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SIMULTANEOUS EQUATIONS, FOR WHICH THE ITERATIVE PROCESS IS CONVERGENT

by

VASILE PETEANU

Cluj

We consider a plane, convex domain, F its frontier and a direction Δ in the figure's plane.

Definition 1. *We call a symmetry curve of the F figure, attached to the Δ direction, the geometrical place of the chords' centres of the F figure, parallel to Δ.*

We note with $F\Delta$ the symmetry curve of the F figure, attached to the Δ direction. Since F is convex, $F\Delta$ will surely be a continous curve. Let Δ_1 and Δ_2 be two orthogonal directions in the plane of the F figure.

Definition 2. *A point belonging to the $F\Delta_1$ and $F\Delta_2$ curve's intersection, and which is interior to F, will be called a generalized centre of symmetry, attached to the $\Delta_1\Delta_2$ system.*

The existence of such a centre is evident. We observe immediatly that if F has a symmetry centre in its usual meaning, it is also a generalized centre of symmetry, attached to any system of axis. Further on, we shall consider the plane related to an orthogonal chartesian system, in which the Ox axis is parallel with Δ_1 and Oy with Δ_2. Then, $F\Delta_2$ may be interpreted as being the graphical representation of the function $y = f(x)$, and $F\Delta_1$ as being the graphical representation of an another $x = g(y)$ function, the functions $f(x)$ and $g(y)$ being defined on the $[a_0, b_0]$ interval, respectively $[c_0, d_0]$, where

$$a_0 = \inf \{x | M(x, y) \in F\}$$
$$b_0 = \sup \{x | M(x, y) \in F\}$$
$$c_0 = \inf \{y | M(x, y) \in F\}$$
$$d_0 = \sup \{y | M(x, y) \in F\}$$

We define a proper solution of the system

$$y = f(x)$$
$$(1) \qquad x = g(y)$$

any solution of the system, which is not to be found on F. Any proper solution of the system (1) is a generalized centre of symmetry of F, attached to the reference system and reciprocally.

THEOREM 1. *The set of proper solutions of the system* (1) *is convex.*

Proof. The fact that $O_1(x_1, y_1)$ and $O_2(x_2, y_2)$ are proper solutions of the (1) system, implies the existence of four strictly positive values $\alpha_1, \beta_1, \alpha_2, \beta_2$, so that the points

$$(2) \qquad \begin{matrix} M_1(x_1+\alpha_1, y_1) & M_3(x_1, y_1+\beta_1) & M_5(x_2+\alpha_2, y_2) & M_7(x_2, y_2+\beta_2) \\ M_2(x_1-\alpha_1, y_1) & M_4(x_1, y_1-\beta_1) & M_6(x_2-\alpha_2, y_2) & M_8(x_2, y_2-\beta_2) \end{matrix}$$

are to be found on F. In order to simplify the calculus and without loss in generality, we can take $x_1 = 0$, $y_1 = 1$, $x_2 = 1$, $y_2 = 0$. A necessary condition for the points (2), which derives from the convexity of F, is that the M_1 and M_7 points must be situated in an opposed semiplane to the O_1 and O_2 points, relative to the straight line M_3M_5, and respectively the M_4 and M_6 points are to be found in the semiplane opposed to the O_1 and O_2 points, relative to the M_2M_8 straight line. These conditions are expressed by the following inequalities:

$$\frac{\alpha_1}{1+\alpha_2} + \frac{1}{1+\beta_1} - 1 \geqq 0 \qquad \frac{\beta_1}{1+\beta_2} + \frac{1}{1+\alpha_1} - 1 \geqq 0$$

$$\frac{\beta_2}{1+\beta_1} + \frac{1}{1+\alpha_2} - 1 \geqq 0 \qquad \frac{\alpha_2}{1+\alpha_1} + \frac{1}{1+\beta_2} - 1 \geqq 0$$

The preceding inequalities can also be written:

$$(3) \qquad \begin{matrix} \dfrac{\alpha_1}{1+\alpha_2} \geqq \dfrac{\beta_1}{1+\beta_1} & \dfrac{\alpha_2}{1+\alpha_1} \geqq \dfrac{\beta_2}{1+\beta_2} \\ \dfrac{\beta_1}{1+\beta_2} \geqq \dfrac{\alpha_1}{1+\alpha_1} & \dfrac{\beta_2}{1+\beta_1} \geqq \dfrac{\alpha_2}{1+\alpha_2} \end{matrix}$$

The inequalities (3) can be simultaneously verified only if $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, and their consequence is that the M_3, M_7, M_1 and M_5 points, respectively M_2, M_6, M_4 and M_8 are colinear. But in this case, F contains two parallel segments, and any point of the O_1O_2 segment has the property to be a generalized symmetry centre, attached to the reference system, fact that demonstrates the theorem.

Lemma: *If*

$$x_1 = g(y_3) \qquad x_4 = g(y_2)$$
$$y_1 = f(x_3) \qquad y_4 = f(x_2)$$

and

$$x_1 \leqq x_2 \leqq x_3 \leqq x_4$$
$$y_1 \leqq y_2 \leqq y_3 \leqq y_4$$

then

$$x_1 = x_2 \qquad x_3 = x_4$$
$$y_1 = y_2 \qquad y_3 = y_4$$

Proof. There exists the strictly positive $\alpha_1, \beta_1, \alpha_2, \beta_2$ values, so that the $A(x_1-\alpha_1, y_3)$, $B(x_2, y_4+\beta_1)$, $C(x_4+\alpha_2, y_2)$, $D(x_3, y_1-\beta_2)$ points are to be found on F. Let $y = f_1(x)$ and $x = g_1(y)$ be the equations of the symmetry curve of the $ABCD$ quadrilateral, attached to the Oy direction respectively Ox. The CM_3, M_3M_4 and M_4D segments compose the graphical representation of the first equation, and the AM_1, M_1M_2 and M_4D segments compose the second equation's graphical representation, where $M_1 = M_1(x_2, y_5)$ $M_2 = M_2(x_3, y_0)$, $M_3 = M_3(x_0, y_3)$ and $M_4 = M_4(x_5, y_2)$ with $y_0 \leqq y_1$, $y_4 \leqq y_5$, $x_0 \leqq x_1$, $x_4 \leqq x_5$. Since the function f_1 is continous, there are two values x' and x'' belonging to the $[x_2\ x_3]$ interval, so that $f_1(x') = y_3$ and $f_1(x'') = y_2$. From the inequalities

$$(4) \qquad \begin{aligned} & x_1 - \alpha_1 - g_1(y_3) < 0 \\ & x' - g_1(y_3) \geqq 0 \\ & x'' - g_1(y_2) \leqq 0 \\ & x_4 + \alpha_2 - g_1(y_2) > 0 \end{aligned}$$

and according to the continuity of the f_1 and g_1 functions, it results the existence of three generalized symmetry centres, attached to the reference system. As a consequence of the Theorem 1, the only possible case is

$$x_0 = x_1 = x_2 \qquad x_3 = x_4 = x_5 \qquad \alpha_1 = \alpha_2$$
$$y_0 = y_1 = y_2 \qquad y_3 = y_4 = y_5 \qquad \beta_1 = \beta_2$$

a case in which the F figure allows a segment of generalized centres of symmetry attached to the reference system.

THEOREM 2. *For the system (1), the iterative process is convergent to a proper solution, for any initial approximation $M_0(x_0, y_0)$ in the interior of the F convex figure.*

Proof: Let

$$(5) \qquad \begin{aligned} y_i &= f(x_{i-1}) \\ x_i &= g(y_{i-1}) \end{aligned}$$

the iteration of order i. We note

$$(6) \qquad \begin{aligned} a_i &= \inf_{c_{i-1} \leqq y \leqq d_{i-1}} g(y), & b_i &= \sup_{c_{i-1} \leqq y \leqq d_{i-1}} g(y) \\ c_i &= \inf_{a_{i-1} \leqq x \leqq b_{i-1}} f(x), & d_i &= \sup_{a_{i-1} \leqq x \leqq b_{i-1}} f(x) \end{aligned}$$

Evidently

$$(7) \qquad \begin{aligned} a_{i-1} \leqq a_i \leqq x_i \leqq b_i \leqq b_{i-1} \\ c_{i-1} \leqq c_i \leqq y_i \leqq d_i \leqq d_{i-1} \end{aligned} \qquad i = 1, 2, \ldots$$

In the iterative process, there are two possibilities : or $\lim\limits_{i\to\infty} (b_i - a_i) = 0$ or $\lim\limits_{i\to\infty} (b_i - a_i) \neq 0$. In the first case, the proceeding converges. We shall analyse the second case. Let $a^* = \lim\limits_{i\to\infty} a_i$ and $b^* = \lim\limits_{i\to\infty} b_i$. The possibility that $\lim\limits_{i\to\infty} (d_i - c_i) = 0$ is excluded, because in this case $g(y)$ would take an infinity of values in a point, contrary to the definition. We note that $c^* = \lim\limits_{i\to\infty} c_i$ and $d^* = \lim\limits_{i\to\infty} d_i$. Evidently, there are the x_2 and x_3 values in the $[a^*, b^*]$, interval, and y_2 and y_3 in the $[c^*, d^*]$ interval, so that $f(x_2) = d^*$ $f(x_3) = c^*$, $g(y_2) = b^*$ and $g(y_3) = a^*$. The preceding lemna, specifies that here $x_2 = a^*$, $x_3 = b^*$, $y_2 = c^*$, $y_3 = d^*$. According to the Theorem 1, any point of the $O_1(a^*, d^*)$ $O_2(b^*, c^*)$ segment is a proper solution of the system 1. The convergence of the iterative process (5) is also provided in this case, fact that demonsrates the theorem.

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UN ALGORITHME POUR RÉSOUDRE CERTAINS PROBLÈMES DE PROGRAMMATION MATHÉMATIQUE

par

F. RADÓ

à Cluj

1. Dans le livre [5] se trouve énoncé le problème suivant de *programmation linéaire à contraintes conditionnelles* :

On considère l'ensemble des vecteurs $x = (x_1, x_2, \ldots, x_n)$ à composantes réelles, qui vérifient les conditions

$$(1) \qquad x_j \geqq 0, \quad j = 1, 2, \ldots, n$$

$$(2) \qquad D_i(x) = \sum_{j=1}^{n} d_{ij}\, x_j - d_i \geqq 0, \quad i = 1, 2, \ldots, m$$

$$(3) \qquad A_k(x) = \sum_{j=1}^{n} a_{kj}\, x_j - a_k < 0 \Rightarrow B_k(x) = \sum_{j=1}^{n} b_{kj}\, x_j - b_k \geqq 0, \quad k = 1, \ldots, p.$$

Déterminer le vecteur x, pour lequel la fonction

$$(4) \qquad f(x) = \sum_{j=1}^{n} c_j\, x_j$$

prend sa valeur minime (les nombres a_{kj}, b_{kj}, d_{ij}, a_k, b_k, d_i, c_j sont donnés).

Divers problèmes d'économie conduisent à ce modèle mathématique, qui généralise la programmation linéaire. Dans le travail [1], un problème de minimum relatif au temps de fabrication d'un produit qui passe par une série de machines, dans un ordre déterminé, a été posé sous la forme énoncée. Les problèmes de planification en temps de la production (sequencing prob-