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BALANCE EQUATIONS FOR THE VECTOR FIELDS DEFINED ON ORIENTABLE MANIFOLDS.

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Introduction. In the continuum mechanics and the nonequilibrium thermodynamics the balance equations are the main instruments for modelling the macroscopic bodies as continuous media [5][1]. The derivation of the balance equations is possible using Liouville equation. The kinetic and the balance equations for the collisional invariants (mass, momentum, energy) can be deduced under some restrictive conditions [2]-[4]. But Liouville equation is a conservation condition for the probability in the (microscopic) phase space, which is a particular form of balance equation. Thus, the microscopic derivation of the macroscopic properties of the thermodynamic systems is equivalent to an establishing a relation between the balance equation in the phase space and the balance equations in tridimensional Euclidian space. The existence in the phase space of other balance equations in addition to the Liouville equation could simplify and generalize some of the results in this domain. We shall show that a balance equation exists in the phase space for any scalar physical quantity. We consider the general case of a time-dependent vector field defined on an orientable manifold.

§ 1. **Definition.** We take over the definitions and the notations from [1]. Consider an orientable n-dimensional manifold M and a volume form $\Omega \in \Omega^n(M)$. If $\boldsymbol{R}$ is the temporal axis, then the extended phase space is the product manifold $\boldsymbol{R} \times M$. The mappings

$$\pi_2 : \boldsymbol{R}\times M \to M\ ;\ (s, m) \mapsto m \qquad j_s : M \to \boldsymbol{R}\times M\ ;\ m \mapsto (s, m)$$

are defined so that $\pi_2 \circ j$ is an identity on M. A smooth map $X : \boldsymbol{R}\times M \to TM$ is called a *time-dependent vector field* if $\tau_M \circ X(t, m) = m$ for any $(s, m) \in \boldsymbol{R}\times M$, where τ_M is the projection of the tangent bundle TM. A vector field $\tilde{X} \in \mathscr{X}(\boldsymbol{R}\times M)$, called a *suspension* of X, can be constructed by means of X in the following way:

$$\tilde{X} = \underline{t} + Tj_s \circ X : \boldsymbol{R}\times M \to T\boldsymbol{R}\times TM\ ;\ (s, m) \mapsto ((s, l), X(s, m)), \tag{1}$$

where $\underline{t} : \boldsymbol{R}\times M \to T\boldsymbol{R}\times TM\ ;\ (s, m) \to ((s, l), 0)$.

For a smooth real function $f \in \mathscr{F}(\boldsymbol{R}\times M)$, define $f_X = f \circ j_s \in \mathscr{F}(M)$ for any $s \in \boldsymbol{R}$. Using the additive property of the Lie derivative, we obtain from (1)

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(2) $$\dot{f}=L_{\tilde{X}}f=\partial_t f+(L_{X_s}f_s)\circ\pi_2,$$

where the notation $\partial_t f$ is used, and $\dot{f}\in\mathcal{F}(\boldsymbol{R}\times M)$ is the variation rate of f along the integral curves of $\tilde{X}$.

Since the temporal axis is an orientable manifold, the extended phase space is also orientable with the volume form $\tilde{\Omega}=\tilde{d}t\wedge\pi_2^*\Omega$, where

$$t:\ \boldsymbol{R}\times M\rightarrow\boldsymbol{R}\times M\ ;\ (s,m)\mapsto(s,0)$$

and $\tilde{d}$ is the exterior derivative on $\boldsymbol{R}\times M$. Taking into account that $L_t(\tilde{d}t)=0$ and using (1), we have the relation

(3) $$L_{\tilde{X}}\tilde{\Omega}=\tilde{d}t\wedge\pi_2^*(L_{X_t}\Omega),$$

which can be also written as $\mathrm{div}_{\tilde{\Omega}}\tilde{X}=(\mathrm{div}_\Omega X_t)\circ\pi^2$.

§ 2. Classical mechanics. In the classical mechanics M is a symplectic manifold with the structure given by a closed nondegenerate two-form ω. In this case M is orientable with the volume form $\Omega=\omega^{n/2}$. For a Hamiltonian function $H\in\mathcal{F}(\boldsymbol{R}\times M)$, the time-dependent Hamiltonian vector field is defined as $X_H:\ \boldsymbol{R}\times M\rightarrow TM\ ;\ (s,m)\mapsto X_s(m)$, where $X_s=(dH_s)^\#\in\mathcal{X}(M)$. Liouville's theorem states that Ω is invariant with respect to X_s:

(4) $$L_{X_s}\Omega=0\quad\text{or}\quad\mathrm{div}_\Omega X_x=0.$$

From (3) it follows that Liouville's theorem can be also written as

(5) $$L_{\tilde{X}}\tilde{\Omega}=0.$$

The probability density is a positive smooth function $\rho\in\mathcal{F}(\boldsymbol{R}\times M)$ which, integrated on a domain $\tilde{D}\subset\boldsymbol{R}\times M$, supplies some quantitative information on the probability so that the state be included into $\tilde{D}$. Liouville equation expresses the condition that the form $\rho\tilde{\Omega}$ is invariant with respect to $\tilde{X}$

(6) $$L_{\tilde{X}}(\rho\tilde{\Omega})=0.$$

For given $\tilde{X}$ and $\tilde{\Omega}$ and for some initial appropriate conditions, the probability density ρ can be determined from (6). Therefore Liouville's theorem (5) and Liouville equation (6) describe different properties and they must not be confounded. If X is Hamiltonian, then (5) holds and we have $L_{\tilde{X}}(\rho\tilde{\Omega})=(L_{\tilde{X}}\rho)\tilde{\Omega}$. Because of (2), the Liouville equation (6) takes the usual form

(7) $$\dot{\rho}=\partial_t\rho+(L_{X_s}\rho_s)\circ\pi_2=0.$$

Since $\rho\tilde{\Omega}$ is an external form of maximal rank on $\boldsymbol{R}\times M$, then (6) becomes

(8) $$\tilde{d}i_{\tilde{X}}(\rho\tilde{\Omega})=0.$$

Integrating (8) on a domain $\tilde{D}\subset\boldsymbol{R}\times M$ and applying Stokes' theorem, we obtain that the probability flux through the boundary of $\tilde{D}$ vanishes. Hence, Liouville equation is equivalent to the probability conservation and it takes the form of the balance equation (8) in the extended phase space.

Consider a smooth real function $f\in\mathcal{F}(\boldsymbol{R}\times M)$ and assume that a probability density ρ satisfying (6) is known. Then we have

$$\dot{f}\rho\tilde{\Omega}=(L_{\tilde{X}}f)\rho\tilde{\Omega}+fL_{\tilde{X}}(\rho\tilde{\Omega})=L_{\tilde{X}}(f\rho\tilde{\Omega}).$$

Since $f\rho\tilde{\Omega}$ is an exterior form of maximal rank on $\boldsymbol{R}\times M$, it follows that f satisfies the balance equation

$$\dot{f}\rho\tilde{\Omega}=\tilde{d}i_{\tilde{X}}(f\rho\tilde{\Omega}). \tag{9}$$

In this case the contribution of each state to the integral on a domain $\tilde{D}\subset\boldsymbol{R}\times M$ is weighted by ρ. In contrast with (8), the flux of f through the boundary of D in (9) does not always vanish and it is balanced by the generation of f inside $\tilde{D}$.

§ 3. **Statistical mechanics.** In the statistical mechanics only balance equations in the reduced phase space M for a given moment $s\in\boldsymbol{R}$ are considered, and not those in the extended phase space like (8) and (9). The terms of (5) can be written as

$$L_{\tilde{X}}(f\rho\tilde{\Omega})=dt\wedge\pi_2^*\{[\partial_t(f\rho)]_s\Omega+L_{X_s}(f_s\rho_s\Omega)\},\ \dot{f}\rho\tilde{\Omega}=dt\wedge\pi_2^*(\dot{f}_s\rho_s\Omega).$$

Then (9) becomes a balance equation on M :

$$(\partial_t f)_s\rho_x\Omega+di_{X_s}(f_s\rho_s\Omega)=[\dot{f}_s\rho_s-f_s(\partial_t\rho)_s]\Omega. \tag{10}$$

The meaning of these terms follows from the integration of (10) on a domain $D\subset M$ constant in time. The first term in the left-hand side is equal to the variation of the amount of f contained inside D. The next term gives the flux of f through the boundary of D owing to the flow defined by X_x. Note that in (10) the microscopic flux from the thermodynamic balance equation is absent because no average allowing the definition of a thermal motion has been made. The right-hand side of (10) gives the amount of f which is generated inside D owing to interior or exterior causes. If we take $f=1$ in (10), then

$$(\partial_t\rho)_s+di_{X_s}(\rho_s\Omega)=0. \tag{11}$$

This is Liouville equation (7) written on M.

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GROUP-INVARIANT METHODS IN THE THEORY OF PROJECTIVE MAPPINGS OF SPACE-TIME MANIFOLDS.

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§ 1. **Introduction.** A projective transformation of a pseudo-Riemannian manifold M^n is an automorphism of the projective structure which transforms the geodesic lines in M^n again into geodesics. The projective transformations systematically occur when we search some symmetries of equations of mathematical physics. It suffices to mention that Lie algebra of infinitesimal point symmetries of Korteveg-de Vries equation is a subalgebra of the projective (more exactly, affine) Lie algebra, and Riccati equation, by Iboragimov's expression, is "an original realization" of the projective group on a straight line. This property may be explained by the fact that the maximal group of point symmetries of dynamic Newton's equations is a projective group acting in a 4-dimensional flat space-time. This result has been obtained within the frame-work of a geometrical approach based on Lie and Cartan's ideas.

The goal of this paper is a development of the method of the theory of automorphisms of some geometrical structures and also a special technique of integrating equations on some manifolds and their application to the group analysis of differential equations of the mathematical models of physics and mechanics. The main idea is a consistent consideration of the symmetries of differential equations as automorphisms of some geometrical structures, in particular, as automorphisms of the projective structures, i.e., projective transformations. This approach would make a contribution in the geometry of the differential equations and group-invariant methods in physics, unifying and reviving on a new level E. Cartan's and S. Lie's ideas and continuing the fundamental investigations of T. Levi-Civita, G. Fubini and A. Z. Petrov.

Lie was seeking to give an explicit geometrical character to the symmetries of the differential equations. Cartan has created his theory of the projectively connected manifolds, persistently stressing its significance for the investigation of the differential equations [5][1]. The methods of differential geometry, in particular, the methods of Cartan's theory, give a systematical approach to the determination of the local and non-local symmetries for a wide classes of ordinary and partial differential equations and finding their solutions.

The newest geometrical methods in theoretical physics have been inpetuously penetrated in last ten years. The modern physical field theory acts in multidimensional curved Lorentzian manifold (space-time). The paths of motion of test bodies which are a main source of information about the structure of the physical fields are defined by the geodesic curves.

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