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To Professor TIBERIU POPOVICIU
in honour of the 60^{th} *anniversary of his birthday*

ON BERNSTEIN POWER SERIES

by

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1. Introduction

In the first part of the paper, a method for the construction of a sequence of positive linear operators is indicated. The generality of this method will result from several examples and some well-known operators will be obtained. Similar methods were studied in [3], [4] and [10].

In the second part, the operators of Szász-Mirakyan and Baskakov are considered, and we establish the analogy between these operators and Bernstein operator. Next, by using both the theory of the convex functions relying on the divided difference notion, substantiated by T. POPOVICIU, and likewise his results concerning the remainder in the approximation linear formulae [5] — [7], we give new properties of the above operators. Moreover it can be noticed that these properties are similar to those of the Bernstein operator; they are related to the monotony of the sequence of operators taking into account the shape of the function and to the simple form of the remainder. In the case of the Bernstein operator these properties were established by O. ARAMĂ [1].

2. Notation and definitions

Further we use the following symbols:

ω is the set of natural numbers.

$[x_1, x_2, \ldots, x_{k+2}; f]$ is the $k+1$ — order divided difference of the function f on the knots $x_1, x_2 \ldots, x_{k+2}$.

$Q[a, b]$ is the set of the functions which are defined and bounded on the interval $[0, \infty)$ and which are continuous on the interval $[a, b]$, continuous to the left on $x = a$ and continuous to the right on $x = b$.

$\{L_n\}$, $n \in \omega$, is a sequence of operators defined on $Q[a, b]$.

$\Delta_1 L_n = L_{n+1} - L_n$

$V_D[f]$ is the variation of f, defined as the number of changes of sign of the function f as x varies across its domain, D.

Moreover we recall the following definitions (see [5] — [7]).

Definition 1. *A real function f is called convex, non-concave, polynomial, non-convex respectively concave of k-order, on the interval $[a, b]$, if*

$$[x_1, x_2, \ldots, x_{k+2}; f] > 0, \geqslant 0, = 0 \leqslant 0 \text{ resp. } < 0,$$

for any system of $k + 2$ knots from $[a, b]$.

Definition. 2. *A functional T which is defined on $C[a, b]$ is of the exactness degree k, and in this case we say that T is in $\mathcal{E}_k$, if*

$$T[x^j] = 0, \ j = 0, 1, \ldots, k, \text{ and } T[x^{k+1}] \neq 0.$$

Definition 3. *A functional T defined on $C[a, b]$ has the simple form of k-order, and in this case we shorten by $T \in \mathcal{A}_k$, if for all f in $C[a, b]$, we have*

$$T[f] = C\,[x_1, x_2, \ldots, x_{k+2}; f],$$

where $C \neq 0$ is independent of f and the distinct knots $x_1, x_2, \ldots, x_{k+2}$ generally depend by the choice of f.

We use the following results in the next sections.

THEOREM I (T. POPOVICIU). *Let $C^*[a, b]$ the conjugate space of $C[a, b]$ and let $T \in C^*[a, b]$. The functional T is in $\mathcal{A}_k$ if and only if $T \in \mathcal{E}_k$, and $T[f] \neq 0$ for any convex function of k-order on $[a, b]$ ($f \in C[a, b]$).*

THEOREM II (P. P. KOROVKIN). *We consider a continuous non-decreasing function $\chi(x)$ on the interval $[a, b]$. Let $a^* = \chi(a)$ and $b^* = \chi(b)$. Let $f \in Q[a^*, b^*]$ and let $\{L_n\}$, $n \in \omega$, be a sequence of linear positive operators defined on $Q[a,^* b^*]$. If this sequence satisfies the conditions*

$$\lim_{n\to\infty} L_n[1; x] = 1$$

$$\lim_{n\to\infty} L_n[t; x] = \chi(x)$$

$$\lim_{n\to\infty} L_n[t^2; x] = [\chi(x)]^2,$$

uniformly on the interval $[a, b]$, then the sequence of functions $\{L_n[f; x]\}$, $n \in \omega$, converges uniformly on $[a, b]$ to $f[\chi(x)]$.

This statement is a generalization of small weight of the theorems which were given by Bohman, Korovkin and for a particular sequence of operators by T. POPOVICIU [8]. The proof of the above theorem is essentially the same with Korovkin's proof. The fact that $\chi(x)$ is a non-decreasing function will be used for the proof that when $x \in [a, b]$ then $\chi(x)$ is a continuity point of f.

3. A method for constructing linear positive operators

Let us consider two real functions $\alpha(x)$ and $g(x)$ which are holomorphic functions defined in the diskes $|x| < R_1$ and $|x| < R_2$. Also we suppose that the coefficients of the corresponding developments in power series are non-negative and that $\alpha(0) \neq 0$. Let $\lambda(n)$ and $\lambda_1(n)$, $n \in \omega$, positive functions. We define the sequence $\{\alpha_n\}$, $n \in \omega$, by the relation

$$\alpha_n(x) = \exp \lambda_1(n) \int_0^x \alpha'(s)\, g(s) ds, \ x \in [0, R), \ R = \min (R_1, R_2). \tag{1}$$

In this case the function $\alpha_n(x)$ admits a developpement in power series with the convergence radius equal to R, that is

$$\alpha_n(x) = \sum_{\nu=0}^{\infty} c_{n\nu} x^\nu$$

and the coefficients $c_{n\nu}$ are non-negative.

In order to construct sequences of operators, we assume as verified the conditions:

1. $\lim\limits_{n\to\infty} \lambda(n) = \infty$, $\lim\limits_{n\to\infty} \lambda_1(n) = \infty$

2. $\lim\limits_{n\to\infty} \dfrac{\lambda_1(n)}{\lambda(n)} = 1.$

We consider the operator L_n which is defined, on the class of functions $Q[0, \delta]$, $\delta > 0$, by the relation

$$L_n[f; x] = \frac{1}{\alpha_n(x)} \sum_{\nu=0}^{\infty} c_{n\nu} x^\nu f\left[\frac{\nu}{\lambda(n)}\right]. \tag{2}$$

This operator is obviously linear and positive for $x \geqslant 0$. Taking into account the identity

$$1 = \frac{1}{\alpha_n(x)} \sum_{\nu=0}^{\infty} c_{n\nu} x^\nu, \ x \in [0, R),$$

we give

$$L_n[1\,;x] = 1, \quad x \in [0, R). \tag{3}$$

Further we consider the function

$$\tau(x) = x\alpha'(x)\,g(x),$$

and by our conditions for α and g it results that τ is an absolutely monotone function on the interval $[0, R)$. Using the above remark we have the following.

THEOREM 1. *Let $a \in (0, R)$ and let $a^* = \tau(a)$. If $f \in Q[0, a^*]$, then the sequence of functions $\{L_n[f\,;x]\}$, $n \in \omega$, converges uniformly on the interval $[0, a]$ to the function $f[\tau(x)]$.*

Proof. We shall show that the sequence of linear positive operators $\{L_n\}, n \in \omega$, verifies the conditions of Korovkin's theorem. It is clear that τ is a non-decreasing function. Let $x \in [0, a]$; in this case we observe that $\tau(x) \in [0, a^*]$. For $f(t) = t$ we have

$$L_n[t\,;x] = \frac{1}{\alpha_n(x)}\sum_{\nu=0}^{\infty} c_{n\nu}\frac{\nu}{\lambda(n)}x^\nu = \frac{x\alpha_n'(x)}{\lambda(n)\alpha_n(x)}.$$

On the other hand from the differentiation of (1), we obtain

$$\frac{\alpha_n'(x)}{\alpha_n(x)} = \lambda_1(n)\,\alpha'(x)\,g(x)$$

which enables us to give

$$L_n[t\,;x] = \frac{\lambda_1(n)}{\lambda(n)}\tau(x). \tag{4}$$

When $f(t) = t^2$, we have

$$L_n[t^2\,;x] = \frac{1}{\alpha_n(x)}\sum_{\nu=0}^{\infty} c_{n\nu}\frac{\nu^2}{\lambda^2(n)}x^\nu = \frac{x^2\alpha_n''(x) + x\alpha_n'(x)}{\lambda^2(n)\,\alpha_n(x)}.$$

Finally, since

$$\frac{\alpha_n''(x)}{\alpha_n(x)} = \lambda_1^2(n)\,x'^2(x)\,g^2(x) + \lambda_1(n)\,\alpha'(x)\,g'(x) + \lambda_1(n)\,\alpha''(x)\,g(x),$$

we can prove that

$$L_n[t^2\,;x] = \left[\frac{\lambda_1(x)}{\lambda(n)}\tau(x)\right]^2 + \frac{\lambda_1(n)}{\lambda^2(n)}\left[x^2\alpha'(x)\,g'(x) + x^2\alpha''(x)\,g(x) + \tau(x)\right]. \tag{5}$$

For $n \to \infty$, taking into account the relations (3), (4) and (5) as well as the conditions 1. and 2., it follows that

$$\lim_{n\to\infty} L_n[1\,;x] = 1$$

$$\lim_{n\to\infty} L_n[t\,;x] = \tau(x)$$

$$\lim_{n\to\infty} L_n[t^2\,;x] = [\tau(x)]^2,$$

and because for any $x \in [0, a]$ the points $\tau(x)$ are continuity points for the function f, according to Korovkin's theorem our assertion is proved.

This method is a generalization of Kesava Menon's method for the construction of linear positive operators. In his method we must have

$$\lambda(n) = \lambda_1(n) = n,\ g(x) = 1,$$

and $\{\alpha_n(x)\}$, $n \in \omega$, is the sequence obtained by substituting $\ln\alpha(x)$ in $\alpha(x)$, in other words we give

$$\alpha_n(x) = [\alpha(x)]^n.$$

Relating to the class of operators L_n, defined by (2), it occurs the following property.

THEOREM 2. *The operator L_n is a variation-diminishing operator, that is*

$$V_{[0,a]}\{L_n[f\,;x]\} \leqslant V_{[0,\infty)}[f(x)].$$

Proof. It is known [2], that if

$$\psi(x) = \sum_{\nu=0}^{\infty} a_\nu x^\nu$$

is convergent for $x \in (-\delta, \delta)$, then for $\gamma < \delta$ we have

$$V_{[0,\gamma]}[\psi(x)] \leqslant V[\{a_\nu\}_{\nu\in\omega}],$$

where $V[\{a_\nu\}_{\nu\in\omega}]$ is the variation of the sequence $\{a_\nu\}_{\nu\in\omega}$.
Since

$$\frac{c_{n\nu}}{\alpha_n(x)} \geqslant 0,\ x \in [0, a],$$

it results that

$$V_{[0,a]}\{L_n[f\,;x]\} \leqslant V\left[\left\{f\left(\frac{\nu}{\lambda(n)}\right)\right\}_{\nu\in\omega}\right] \leqslant V_{[0,\infty)}[f(x)],$$

and the theorem is proved.

A theorem of this type was given by I. J. SCHOENBERG in the case of the Bernstein operator [9].

Remark. Let the functionals R_n, $n \in \omega$, be defined by

$$R_n[f] = f[\tau(x)] - L_n[f; x], \quad f \in Q[0, a^*].$$

If $\lambda(n) = \lambda_1(n)$, then the functionals $\Delta_1 L_n$ and R_n are in $\mathcal{E}_1$. Using (3), (4) and (5) the proof of this assertion is immediate.

In what follows we shall try to indicate a concrete method of getting some well-known operators. Having the above defined function α, let $g = 1$, and $\lambda(n) = \lambda_1(n) = n$. It is obvious that

$$\alpha_n(x) = e^{n\alpha(x)}$$

and

$$\tau(x) = x\alpha'(x).$$

We make mention of the following particular cases:

I. For $\alpha(x) = x$, we obtain the Szász-Mirakyan operator, defined by

$$S_n[f; x] = e^{-nx} \sum_{\nu=0}^{\infty} \frac{(nx)^\nu}{\nu!} f\left(\frac{\nu}{n}\right). \tag{6}$$

For $f \in Q[0, a]$, $a > 0$, the sequence $\{S_n[f; x]\}$, $n \in \omega$, converges uniformly on $[0, a]$ to $f(x)$.

II. If $\alpha(x) = \ln \frac{1}{1-x}$, the operator has the form

$$L_n[f; x] = (1-x)^n \sum_{\nu=0}^{\infty} \binom{n+\nu-1}{\nu} x^\nu f\left(\frac{\nu}{n}\right). \tag{7}$$

For any $f \in Q[0, a^*]$, $a^* = \frac{a}{1-a}$, $0 < a < 1$, the sequence $\{L_n[f; x]\}$, $n \in \omega$, converges uniformly on $[0, a]$ to $f\left(\frac{x}{1-x}\right)$.

III. By means of the operator defined by (7), Baskakov's operator is obtained, replacing x by $\frac{x}{1+x}$, that is

$$K_n[f; x] = L_n\left[f; \frac{x}{1+x}\right] = \sum_{\nu=0}^{\infty} \binom{n+\nu-1}{\nu} \frac{x^\nu}{(1+x)^{n+\nu}} f\left(\frac{\nu}{n}\right). \tag{8}$$

If $f \in Q[0, a]$, $a > 0$, then the corresponding sequence converges uniformly on $[0, a]$ to $f(x)$.

4. Some properties of the linear positive operators

THEOREM 3. a) *If the function f defined on $[0, \infty)$ is on this interval respectively convex, non-concave, polynomial, non-convex, concave of 1-order, then the sequence of operators $\{S_n\}$, $n \in \omega$, defined by (6), is respectively decreasing, non-increasing, stationary, non-decreasing, increasing.*

b) *If f is continuous on $[0, \infty)$, we have the following equalities*

$$\Delta_1 S_n[f; x_0] = -\frac{x_0}{n(n+1)} [\zeta_{1n}, \zeta_{2n}, \zeta_{3n}; f]$$

$$f(x_0) - S_n[f; x_0] = -\frac{x_0}{n} [\eta_{1n}, \eta_{2n}, \eta_{3n}; f]$$

where $\zeta_{in}, \eta_{in} \in [0, \infty)$, $i = 1, 2, 3$, and x_0 is a fixed point in $[0, \infty)$.

Proof. The assertion a) was proved in [2]. Let x_0 be a fixed point in $[0, \infty)$. According to the remark and to the first assertion we see that the functional $\Delta_1 S_n$ verifies the conditions of Popoviciu's theorem. Thus, this functional has a simple form of 1-order, namely

$$\Delta_1 S_n[f] = C_n(x_0) [\zeta_{1n}, \zeta_{2n}, \zeta_{2n}; f], \quad f \in C[0, \infty). \tag{9}$$

The value of $C_n(x_0)$ can be determined by particularizing convenably the function f (see [7]).

For $f(t) = t^2$, we obtain

$$S_n[t^2; x_0] = \frac{x_0 e^{-nx_0}}{n} \sum_{\nu=0}^{\infty} \frac{n^\nu(\nu+1)}{\nu!} x_0^\nu = x_0^2 + \frac{x_0}{n}.$$

Consequently

$$\Delta_1 S_n[t^2] = C_n(x_0) = -\frac{x_0}{n(n+1)}.$$

Substituting $C_n(x_0)$, thus determined, in (9) we obtain the first affirmation from b).

Similarly, we can speak about the simple form of the functional defined by

$$R_n[f; x_0] = f(x_0) - S_n[f; x_0], \quad f \in C[0, \infty).$$

Indeed, this functional takes negatives values for any convex function of 1-order, and R_n is in $\mathcal{E}_1$. Let

$$R_n[f] = A_n(x_0) [\eta_{1n}, \eta_{2n}, \eta_{3n}; f],$$

and it is readily verified that

$$R_n[t^2] = A_n(x_0) = -\frac{x_0}{n},$$

and thus the theorem is proved.

For the Bernstein operator an analogous result was obtained by O. ARAMĂ [1].

Next, in order to study the Baskakov operator we shall prove the following results.

Lemma. *Let f be defined on* $[0, \infty)$. *Then for the class of linear positive operators L_n, defined by* (7) *the functional $\Delta_1 L_n[f; x_0]$ has the following form*

$$\Delta_1 L_n[f; x_0] = (1 - x_0)^n \sum_{\nu=1}^{\infty} \Lambda_{\nu n}[f] x_0^\nu,$$

where

$$\Lambda_{\nu n}[f] = -\frac{1}{n(n+1)} \binom{n+\nu}{\nu-1} \left[\frac{\nu-1}{n+1}, \frac{\nu}{n+1}, \frac{\nu}{n}; f\right].$$

Proof. We have

$$\Delta_1 L_n[f; x_0] = (1 - x_0)^n (1 - x_0) \sum_{\nu=0}^{\infty} \binom{n+\nu}{\nu} f\left(\frac{\nu}{n+1}\right) x_0^\nu -$$

$$- (1 - x_0)^n \sum_{\nu=0}^{\infty} \binom{n+\nu-1}{\nu} f\left(\frac{\nu}{n}\right) x_0^\nu =$$

$$= -(1 - x_0)^n \sum_{\nu=0}^{\infty} \binom{n+\nu}{\nu} \left[\frac{\nu}{n+\nu} f\left(\frac{\nu-1}{n+1}\right) - f\left(\frac{\nu}{n+1}\right) + \frac{n}{n+\nu} f\left(\frac{\nu}{n}\right)\right] x_0^\nu =$$

$$= -(1 - x_0)^n \sum_{\nu=1}^{\infty} \frac{1}{n(n+1)} \binom{n+\nu}{\nu-1} \left[\frac{\nu-1}{n+1}, \frac{\nu}{n+1}, \frac{\nu}{n}; f\right] x_0^\nu.$$

Putting

$$\Lambda_{\nu n}[f] = -\frac{1}{n(n+1)} \binom{n+1}{\nu-1} \left[\frac{\nu-1}{n+1}, \frac{\nu}{n+1}, \frac{\nu}{n}; f\right]$$

the lemma is proved.

THEOREM 4. a) *If the function f defined on* $[0, \infty)$ *is on this interval respectively convex, non-concave, polynomial, non-convex, concave of 1-order, then the sequence of operators $\{L_n\}$, $n \in \omega$, defined by* (7), *is respectively decreasing, non-increasing, stationary, non-decreasing, increasing.*

b) *If f is continuous on* $[0, \infty)$, *we have the following equalities*

$$\Delta_1 L_n[f; x_0] = -\frac{x_0(1 - x_0)^{-2}}{n(n+1)} [\mu_{1n}, \mu_{2n}, \mu_{3n}; f]$$

$$f\left(\frac{x_0}{1 - x_0}\right) - L_n[f; x_0] = -\frac{x_0(1 - x_0)^{-2}}{n} [\xi_{1n}, \xi_{2n}, \xi_{3n}; f]$$

where μ_{in}, $\xi_{in} \in [0, \infty)$, $i = 1, 2, 3$, and x_0 is a fixed point in this interval.

Proof. The first assertion is an easy consequence of the above lemma. Because of the fact that for fixed x_0 in the interval $[0, \infty)$, the set $\mathcal{E}_1$ contains the functionals defined by $\Delta_1 L_n[f; x_0]$ and by $f\left(\frac{x_0}{1 - x_0}\right) - L_n[f; x_0]$, and in view of a), it is easy to verify that both are in $\mathcal{A}_1$. Thus we can write

$$\Delta_1 L_n[f; x_0] = B_n(x_0) [\mu_{1n}, \mu_{2n}, \mu_{3n}; f].$$

For $f(t) = t^2$, and taking into consideration that

$$\Lambda_{\nu n}[t^2] = -\frac{1}{n(n+1)} \binom{n+\nu}{\nu-1},$$

it follows

$$\Delta_1 L_n[t^2; x_0] = B_n(x_0) = -\frac{x_0(1 - x_0)^{-2}}{n(n+1)}$$

and we derive the first equality. Further we give that for

$$R_n[f] = f\left(\frac{x_0}{1 - x_0}\right) - L_n[f; x_0] = E_n(x_0) [\xi_{1n}, \xi_{2n}, \xi_{3n}; f]$$

we obtain

$$R_n[t^2] = E_n(x_0) = -\frac{x_0(1 - x)^{-2}}{n}.$$

From this theorem the following remarkable property for the Baskakov operator is resulting.

THEOREM 5. a) *If the function f defined on* $[0, \infty)$ *is on this interval respectively convex, non-concave, polynomial, non-convex, concave of 1-order, then the sequence of operators $\{K_n\}$, $n \in \omega$, defined by* (8), *is respectively decreasing, non-increasing, stationary, non-decreasing, increasing.*

b) *If f is continuous on* $[0, \infty)$, *we have the following equalities*

$$\Delta_1 K_n[f; x_0] = -\frac{x_0(x_0 + 1)}{n(n+1)} [\varkappa_{1n}, \varkappa_{2n}, \varkappa_{3n}; f]$$

$$f(x_0) - K_n[f; x_0] = -\frac{x_0(x_0 + 1)}{n} [\varepsilon_{1n}, \varepsilon_{2n}, \varepsilon_{3n}; f]$$

where $\varkappa_{in}$, $\varepsilon_{in} \in [0, \infty)$, $i = 1, 2, 3$, and x_0 is a fixed point in $[0, \infty)$.

Proof. The above results follow immediately from theorem 4, taking into account the equality

$$K_n[f; x_0] = L_n\left[f; \frac{x_0}{1 + x_0}\right].$$

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Hommage à M. T. POPOVICIU
à l'occasion de son 60-e anniversaire

L'APPLICABILITÉ PROJECTIVE D'UNE CONGRUENCE DE DROITES EN ELLE-MÊME

par
FROIM MARCUS
à Iassy

1. On a étudié jusque maintenant les déformations projectives des surfaces en elles-mêmes [1] mais pas encore le même problème pour les congruences de droites.

Le présent travail, et un autre de prochaine publication, s'occupent de ce problème.

D'après Cartan, une congruence de droites est généralement projectivement indéformable de deuxième ordre, et celles qui sont déformables, dépendent d'une fonction arbitraire de deux variables. Pour reconnaître si deux congruences sont projectivement applicables nous avons le critérium de FUBINI [2] et de A. TERRACINI [6].

D'après FUBINI [2] *deux surfaces S et S' sont les nappes focales de deux congruences de droites applicables (de deuxième ordre) si les asymptotiques u, v se correspondent et s'il existe une fonction $\rho = \rho(u, v)$ telle que les deux quantités*

$$(\beta\rho)_v - \left(\frac{\gamma}{\rho}\right)_u,$$

$$\rho^2\left(-L + \rho\beta_u - \frac{1}{2}\rho^2\beta^2 + 2\beta\rho_u\right) + \left(M - \frac{\gamma_v}{\rho} + \frac{1}{2}\frac{\gamma^2}{\rho^2} + 2\gamma\frac{\rho_v}{\rho^2}\right), \tag{1.1}$$

aient des valeurs égales sur S et S'.

(β, γ, L, M sont les expressions bien connues de la théorie des surfaces).