\left(\frac{x_1}{r}, \dots, \frac{x_{n-1}}{r}\right), \qquad (x_1, x_2, \dots, x_{n-1}) \in rV. \tag{6}$$

Suppose that t is a given positive number. We shall show that if the coefficient r in the above homothety tends to infinity, then any piece of the surface (6) of diameter $\leq t$ tends to the tangent hyperplane in one of the points of this piece. Let be rx', rx'' two points on (6) such that $\rho(rx', rx'') =$

$= r\rho(x', x'') \leq t$. If h denotes the distance from the point rx'' to the hyperplane H, tangent in the point rx' to (6), then we have

$$\frac{h}{\rho(rx', rx'')} = \frac{\frac{h}{r}}{\rho(x', x'')}, \tag{7}$$

and if we fix $\rho(rx', rx'')$ to be t, then $\rho(x', x'') = \dfrac{t}{r}$, i.e. $\rho(x', x'') \to 0$, if $r \to \infty$. But $\dfrac{h}{r}$ is the distance of x'' from the hyperplane H_1, which is tangent in x' to (5), and therefore we have $\dfrac{\frac{h}{r}}{\rho(x', x'')} \to 0$ as $\rho(x', x'') \to 0$ and from (7) it follows that $\dfrac{h}{t} \to 0$ as $r \to \infty$, i.e. $h \to 0$ as $r \to \infty$. Because the surface S is compact, we may assert that h converges to 0 uniformly with $r \to \infty$.

Denote now by t the diameter of the set $\bigcup_{i=1}^{n+1} K_i$. Suppose that the set Ω is not bounded, and let $\{c^\nu\}_{\nu=1}^{\infty}$ be a sequence of centres of spheres, which tends to infinity. Let $p_i^\nu \in K_i$, $i = 1, 2, \dots, n+1$ be the points determining the sphere with the center c^ν. Without loss of generality we may suppose, that $\{p_i^\nu\}_{\nu=1}^{\infty}$, $i = 1, 2, \dots, n+1$ are convergent, and the limits are $q_i \in K_i$, $i = 1, 2, \dots, n+1$. From our above considerations it follows that the points p_i^ν, $i = 1, 2, \dots, n+1$ tend to a hyperplane, i.e. q_i, $i = 1, 2, \dots, n+1$ are in a hyperplane; but this contradicts the hypothesis, that K_i, $i = 1, 2, \dots, n+1$ are independent sets.

(ii) Suppose that $p_i \in K_i$, $i = 1, 2, \dots, n+1$ and $p_i^\nu \in K_i$, $p_i^\nu \to p_i$, $\nu \to \infty$, $i = 1, 2, \dots, n+1$. Then $c_\nu = c(p_1^\nu, p_2^\nu, \dots, p_{n+1}^\nu) \to c = c(p_1, p_2, \dots, p_{n+1})$. Because the sequence $\{c_\nu\}^\infty$, is bounded by (i), it has a cluster point c'. Without loss of generality we may suppose that $c^\nu \to c'$. If $c' \neq c$, then the sphere with center c' and the radius $\rho = \lim \rho^\nu$ where ρ^ν is the radius of the sphere determined by p_i^ν, $i = 1, 2, \dots, n+1$, will contain the points $p_1, p_2, \dots, p_{n+1}$. This is a contradiction with the hypothesis that K_i, $i = 1, 2, \dots, n+1$ are strictly independent sets and (ii) is proved.

Let be B a ball which contains Ω. A mapping φ of B into itself will be defined as follows: If $q \in B$ we denote by $p_i(q)$ the element of the best approximation to q from the set K_i, $i = 1, 2, \dots, n+1$. It is well known from the theory of best approximation (see for instance [2], Proposition 2.3), that $p_i(q)$ is uniquely determined and depends continuously on q. Let be

$$\varphi(q) = c(p_1(q), p_2(q), \dots, p_{n+1}(q))$$

the centre of the sphere determined by the points $p_i(q)$, $i = 1, 2, \ldots, n+1$. From (ii) it follows that $\varphi(q)$ is a continuous function of $p_i(q)$, $i = 1, 2, \ldots, n+1$ and therefore also of q. By Brouwer's fixed point theorem φ will have at least one fixed point, say p, i.e. a point for which

$$\varphi(p) = p.$$

The point p has the property required by the theorem and the proof is completed.

Corollary 1. *Let $K_1, K_2, \ldots, K_{n+1}$ be independent convex and compact sets in $\mathbf{R}^n$, the n-dimensional Euclidean space. Then this family of convex sets admits a supporting sphere.*

Proof. Really, an independent family of sets in the Euclidean space $\mathbf{R}^n$ will also be strictly independent.

The Corollary 1 is in fact the existence part of our Theorem 1 in [3]. The second part of this theorem asserts the unicity of the supporting sphere.

THEOREM 3. *Let K_1, K_2, K_3 be independent convex and compact sets in the Minkowski space X^2 with a strictly convex unit sphere of class C^1. Then this family admits exactly one supporting sphere.*

Proof. In X^2 the independence and strict independence are equivalent notions. This follows from Theorem 2 in [4], which asserts that if $p_1p_2p_3$ is a triangle in X^2 then there exists exactly one triangle $p_1'p_2'p_3'$ with the same orientation as $p_1p_2p_3$ and with parallel sides, which is inscribed in the unit sphere S of X^2 (S was supposed to be a strictly convex and closed arc of class C^1). If we suppose that the independent points p_1, p_2 and p_3 determine two spheres, S_1 and S_2, then because S_2 can be obtained from S_1 by a translation and a central homothety with a positive coefficient, p_1, p_2 and p_3 will be mapped into p_1', p_2' and p_3' in S_2 and the triangles $p_1p_2p_3$ and $p_1'p_2'p_3'$ will then be of the same orientation, and will have parallel sides and are different. But this is a contradiction.

From Theorem 2 follows then the existence of a supporting sphere S for K_1, K_2 and K_3. We shall show that S is the only supporting sphere. Suppose the contrary, there exists two supporting spheres S' and S''. Denote $q_i' = S' \cap K_i$ and $q_i'' = S'' \cap K_i$, $i = 1, 2, 3$. The segments $q_i'q_i''$ are in K_i, $i = 1, 2, 3$. S' and S'' have at mots two common points. Consider that s_1 and s_2 are distinct common points of S' and S''. The straight line determined by s_1 and s_2 will intersect the segments $q_i'q_i''$, $i = 1, 2, 3$ and therefore also the sets K_i, $i = 1, 2, 3$, which is a contradiction. Similarly may be obtained a contradiction also in the other relative positions of S' and S''. This proves the theorem.

THEOREM 4. *Let $\mathcal{K}$ be a family of compact, independent convex sets in the Minkowski space X^2, with a strictly convex unit sphere S of the class C^1.*

Suppose that the number of members in $\mathcal{K}$ is ≥ 5. If for any three members in $\mathcal{K}$ there exists a supporting sphere of a given radius r, then there exists a supporting sphere of radius r for all the family $\mathcal{K}$.

This theorem is a generalisation of Theorem 2 in [3], and it will be proved by an appropriate extension of the proof in the cited paper. We begin with the proof of the following lemma:

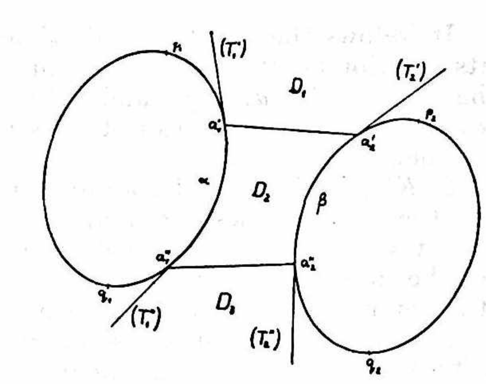

Fig. 1.

Lemma 2. *Let K' and K'' be two compact, disjoint, convex sets in the space X^2, such that $\rho(K', K'') < 2r$. Then there exist at most two supporting spheres of radius r for these two sets.*

Proof. Suppose the contrary: There exist three distinct supporting spheres S_1, S_2, S_3 of radius r for K' and K''. Let S_1 and S_2 be given and ask for the possible positions of S_3. Suppose that S_1 and S_2 are as in the Fig. 1. and ask for the possible positions of S_3. If S_1 and S_2 have a nonvoid intersection the proof is analogous.

Suppose that $a_i' = K' \cap S_i$, $a_i'' = K'' \cap S_i$, $i = 1, 2, 3$. From the definition of the supporting sphere it follows that a_1' and a_1'' (see Fig. 1.) must be on the arc $p_1 \alpha q_1$ and a_2' and a_2'' must be on the arc $p_2 \beta q_2$ (where p_1 and p_2, respectively q_1 and q_2 are the contact points of the common exterior supporting lines of S_1 and S_2 having S_1 and S_2 on the same sides). The segments $a_1' a_2'$ in K' and $a_1'' a_2''$ in K'' are disjoint because K' and K'' are disjoint. Denote by (T_i') (respectively, by (T_i'')) the tangent to S_i from a_i' (respectively a_i'') as in Fig. 1. From the definition of the supporting sphere and the convexity of K' and K'', it follows that a_3' and a_3'' must be in the domain between the lines $(T_1')a_1' \alpha a_1''(T_1'')$ and $(T_2')a_2' \beta a_2''(T_2'')$. The segments $a_1'a_2'$ and $a_1''a_2''$ determine the domains D_1, D_2 and D_3. If $a_3'' \in D_1$, then the segments $a_3''a_1''$ and $a_3''a_2''$ in K'' intersect either the segment $a_1'a_2'$ or one of the spheres S_1 or S_2 in a point distinct from $a_i''(i = 1, 2)$, which is a contradiction. Similarly, a_3' cannot be in D_3.

It is obvious that if S_3 has a common point with the domain D determined by the line segments p_1p_2, q_1q_2 and the arcs $p_1a_1' \alpha a_1''q_1$ and $p_2a_2' \beta a_2''q_2$, then S_3 or its interior has common points with the set $p_1p_2 \cup q_1q_2$. If $S_3 \cap p_1p_2 \neq \emptyset$ and $S_3 \cap q_1q_2 \neq \emptyset$, then S_3 is tangent to p_1p_2 and to q_1q_2.

Suppose now that $a_3' \in D_1$ and $a_3'' \in D_2 \cup D_3$. Then a_3' or a_3'' is in D, because $\rho(a_3', a_3'') \leq 2r$. The line segments $a_1' a_3'$ and $a_2' a_3'$ belong to K' and because K' has no points in the interior of S_3, it follows either that a_3' is on the line $a_1' a_2'$ or that S_3 is „above" the triangle $a_1' a_2' a_3'$. But in the second case S_3 cannot have points in $D_2 \cup D_3$.

By a similar reasoning it follows that if $a_3'' \in D_3$ and $a_3' \in D_1 \cup D_2$, then a_3'' is on the line segment $a_1'' a_2''$.

It follows then that a_3' and a_3'' are in D_2. Because S_3 cannot contain in its interior points of the segments $a_1' a_2'$ or $a_1'' a_2''$, it follows that we must have $a_1' = p_1$, $a_2' = p_2$ and $a_1'' = q_1$, $a_2'' = q_2$, i.e. $D_2 = D$. Moreover, a_3' must be in the interior of the segment $p_1 p_2$ and a_3'' in the interior of the segment $q_1 q_2$.

If K' (or K'') would contain a point p in the infinite strip determined by the parallel lines p_1p_2 and q_1q_2, then from the fact that S_i are of class C^1, it would follow that pa_i' (or respectively pa_i'') would contain points of the interior of S_i, which is not possible. But then $\rho(K', K'') \geq 2r$ and the lemma is proved by contradiction.

Proof of Theorem 4. Consider five members of $\mathcal{K}$: $K_1, K_2, \ldots, K_5$. The sets K_1, K_2, K_3 have a supporting sphere S of radius r. Because S is strictly convex it follows that either $\rho(K_1, K_2) < 2r$ or $\rho(K_1, K_3) < 2r$. Without loss of generality, we may suppose $\rho(K_1, K_2) < 2r$. Then by the Lemma 2, K_1 and K_2 have at most two supporting spheres S_1 and S_2 of radius r. For a given i, $3 \leq i \leq 5$ the supporting sphere of K_1, K_2, K_i must coincide either with S_1 or with S_2. We may suppose then that S_1 is the supporting sphere for K_1, K_2, K_3, K_4 and S_2 is the supporting sphere for K_1, K_2, K_5. We have either $\rho(K_5, K_1) < 2r$ or $\rho(K_5, K_2) < 2r$. Suppose for ex. $\rho(K_5, K_1) < 2r$. Let S_1' and S_2' be the only possible supporting spheres of radius r for the sets K_5, K_1 (see Lemma 2.). As above, one of these supporting spheres, say S_1', must be a supporting sphere for the sets K_1, K_5, K_i, K_j, where i and j are two given different indices, $2 \leq i, j \leq 4$. But then for K_1, K_i and K_j are supporting spheres both S_1 and S_1'. By Theorem 3 it follows that these spheres coincide and because S_1 is a supporting sphere for K_1, K_2, K_3, K_4 and S_1' is a supporting sphere also for K_5, it follows that these five sets have a common supporting sphere. The same holds for K_1, K_2, K_3, K_4 and K_i, with each K_i in $\mathcal{K}$ and the theorem follows.

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SOME REMARKS CONCERNING THE MINIMAL ATTRACTORS OF CONTINUOUS MAPPINGS

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1. Let $(X, \mathcal{G})$ be a topological space, $\varphi : X \to X$ be a given continuous mapping from X into itself, A be any subset of X, e_A be its characteristic function (so that $e_A x = 1$ if $x \in A$, and $e_A x = 0$ if $x \in X - A$), and define, for any point of X, the real set functions $\underline{p}_x$ and $\overline{p}_x$ by

$$\left.\begin{aligned} \underline{p}_x A &= \liminf_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} e_A(\varphi^k x), \\ \overline{p}_x A &= \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} e_A(\varphi^k x) \end{aligned}\right\} \text{ for any subset } A \text{ of } X.$$

If $\underline{p}_x A = \overline{p}_x A$, the common value of both sides can be interpreted as „the probability of finding the moving point $\varphi^k x$ (with x fixed, and $k = 0, 1, 2, \ldots$) in the set A". For similar concepts in topological dynamics, see [2], pp. 361–362.

A closed subset F, of X will be called an *attractor of the point* x if

$$\forall (G \in \mathcal{G}) [G \supseteq F \Rightarrow \underline{p}_x G = 1].$$

An attractor F will be said to be a *minimal attractor of the point x if* it does not contain any attractor of x which is a proper subset of F.

These notions can be generalized. A closed subset, F, of X will be called and *attractor of the arbitrary subset S of X* if F is an attractor of each point x contained in S. The set F is said to be a *minimal attractor of S* if it is an attractor of S and does not contain any attractor of S which is a proper subset of F.