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ON THE MISRA-PRIGOGINE-COURBAGE THEORY OF IRREVERSIBILITY 2. THE EXISTENCE OF THE NONUNITARY SIMILARITY

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Abstract: Stochastic processes and dynamical systems in measure spaces are defined as classes of random variables in the Doob sense. Markov processes which are ergodic into a "strong" sense are shown to be suitable models for the thermodynamic irreversibility. These processes are isomorphic, in the Doob sense, with Kolmogorov dynamical systems into the space of trajectories. In this approach, we show that the Misra-Prigogine-Courbage theory of irreversibility can be formulated as a change of representation, from strong ergodic Markov processes to dynamical systems into the space of trajectories. The physical meaning is that all strong ergodic Markov processes, describing experimentally observed irreversibility, can be formally presented as unitary "superdynamics".

1. RECENT DEVELOPMENTS OF THE MPC-THEORY

The paper of Misra, Prigogine and Courbage, entitled *From Deterministic Dynamics to Probabilistic Description,* [18] was followed by many attempts to obtain thermodynamic irreversibility using dynamical systems as models of physical processes. Herein after, such approaches will be referred to as MPC-theory. The irrevesibility models are given by "strong" Markov semigroups of operators (i.e. with an unique fixed point attracting the whole definition function space and, consequently yielding H-theorem and increasing entropy law). The aim of the MPC-Theory is to find suitable ("unstable" or "chaotic") dynamical systems whose evolution group of operators (Frobenius-Perron or Koopman), $\{U_t\}_{t\in\mathbb{R}}$, may be related to strong Markov semigroups, $\{K_t\}_{t\geq 0}$, by the intertwining relation

$$K_t\Phi = \Phi U_t,\ t \geq 0.$$

In [18], pp.13-17, Φ is an invertible mapping between the corresponding L^2 function spaces and $\{U_t\}_{t\in\mathbb{R}}$ the Frobenius-Perron group of operators corresponding to a Bernoulli shift (the symbolic dynamical system associated with the "baker mapping", i.e. a Bernoulli translation on a space of double infinite sequences). The relation (1) is said to be a *nonunitary similarity.* When Φ is not invertible, (1) defines a *coarse graining* projection, which was proved for the larger class of K-systems (i.e. dynamical systems carring a given partition into the finest one-point partition for $t \to \infty$ and into the coarsest one-cell partition for $t \to -\infty$). The model example is a square mapping obtained by projecting "baker mapping" onto the cells of a K-partition [3, 7, 19].

While the coarse-graining rises no problems (irreversibility obtained by projecting a dynamical system could be interpreted as a loss of information in passing from microscopic to macroscopic description), the "similarity seems to be paradoxical because all dynamical systems preserve the entropy constant ([14], chap.9). To avoid that, we note that the irreversible evolution operators K_t act in $L^2(Y)$ (a space of functions defined in a state space$Y$, where the usual reversible description is done by Liouville equation). The reversible dynamical system acts itself in a function space (for instance the space of real functions $Y^{\mathbb{R}}$) and the corresponding operators U_t in $L^2(Y^{\mathbb{R}})$ and does not provide a model of any classical statistical mechanics system. Through (1) only a formal "dynamical" representation (in $Y^{\mathbb{R}}$) of irreversible process is aimed, which does not contradict the entropy law of classical system evolving in Y. That is why, most recent developments of MPC-theory go backwards, *from probabilistic description to deterministic dynamics.* So, Antoniou and Gustafson [2] try to answer the more physical relevant question "when experimentally observed strong Markov semigroups, like diffusion or chemical reactions, can be *lifted* to some unitary *superdynamics*". These approaches are made, following the theory of Sz.-Nagy and Foiaş [24], in terms of "positive unitary dilations". Results are obtained for "exact Markov semigroup" associated with square maps (using also the Rokhlin theorem on "natural extension of exact dynamical systems" [21]) in [2] and for more generally stationary Markov processes (by the extension of probability measures) in [5, 6].

In a previous paper [23], we have proved the existence of positive unitary dilations for all stationary Markov processes. This result comes from the idea that unambiguous comparation between dynamical systems and stochastic processes is possible when both are defined as random variables in suitable "phase" spaces. Now we prove that nonunitary similarity also exists for all strong-ergodic Markov processes defined by Gardiner [9].

2. DEFINITIONS AND PRELIMINARY RESULTS

Some definitions and three Lemmas from [23] are necessary in order to prove the Theorem from the next section.

Both stochastic processes and dynamical systems are equivalence classes of random variables. The spaces necessary to define random variables are: the complete probability space $(\Omega, \mathcal{A}, P)$, and the measurable space $(X, \mathcal{B})$, where $\mathcal{A}$ and $\mathcal{B}$ are the corresponding σ-algebra and P is a probability measure.

A random variable is a $(\mathcal{A}, \mathcal{B})$ measurable function $\eta : \Omega \to X$. The measurability condition is, usually, written as $\{\eta \in B\} \in \mathcal{A}$, $\forall B \in \mathcal{B}$,where $\{\eta \in B\} = \{\omega \in \Omega \mid \eta(\omega) \in B\}$. Hence the preimages through η of the elements of $\mathcal{B}$ are elements of $\mathcal{A}$. A measurable function is, necessarily, surjective because $\{\eta \in X\} \in \mathcal{A}$.

The distribution of the random variable η is the measure on $\mathcal{B}$ given by

$$P_\eta(B) = P(\{\eta \in B\}),\ \forall B \in \mathcal{B}.$$

The space $(\Omega, \mathcal{A}, P)$ is usually called the *basic probability space* and the sets $A \in \mathcal{A}$ are referred to as *events* [26]. Since the measure P_η satisfies the equalities $P_\eta(X) = P(\{\eta \in X\}) = P(\Omega) = 1$, it follows that $(X, \mathcal{B}, P_\eta)$ is also a probability space, called the *phase space*. For the sake of simplicity X is also called the phase space. The sets $B \in \mathcal{B}$ are called the *realizations* of the random variable.

The useful probabilistic description is done by *the expectations* of quantities defined by composed functions of random variables, i.e. the Lebesgue integral

$$Mf(\eta) = \int_\Omega f(\eta(\omega))P\,(d\omega)\,. \tag{1}$$

Due to a change of variables formula, it follows that the random variables whose distribution are identical have identical expectations [16], p.180.

Two probability spaces are isomorphic, $(\Omega_1, \mathcal{A}_1, P_1) \sim (\Omega_2, \mathcal{A}_2, P_2)$, if there exists a bijective and bimeasurable mapping $\Theta : \mathcal{A}_1 \longmapsto \mathcal{A}_2$ which preserves the probability measure, i.e. $P_1(A_1) = P_2(\Theta(A_1))$ ([8], §I,1).

Let $\mathcal{A}_\eta = \sigma\{\{\eta{\in}B\} \mid B \in \mathcal{B}\}$ be the σ-algebra on Ω generated by sets $\{\eta \in B\}$. The $(\mathcal{A}, \mathcal{B})$ measurability of η implies $\mathcal{A}_\eta \subset \mathcal{A}$. The set Ω endowed with the σ-algebra $\mathcal{A}_\eta$ and the measure $P_{\mathcal{A}_\eta}$, defined as the restriction of P to $\mathcal{A}_\eta$ is referred to as the *minimal probability space*, $(\Omega, \mathcal{A}_\eta, P_{\mathcal{A}_\eta})$, of the random variable η. By the construction of the minimal probability space the mapping $\bar{\eta} : \mathcal{B} \longmapsto \mathcal{A}_\eta$, $\bar{\eta}(B) = \{\eta \in B\} = \bar{A}$, $\forall B \in \mathcal{B}$, is bijective, bimeasurable and measure preserving, thus we have $(\Omega, \mathcal{A}_\eta, P_{\mathcal{A}_\eta}) \sim (X, \mathcal{B}, P_\eta)$. From this isomorphism it follows that an equivalence class of random variables is defined by a given phase space $(X, \mathcal{B}, P_{\mathcal{B}})$. Consequently, when a random variable is defined by "a probability space called phase space" [3], pp.178, 188, its meaning is this equivalence class.

A random function in the Doob sense is a random variable valued into a functions space,

$$\eta : \Omega \longmapsto Y^\Lambda, \;\; \eta(\omega) \in Y^\Lambda, \; \omega \in \Omega.$$

For fixed ω, the graph of the function $y^\omega : \Lambda \longmapsto Y$, $\eta(\omega) = y^\omega$, is a *trajectory* and its values, $y^\omega(\lambda) = y_\lambda$, are points in Y. In order to avoid misunderstandings, we call Y the *state space* and use the "phase space" for $X = Y^\Lambda$ only. Thus the realizations of the random function η are sets of trajectories in Y^Λ. For fixed λ, the function $\eta_\lambda : \Omega \longmapsto Y$, $\eta_\lambda(\omega) = y^\omega(\lambda)$, is a random variable whose phase space coincides with the state space Y. If $\Lambda \subseteq \mathbb{R}$ and λ means time, then the random function is a *stochastic process.* If $\Lambda \subseteq \mathbb{R}^d$ the random function is a *d-dimensional random field.*

In order to define the distribution P_η in infinite dimensional function spaces one uses the joint distributions of finite dimensional random vectors $(\eta_{\lambda_1}, ..., \eta_{\lambda_n})$ on Y^n. They are called *finite dimensional distributions* and are measures on the σ-algebra $\mathcal{B}^n$ of Y^n defined by

$$P_{\lambda_1 \ldots \lambda_n}(B) = P(\{\eta_{\lambda_1} \in B_1, ..., \eta_{\lambda_n} \in B_n\}), \; B \in \mathcal{B}^n, \; B_i \in \mathcal{B}. \qquad (2)$$

When (2) satisfy the *consistency conditions* $P_{\lambda_{i_1}, ..., \lambda_{i_n}}(B_{i_1} \times ... \times B_{i_n}) = P_{\lambda_1 \ldots \lambda_n}(B_1 \times ... \times B_n)$, for any permutation $\{i_1, ..., i_n\}$ of $\{1, ..., n\}$ and any $B_1, ..., B_n \in \mathcal{B}$, and $P_{\lambda_1 \ldots \lambda_n \lambda_{n+1}}(B_1 \times \ldots \times B_n \times Y) = P_{\lambda_1 \ldots \lambda_n}(B_1 \times ... \times B_n)$, the 'Kolmogorov Theorem on finite dimensional distributions' ensures the existence of a probability measure obeying $P_\eta(C_n) = P_{\lambda_1 \ldots \lambda_n}(B)$, for any $C_n \in \mathcal{B}^\Lambda$, where $\mathcal{B}^\Lambda$ is the smallest σ-algebra containing all the cylindrical sets, $C_n = \{y_\lambda | (y_{\lambda_1}, ..., y_{\lambda_n}) \in B, B \in \mathcal{B}^n, \lambda_1, \ldots, \lambda_n \in \Lambda\}$, and $\mathcal{B}^n$ is the σ-algebra in $\mathbb{R}^\Lambda$. Then an equivalence class of random functions in the Doob sense η is defined, up to an isomorphism by the space $(\mathbb{R}^\Lambda, \mathcal{B}^\Lambda, P_\eta)$ [12], p.166.

Lemma 1. *If $\eta : \Omega \longmapsto Y^I$, $I \subseteq \mathbb{R}$, is a stochastic process, defined on the state space $(Y, \mathcal{B})$, $Y \subseteq \mathbb{R}$, density of the n-dimensional distributions are functions $p \in L^1(Y^n)$, $Y \subseteq \mathbb{R}$, given by*

$$p(y_1, \lambda_1; ...; y_n, \lambda_n) = M[\delta(y_1 - \eta_{\lambda 1}(\omega)) ... \delta(y_n - \eta_{\lambda n}(\omega))]. \;\square \qquad (3)$$

In (3), δ is the the Dirac functional and M the expectation defined by (1). The generalization for $Y \subseteq \mathbb{R}^d$, $d \geq 2$ is straightforward ([13], p.209).

The conditional probability density is defined by

$$p(y_1,t_1;...;y_r,t_r \mid y_{r+1},t_{r+1};...;y_n,t_n) = \\ = p(y_1,t_1;...;y_n,t_n)/p(y_{r+1},t_{r+1};...;y_n,t_n).$$

Markov processes depend only on one of the earlier states and not on the whole process history. By the Kolmogorov Theorem 1.1, Markov processes are uniquely defined, in the sense of Doob, by the 1-dimensional density and the two states conditional density (called *transition probability density*). Transition probabilities obey the *Chapman-Kolmogorov equation*

$$p(y_1,t_1 \mid y_3,t_3) = \int_{\mathbf{R}} p(y_1,t_1 \mid y_2,t_2)p(y_2,t_2 \mid y_3,t_3)dy_2. \tag{4}$$

Stationary Markov processes are defined by $p(y_1,t_1 \mid y_2,t_2) = p_s(y_1,t_1 - t_2 \mid y_2)$ and $p(y,t) = p_s(y)$.

The stationary Markov operator of kernel type is the linear operator $K^\tau : L^1(Y) \longmapsto L^1(Y)$ defined by

$$K^\tau f(y) = \int_Y p(y,\tau \mid y_0)f(y_0)dy_0, \text{ where } \tau = t_1 - t_2,\ \tau \geq 0. \tag{5}$$

The operators K^τ preserve the positivity and are isometric, i.e. they have the properties: (M1) $K^\tau f \geq 0$, $\forall f > 0$, $f \in L^1(Y)$ and (M2) $\|K^\tau f\|_{L^1} = \|f\|_{L^1}$, $\forall\ f \in L^1(Y)$. The evolution of densities is given by $p(y_1,t_1) = \int_Y p(y_1,t_1 - t_2 \mid y_2,t_2)p(y_2,t_2)dy_2 = (K^{t_1-t_2}p)(y_1,t_1)$. By the Chapman-Kolmogorov equation (4) the operators (5) have the properties $K_{\tau_1+\tau_2}f = K_{\tau_2}K_{\tau_1}f$, $K_0 f = f$, $\forall\ f \in L^1(Y)$ and $\forall\ \tau_1,\tau_2 \geq 0$, which define the semigourp $\{K_\tau\}_{\tau\geq 0}$ of Markov operators.

A stationary Markov process is *strongly ergodic* ([9], p. 60) if $p_s(y_1,t_1 - t_2 \mid y_2) \longrightarrow p_s(y_1)$, for $t_1 - t_2 \longrightarrow \infty$. Correspondingly, the strongly ergodic Markov operators obey

$$\|K_\tau p - p_s\|_{L^1} \to 0, \text{ for } \tau \to \infty, \tag{6}$$

i.e. they are just the "strong Markov semigroups" [13] or "irreversible semigroups" [1] used as model of irreversibility in Misra-Prigogine-Courbage theory. For them it was easy shown that the *Gibbs entropy*, $-\int p(y,t)\log p(y,t)dy$, (and, also, any convex functional of p) monotonically increases to the maximum value corresponding to thermodynamical equilibrium [18].

Lemma 2. *The strong Markov semigroups $\{K_\tau\}_{\tau\geq 0}$, have the properties of mixing,*

$$\int_0^1 g(y)(K^\tau f)(y)dy \to \int_0^1 g(y)dy \int_0^1 f(y')dy',$$

and exactness,

$$\|K_\tau f - 1\|_{L^1} \to 0, \text{ for } \tau \to \infty ,$$

for any $f, g \in L^1(Y, \mathcal{B}, \mu')$, *where*

$$Y \subseteq \mathbb{R},\ \mu(Y) < \infty,\ \mu'(B) = \left(1 \Big/ \int_Y dy\right) \int_B dy,$$

is the probability measure in $\mathcal{B}$, *and* $p_s(y) \equiv 1$ *is the stationary density with respect to* μ'. □

Usually, *measure dynamical systems on* $(Y, \mathcal{B}, \mu)$ are defined as one-parameter groups of transformations of the state space, $\{S_t \mid S_t : Y \longmapsto Y\}_{t \in \mathbb{R} \text{ (or } \mathbb{Z})}$, which preserve the Lebesgue measure, i.e. $\mu(B) = \mu(S_{-t}B), \forall t \in \mathbb{R}$ (or $\mathbb{Z}$) and $\forall B \in \mathcal{B}$. They are groups of automorphisms on a measure space [8]. Semigroups of endomorphisms on measure spaces are called *semi-dynamical systems*.

If $Y \subseteq \mathbb{R}$, P is an absolutely-continuous measure with respect to the Lebesgue measure μ, if $\mu(S_{-t}B) = 0\ \forall B \in \mathcal{B}$ with $\mu(B) = 0$, and $P(B) = P(S_{-t}B)$, $\forall t \in \mathbb{R}$, then the dynamical system $\{S_t\}_{t \in \mathbb{R}}$ defines a random function isomorphic in the sense of Doob with the state space $(Y, \mathcal{B}, P)$. Thus, the phase space of dynamical systems (i.e. the space of trajectories) is isomorphic to the state space. More generally, this isomorphism defines the *deterministic processes* [11], in opposition with the genuine stochastic processes (where the state space is a projection, at given time value, of the phase space).

Dynamical systems are degenerated Markov processes, in the sense that a point y_0 of every trajectory of the process is transported forward, on the same trajectory, at the point y. The degenerate transition probabilities are given by $p(y, t \mid y_0) = \delta(y - S_t(y_0))$. The corresponding Markov operators (8), called *Frobenius-Perron operators,* are defined by $U_t f(y) = \int_Y \delta(y - S_t(y_0)) f(y_0) dy_0$. The adjoint of the Frobenius-Perron operator, acting on bounded Lebesgue integrable functions on Y whose norm is given by the essential supremum, i.e. $g \in L^\infty(Y)$ ([14], p.43), defined by $\int_Y f(y) U_t^* g(y) dy = \int_Y g(y) U_t f(y) dy$, for any $f \in L^1(Y)$ and $g \in L^\infty(Y)$, is the *Koopman operator*,

$$U_t^* g(y) = g(S(y)),\ \forall\, g \in L^\infty(Y). \tag{7}$$

The measure preserving property $\mu(B) = \mu(S_{-t}B)$ implies

$$(U_t^*)^{-1} f(y) = f(S_{-t}(y)) = U_t f(y),$$

i.e. the operator adjoint to U_t^* is also its inverse. Thus, a measure dynamical system in $(Y, \mathcal{B}, \mu)$ induces an unitary group $\{U_t\}_{t \in \mathbb{R}}$ which invariates the Hilbert space $L^2(Y)$ (Lemma at p. 26 in [8]). This result allows for the thermodynamic behavior of dynamical systems to be described in terms of unitary groups of operators [18].

Lemma 3. *A stationary Markov process can be embedded into the equivalence class of dynamical systems defined by the phase space $(Y^{\mathbb{R}}, \mathcal{B}^{\mathbb{R}}, P_\eta)$, where P_η is the extension by the Kolmogorov Theorem of the measure defined on cylindrical sets by*

$$P_\eta(C_{n+1}) = P_{t_0,\cdots,t_n}(B_0 \times \cdots \times B_n) =$$

$$= \int_{B_0} p_s(y_0)dy_0 \int_{B_1} p_s(y_1, t_1 - t_0 \mid y_0)dy_1 \cdots \tag{8}$$

$$\cdots \int_{B_n} p_s(y_n, t_n - t_{n-1} \mid y_{n-1})dy_n. \ \square$$

The natural representative of this class of dynamical systems is the shift along the trajectories of the Markov process, $\{\Sigma_\tau\}_{\tau\in\mathbb{R}}$, $\Sigma_\tau : Y^I \longmapsto Y^I$, $\Sigma_\tau(y^\omega(t)) = y^\omega(t+\tau)$, which invariates the measure of cylindrical sets

$$P_\eta(C_{n+1}) = P_\eta(\Sigma_\tau(C_{n+1})) = P_\eta(\Sigma_{-\tau}(C_{n+1})).$$

So, the phase space of the stationary Markov process, is also the state space of this dynamical system.

3. THE EXISTENCE OF NONUNITARY SIMILARITY

Following the definition given by Sz.-Nagy and Foiaş [24], pp.10, 31, the group of operators $\{U_\tau\}_{\tau\in\mathbb{R}}$, defined in Hilbert space $L^2(Y^{\mathbb{R}})$, is the *unitary dilation* of the semigroup $\{K_\tau\}_{\tau\geq 0}$, defined in $L^2(Y)$, if they are related by

$$K_\tau = \mathrm{Pr} U_\tau, \ \tau \geq 0, \tag{9}$$

where Pr is the orthogonal projection of $L^2(Y^{\mathbb{R}})$ on $L^2(Y)$ (by its construction $L^2(Y)$ is a sub-space of $L^2(Y^{\mathbb{R}})$). The Theorem 8.1, p.31 in [24] proves that any continuous group of contractions can be dilated to a minimal unitary and continuous group, uniquely defined up to an isomorphism. The property (M2) from § 2 shows that Markov operators are isometric (and thus, contractions) the quoted theorem ensures the existence of unitary dilation.

The dilated group of MPC theory acts on probabilities densities (which are positively defined), so it is also necessary that the dilation preserves the positivity, i.e. for any positive function from $L^2(Y^{\mathbb{R}})$, the value of $\mathrm{Pr}U_\tau$

must be a positive function from $L^2(Y)$. By the Theorem proved in [2], Markov semigroups induced by exact dynamical systems possess unitary dilations to groups induced by K-systems. In [5, 6] the result is extended to constant preserving stationary Markov semigroups (obeying $K_\tau 1 = 1$). The dilations groups are obtained by Rokhlin theorem [21] on natural extension of exact dynamical systems in [2], and extending probability measures in [5, 6].

Now we prove that all stationary Markov semigroups possess unitary dilations. For strongly ergodic and constant preserving Markov processes, we also find the results of Antoniou et al. Moreover, for these processes, we prove the existence of the 'nonunitary equivalence' of the MPC theory.

Theorem 1.

1) *The adjoint Markov semigroups $\{K_\tau^*\}_{\tau\geq 0}$ corresponding to a stationary Markov process possess positive dilation to unitary groups;*

2) *For strongly ergodic and constant preserving Markov semigroups, both $\{K_\tau^*\}_{\tau\geq 0}$ and $\{K_\tau\}_{\tau\geq 0}$ possess positive dilations to unitary groups induced by K-systems;*

3) *Between the operators K_τ of strong ergodic semigroup and those of the dilation group there exists the intertwining relation $K_\tau\Phi = \Phi U_\tau$.*

Proof. 1) From the definition of the adjoint K_τ^*, in $L^2(Y)$, of the Markov operator (5), using the Lemma 1 and the form (3) of the finite-dimensional distributions, we have

$$\begin{aligned}(K_\tau^* f)(y) &= \int_Y p(y',\tau \mid y) f(y')dy' = \tfrac{1}{p(y)}\int_Y p(y',t+\tau\,;\,y,t)f(y')dy' = \\ &= \tfrac{1}{p(y)}\int_Y f(y')dy' \int_\Omega \delta(y'-\eta(t+\tau,\omega))\delta(y-\eta(t,\omega))P(d\omega) = \\ &= \tfrac{1}{p(y)}\int_\Omega f(\eta(t+\tau,\omega))\delta(y-\eta(t,\omega))P(d\omega) = \\ &= \tfrac{1}{p(y)}\int_\Omega f(\Sigma_\tau y^\omega(t))\delta(y-\eta(t,\omega))P(d\omega) = \\ &= \tfrac{1}{p(y)}\int_\Omega U_\tau^* f(y^\omega(t))\delta(y-\eta(t,\omega))P(d\omega),\end{aligned}$$

where $y^\omega(t) = \eta(t,\omega)$. From Lemma 4 it follows that the Markov shift Σ_τ preserves the measure from the space of trajectories. Thus, the Koopman operator, defined by (7), $U_\tau^* f(y^\omega) = f(\Sigma_\tau y^\omega)$, is a unitary operator in $L^2(Y^{\mathbb{R}})$. The previous relation becomes

$$(K_\tau^* f)(y) = (\mathrm{Pr}_y U_\tau^* f)(y), \tag{10}$$

where $\mathrm{Pr}_y : L^2(Y^{\mathbb{R}}) \longmapsto L^2(Y)$, $(\mathrm{Pr}_y f)(y) = \int_\Omega f(y^\omega)P_y(d\omega)$ is the conditional expectation with respect to the measure

$$P_y(d\omega) = \tfrac{1}{p(y)}\delta(y-\eta(t,\omega))P(d\omega).$$

Hence, according to the definition (9), the group $\{U_\tau^*\}_{\tau\in\mathbb{R}}$ is the unitary dilation of the semigroup $\{K_\tau^*\}_{\tau\geq 0}$. Because U_τ^* are also Markov operators, the property (M1) implies the positivity of the dilation.

2) Strongly ergodic Markov processes (6) are mixing, according to Lemma 2. Then the Markov shift $\{\Sigma_\tau\}_{\tau\in\mathbb{R}}$ which preserves the measure (8) is a K-system (see [8], p.181and [22]). Measure preserving property and (10) also imply $(K_\tau f)(y) = (\mathrm{Pr}_y U_\tau f)(y)$. Here, Pr_y has the meaning of a projection on the cells of the K-partition consisting of all the trajectories containing the point (y,t) [2].

3) We define the canonical injection $\mathrm{I}_\omega : L^2(Y) \longmapsto L^2(Y^{\mathbb{R}})$, $\mathrm{I}_\omega f(y^\omega) \equiv f(y), \forall\, y^\omega \in \{y^\omega \mid y^\omega(t) = y, \text{for fixed-}t\}$ (so that $\mathrm{I}_\omega f$ takes constant values on cylindrical sets of the K-partition). Obviously, $\mathrm{Pr}_y \mathrm{I}_\omega = 1$. For each set of the K-partition the relation $\mathrm{I}_\omega \mathrm{Pr}_y = 1$ also holds. By representing the corresponding L^2 functions as limits of linear combinations of step functions one finds that $\mathrm{I}_\omega = (\mathrm{Pr}_y)^{-1} : L^2(Y) \longmapsto L^2(Y^{\mathbb{R}})$. Thus the projection relation (8) becomes $(K_\tau f)(y) = (\mathrm{Pr}_y U_\tau \mathrm{I}_\omega f)(y)$, $f \in L^2(Y)$, and renaming Pr_y by Φ, it just the intertwining relation of the MPC theory, $K_\tau \Phi = \Phi U_\tau$. ■

The formalism of the ergodic statistical mechanics deals with abstract irreversible dynamical systems [1-6, 18]. The use of the unitary operators formalism seems to be a "technical" requirement in developing present day statistical mechanics. The previous theorem shows that realistic irreversible processes with strong ergodicity property (as Brownian motions with reflecting boundaries [11, 15, 20], modelling ideal gases or Ornstein-Uhlenbeck process, useful in, and model example of fluctuation-dissipation theorems [9, 25]) also may be represented by dynamical systems and unitary operators.

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