Observation. Les notions de Q-uniformément continuité et Q-compacité sont indépendantes au moins dans les cas des ensembles bornés au sens de la métrique ou au sens de J. VON NEUMANN. De plus, dans ces cas la continuité d'une application, ajoutée à la Q-compacité de celle-ci, ce qui revient maintenant à la compacité ordinaire, n'entraîne pas la continuité uniforme habituelle. Une telle application, d'après le théorème précédent, ne peut pas être complètement compacte. L'exemple correspondant donné ci-dessous est suggéré par le travail cité de M.. VEINBÈRG.

Soit X l'espace normé des suites bornées de nombres réels, $x = (\xi_i)_{i\in\omega}$, muni de la norme $\|x\| = \sup_{i\in\omega} |\xi_i|$, où ω désigne l'ensemble des nombres naturels dirigé habituellement. Considérons les suites $(x_n)_{n\in\omega}$ et $(y_n)_{n\in\omega}$ où $x_n = (\xi_i^{(n)})_{i\in\omega}$ et $y_n = (\eta_i^{(n)})_{i\in\omega}$, avec

$$\xi_1^{(n)} = \eta_1^{(n)} = \xi_{n+1}^{(n)} = 1, \quad \eta_{n+1}^{(n)} = 1 - \frac{1}{6n} \quad \text{et} \quad \xi_i^{(n)} = \eta_i^{(n)} = 0$$

pour $i \in \omega$, $i \neq 1$ et $i \neq n + 1$. La fonctionnelle réelle A, définie sur X par l'égalité

$$A(x) = \begin{cases} 1 - 6n\,\|x - x_n\|, & \text{pour } \|x - x_n\| < \frac{1}{6n} \\ 6n\,\|x - y_n\| - 1, & \text{pour } \|x - y_n\| < \frac{1}{6n} \\ 0 & \text{pour le reste des éléments,} \end{cases}$$

où $n \in \omega$, est continue et compacte, sans être uniformément continue.

BIBLIOGRAPHIE

[1] Bourbaki N., Topologie générale, ch. II, Paris, 1951.
[2] Kelley J. L., General topology, New York, 1955.
[3] Milionstchikov V. M., *Sur la théorie des équations différentielles $x = f(x, t)$ dans les espaces localement convexes* (en russe). D.A.N. SSSR, **131**, 3, 510–513 (1960).
[4] Neumann J. von, *On complete topological spaces.* Trans. Amer. Math. Soc., 37 1–20 (1935).
[5] Waelbroeck L., Étude spectrale des algèbres complètes, Bruxelles, 1961.
[6] Weil A., Sur les espaces à structure uniforme et sur la topologie générale. Paris, 1937.
[7] Weinberg M. M., Méthodes variationelles dans l'étude des opérations non-linéaires, Moscou, 1956 (en russe).

Reçu le 25. VII. 1966.

MATHEMATICA VOL. 8 (31), 2, 1966, pp. 315 – 333

To Professor TIBERIU POPOVICIU
in honour of the 60^{th} *anniversary of his birthday*

TRANSFORMATIONS OF THE CHEBYSHEV SYSTEMS

by

A. B. NÉMETH
Cluj

0. Introduction and definitions. We denote by $C(S)$ the set of all continuous functions which map S into R the real numbers.

Definition 1. *The functions $\varphi_1, \ldots, \varphi_n$ of the space $C(S)$ form an n-parametric Chebyshev system (or span an n-dimensional Chebyshev or Haar subspace F_n) if every element different of zero of the n-dimensional linear subspace of $C(S)$ spanned by these functions is a function vanishing in at most $n-1$ distinct points of S.*

Definition 2. *The basis $\varphi_1, \ldots, \varphi_n$ of the n-dimensional Chebyshev space F_n is said to be a Markov basis, if for every $0 < k \leq n$, the functions $\varphi_1, \ldots, \varphi_k$ span a k-dimensional Chebyshev space.*

In the present paper we investigate a new kind of iterpolation which is introduced by means of an operator B and a set of Chebyshev spaces as follows:

We consider the sequence of Chebyshev spaces $F_n, F_{n-s}, \ldots, F_{n-\alpha s}$, where s and α are integers for which $s \geq 1$, $0 \leq n - \alpha s < s$, and the operator B having the properties

1. B is linear (additive and homogeneous) on F_{n-is} $i = 0, 1, \ldots, \alpha$.
2. $B(F_{n-is}) = F_{n-(i+1)s}$, $i = 0, 1, \ldots, \alpha$, where $F_{n-(\alpha+1)s} = 0$.
3. If $\varphi \neq 0$, $\varphi \in F_n$ is a function which has at least ks distinct zeros in S, then we have $B^k\varphi \neq 0$ (where by $B^k\varphi$ we denote the result of application of the operator B to the function φ for k times).

We investigate the condition which must be fulfilled by the operator B and by the Chebyshev spaces F_i, $i = n, n-s, \ldots, n-\alpha s$ in order to realise the conditions 1., 2., and 3.

An important case of our interpolation is the case when the Chebyshev spaces F_{n-is}, $i = 1, \ldots, \alpha$ are the subspaces of F_n. The existence of Chebyshev spaces of this kind is always carried aut, when the space F_n has a Markov basis. We shall show that this condition is not fulfilled for every Chebyshev space, not even in the case when S (the domain of definition of elements of this Chebyshev space) is the closed interval $[a, b]$ of the real axis. The property of existence of a Markov basis of a Chebyshev space is in relation to the extensibility of the domain of definition of functions of this Chebyshev space.

In the followings, only case $S = [a, b]$ will be considered.

1. The B-interpolation. Let be F_{n-is}, $i = 0, 1, \ldots, \alpha$ the Chebyshev spaces and B the operator for which condition 1., 2. and 3. of the Introduction are fulfilled. An example of Chebyshev spaces and operator B for which these conditions are valid, are the spaces F_{n-is} spanned by the functions

$$1, x, x^2, \ldots, x^{n-is-1}$$

and the operator $B = D^s$, where D is the differential operator.

The general problem of the B-interpolation may be formulated as follows:

We consider the points x_i, $i = 1, \ldots, n$ in $[a, b]$ the non negative integers k_i, $i = 1, \ldots, n$, and the real numbers y_i, $i = 1, \ldots, n$. To determine the function $\varphi \in F_n$, which has the properties

$$B^{k_i}\varphi(x_i) = y_i, \quad i = 1, \ldots, n. \tag{1}$$

We say that the problem (1) has a solution, if there is a single function $\varphi(x)$ for which the conditions (1) hold.

The B-interpolation problem may be characterized with the pairs (k_i, x_i), $i = 1, \ldots, n$. Really, from the linearity of B follows the

Lemma 1. *The B-interpolation problem* (1) *has solution if and only if the determinant*

$$|B^{k_i}\varphi_j(x_i)|_{i, j = 1, \ldots, n} \tag{2}$$

is different from zero.

Every function $\varphi \in F_n$ has the form

$$\varphi = \sum_{j=1}^{n} c_j\varphi_j \tag{3}$$

where c_j, $j = 1, \ldots, n$ are reals and φ_j, $j = 1, \ldots, n$ form a bazis of

F_n. The interpolation problem (1) has solution if and only if the system of linear equations in c_j, $j = 1, \ldots, n$

$$\sum_{j=1}^{n} c_jB^{k_i}\varphi_j(x_i) = y_i, \quad i = 1, \ldots, n$$

has a single solution, i. e. if the determinant (2) is different from zero.

The problem of the B-interpolation formulated as above has not always solution not even in the case when B is the differential operator, and F_n is the set of polynomials of degree at most $n - 1$. In this case the problem of the B-interpolation is the well known problem of lacunary interpolation which has been introduced by G. D. BIRKHOFF [1]. If the integers k_i, $i = 1, \ldots, n$ are given we shall say that the operatorial orders of the B-interpolation are given.

Definition 3. *The system of the operatorial orders k_i, $i = 1, \ldots, n$ is said to be non degenerate if there could be found a set of points x_i, $i = 1, \ldots, n$ in $[a, b]$ so that the problem* (1) *of the B-interpolation has a solution.*

In this point we shall answer to the question, when the operatorial orders of an B-interpolation problem with the operator B with the properties 1., 2. and 3. may form a non degenerate system. We shall prove the following theorem:

THEOREM 1. *The operatorial orders k_i, $i = 1, \ldots, n$ of a B-interpolation problem with the operator B having the properties* 1., 2. *and* 3., *where $s \geqq 1$, form a non degenerate system if and only if*

$$\prod_{j=0}^{n-1}\left(n - j - p\left\{k_i | k_i > \frac{j}{s}\right\}\right) \neq 0. \tag{4}$$

With $p(A)$ we denote the number of elements of the finite set A.

We prove the necessity of the condition of Theorem 1 applying only the conditions 1. and 2. impose on the operator B. The sufficience of the condition of Theorem 1 may be too proved in more general conditions then the conditions 1., 2. and 3. But these general conditions are not important for us.

Before proving Theorem 1, we prove the following lemma:

Lemma 2. *If the condition* (4) *is fulfilled, then we have*

$$n - j - p\left\{k_i | k_i > \frac{j}{s}\right\} > 0, \quad j = 0, 1, \ldots, n - 1. \tag{5}$$

Proof. We suppose that the Lemma 2 is not true. Then there is at least one j, $0 \leqq j \leqq n - 1$ such that

$$n - j - p\left\{k_i | k_i > \frac{j}{s}\right\} < 0,$$

i. e.

$$n - j < p\left\{k_i | k_i > \frac{j}{s}\right\}.$$

We know that

$$n - p\{k_i | k_i > 0\} \geqq 0.$$

The equality may not be hold, because this would be in contradiction with (4). We show that it is impossible to hold concomitantly the inequalities

$$n - l > p\left\{k_i | k_i > \frac{l}{s}\right\}$$

and

$$n - (l + 1) < p\left\{k_i | k_i > \frac{l+1}{s}\right\}.$$

Really, from these inequalities it follows

$$0 > n - (l+1) - p\left\{k_i | k_i > \frac{l+1}{s}\right\} > p\left\{k_i | k_i = \frac{l+1}{s}\right\} - 1$$

and then

$$-1 \geqq n - (l+1) - p\left\{k_i | k_i > \frac{l+1}{s}\right\} > p\left\{k_i | k_i = \frac{l+1}{s}\right\} - 1$$

and therefore

$$p\left\{k_i | k_i = \frac{l+1}{s}\right\} < 0.$$

But this inequality is in contradiction with the definition of function p. We apply this reasoning for $l = 0, 1, \ldots, j-1$ and conclude that there is at least an integer m, $0 < m < j$ for which we have

$$n - m - p\left\{k_i | k_i > \frac{m}{s}\right\} = 0,$$

which contradicts the condition (4).

Proof of the sufficience of condition of Theorem 1. We suppose that the operatorial orders of the B-interpolation problem are arranged increasingly:

$$k_1 \leqq k_2 \leqq \ldots \leqq k_n.$$

We group the equal operatorial orders in different classes K_l and obtain the classes

(6) $$K_1 < K_2 < \ldots < K_m,$$

where $K_q < K_r$ if for $k_{i_q} \in K_q$ and $k_{i_r} \in K_r$ we have $k_{i_q} < k_{i_r}$.

We introduce the notation

$$\Phi(x_1, \ldots, x_n) = |\varphi_i(x_j)|_{i, j=1, \ldots, n}.$$

The function Φ has the property that if we fix $n-1$ of its variables setting for they distinct values in $[a, b]$, the function Φ becomes, as a function of a single variable, element of the space F_n, having $n-1$ distinct zeros in the fixed values of remaning variables. The proof of the sufficience of Theorem 1, as follows from Lemma 1, may be reduced to the proving that the function

$$B_{x_1}^{k_1} \ldots B_{x_n}^{k_n} \Phi(x_1, \ldots, x_n)$$

is not identical to zero. With $B_{x_i}^{k_i} \Phi(x_1, \ldots, x_n)$ we denote the application of the operator B for k_i times to the function $\Phi(x_1, \ldots, x_n)$ of x_i in which the remaning variables are considered fixed.

We shall prove that this holds by induction with repsect to the sequence (6).

We consider the class $K_m = \{k_i | k_i = k_n\}$. Let be $j_0 = sk_n - 1$. We observe that $j_0 < n - 1$. Really, if it is not true, then we should have $n - 1 \leqq sk_n - 1$, i.e. $\frac{n-1}{s} < k_n$. If the condition (4) holds, then the inequalities (5) are valid. But the obtained inequality is in contradiction with (5) for $j = n-1$. Therefore we have $j_0 < n - 1$.

From the Lemma 2 it follows that

$$n - (sk_n - 1) - p\left\{k_i | k_i > \frac{sk_n - 1}{s}\right\} > 0,$$

i. e.

(7) $$n - (sk_n - 1) > p\{k_i | k_i = k_n\} = \beta_0.$$

We shall prove by induction that the values $x_1, \ldots, x_n$ may be chosen so that we have

(8) $$B_{x_{n-\beta_0+1}}^{k_{n-\beta_0+1}} \ldots B_{x_n}^{k_n} \Phi \neq 0.$$

Let be ν a variable index between 0 and β_0. We set for the variables $x_1, \ldots, x_{n-1}$ distinct values. Then Φ becomes as function of variable x_n element of F_n vanishing at $n-1$ distinct places $x_1, \ldots, x_{n-1}$. By the property 3. of the operator B the function $B_{x_n}^{k_n}\Phi$ of variable x_n is not identical to zero, because in accordance with the inequality (7) it follows that $n - 1 \geqq sk_n$. We fix the variable x_n such that

$$B_{x_n}^{k_n}\Phi \neq 0.$$

For $\nu = 1$ the condition (8) has been proved. If $\beta_0 = 1$ then the condition (8) holds for each operatorial order in K_m. Suppose that $\beta_0 > 1$ and let be the condition (8) fulfilled for $\nu = \sigma - 1$, $1 \leqq \sigma - 1 \leqq \beta_0$, i. e. we suppose that the values $x_1, \ldots, x_n$ may be so chosen that

$$B^{k_n}_{x_{n-\sigma+2}} \ldots B^{k_n}_{x_n}\Phi \neq 0. \tag{9}$$

We consider fixed the points x_i, $i = 1, \ldots, n$, $i \neq n - \sigma + 1$. Then the expression $B^{k_n}_{x_{n-\sigma+2}} \ldots B^{k_n}_{x_n}\Phi$ becomes function of variable $x_{n-\sigma+1}$. Let be x_i, $i = 1, \ldots, n$, $i \neq n - \sigma + 1$, the values for which is valid (9). The function of $x_{n-\sigma+1}$ which we have obtained, accordingly is not identical to zero and is an element of F_n vanishing in $n - \sigma$ distinct points $x_1, \ldots, x_{n-\sigma}$. But $\sigma < \beta_0$, and from (7) it follows that $sk_n < n - \sigma$. From the property 3. of the operator B it follows that applying B k_n times to the function of $x_{n-\sigma+1}$ obtained above, the result will be a function which is not identical to zero, i. e. the point $x_{n-\sigma+1}$ may be so chosen that we have

$$B^{k_n}_{x_{n-\sigma+1}} B^{k_n}_{x_{n-\sigma+2}} \ldots B^{k_n}_{x_n}\Phi \neq 0.$$

It has been proved by induction that the condition (8) may be fulfilled.

Let be now a certain class K_l and an $k_r \in K_l$. We suppose that the points $x_1, \ldots, x_n$ may be chosen so that

$$B^{k_{q+1}}_{x_{q+1}} \ldots B^{k_n}_{x_n}\Phi \neq 0, \tag{10}$$

where $q = \min\{i | k_i > k_r\} - 1$. We introduce the notations $p\{k_i | k_i > k_r\} = \gamma$, $p\{k_i | k_i = k_r\} = \beta_1$. We prove by induction that the points $x_1, \ldots, x_n$ may be chosen so that

$$B^{k_{q-\beta_1+1}}_{x_{q-\beta_1+1}} \ldots B^{k_{q+1}}_{x_{q+1}} \ldots B^{k_n}_{x_n}\Phi \neq 0. \tag{11}$$

Let be ν a variable index $0 \leqq \nu \leqq \beta_1$ and let be $j_1 = sk_r - 1$; we have from the Lemma 2

$$n - (sk_r - 1) - p\left\{k_i | k_i > \frac{sk_r - 1}{s}\right\} > 0,$$

or, applying the notations which have been introduced

$$n - (sk_r - 1) - p\left\{k_i | k_i > \frac{sk_r - 1}{s}\right\} = n - (sk_r - 1) - p\{k_i | k_i \geqq k_r\} =$$
$$= n - (sk_r - 1) - \beta_1 - \gamma > 0,$$

from which

$$n - \beta_1 - \gamma \geqq sk_r. \tag{12}$$

According to the definition of number γ and of index q, this inequality may be written as

$$q - \beta \geqq sk_r. \tag{13}$$

Let be $\nu = 1$. We consider the expression $B^{k_{q+1}}_{x_{q+1}} \ldots B^{k_n}_{x_n}\Phi$ as function of the variable x_q the remaining variables being fixed and equals with those values for which the condition (10) is fulfilled. This function is a not identical to zero element of F_n, vanishing in $q - 1$ distinct points $x_1, \ldots, x_{q-1}$. Applying the operator B k_r times to this function, from property 3. of the operator B and from the inequality $q - 1 \geqq sk_r$ (which follows from (13) because $\beta_1 \geqq 1$) it follows that x_q may be chosen so that

$$B^{k_r}_{x_q} B^{k_{q+1}}_{x_{q+1}} \ldots B^{k_n}_{x_n}\Phi = 0.$$

We suppose that $\beta_1 > 1$, $\nu = \sigma$, $1 \leqq \sigma < \beta_1$, and suppose that the points $x_1, \ldots, x_n$, may be so chosen that

$$B^{k_r}_{x_{q-\sigma+2}} \ldots B^{k_r}_{x_q} B^{k_{q+1}}_{x_{q+1}} \ldots B^{k_n}_{x_n}\Phi \neq 0. \tag{14}$$

Let be now the expression $B^{k_r}_{x_{q-\sigma+2}} \ldots B^{k_r}_{x_q} B^{k_{q+1}}_{x_{q+1}} \ldots B^{k_n}_{x_n}\Phi$ a function of $x_{q-\sigma+1}$, the remaning variables being fixed and equal with those values for which the condition (14) is fulfilled. This function forms a not identical to zero element of F_n vanishing in $q - \sigma$ distinct points $x_1, \ldots, x_{q-\sigma}$. But $\sigma < \beta_1$ and from (13) it follows that $sk_r < q - \sigma$. From the property 3. of the operator B it follows that applying operator B to this function k_r times, the result is a not identical to zero function, and then the value of $x_{n-\sigma+1}$ may be so chosen that

$$B^{k_r}_{x_{q-\sigma+1}} B^{k_r}_{x_{q-\sigma+2}} \ldots B^{k_r}_{x_q} B^{k_{q+1}}_{x_{q+1}} \ldots B^{k_n}_{x_n}\Phi \neq 0.$$

We have proved by induction that the points $x_1, \ldots, x_n$ may be in such a way chosen, to hold the condition (11). Then the necessary condition is proved for the class K_l, which completes our proof by induction with respect to the sequence (6).

Proof cf the necessity of the condition of Theorem 1. We suppose that there is an index l, $0 \leqq l \leqq n - 1$, so that

$$n - l - p\left\{k_i | k_i > \frac{l}{s}\right\} = 0. \tag{15}$$

From this equality it follows that $l + 1 = \min\left\{i | k_i > \frac{l}{s}\right\}$. The function

$$B^{k_1}_{x_1} \ldots B^{k_n}_{x_n}\Phi \tag{16}$$

may be written as

$$B_{x_1}^{k_1} \dots B_{x_l}^{k_l} B_{x_{l+2}}^{k_{l+2}-k_{l+1}} \dots B_{x_n}^{k_n-k_{l+1}} (B_{x_{l+1}}^{k_{l+1}} B_{x_{l+2}}^{k_{l+1}} \dots B_{x_n}^{k_{l+1}} \Phi).$$

Because of operatorial orders arranged increasingly it follows that $k_i - k_{l+1} \geqq 0$ for $i = l+2, \dots, n$. The expression

$$B_{x_{l+1}}^{k_{l+1}} \dots B_{x_n}^{k_{l+1}} \Phi \tag{17}$$

is element of $F_{n-sk_{l+1}}$ for each variable $x_{l+1}, \dots, x_n$. We shall show that this function is identical to zero. Suppose that the variables $x_{l+2}, \dots, x_n$ are distinct. Then (17) as a function of x_{l+1} is a function of $F_{n-sk_{l+1}}$ and if it is not identical to zero it has at most $n - sk_{l+1} - 1$ distinct zeros. But from the inequality $k_{l+1} > \frac{l}{s}$ it follows that

$$n - sk_{l+1} - 1 < n - l - 1$$

and from the form of (17) this function has $n - l - 1$ distinct zeros in $x_{l+2}, \dots, x_n$ and therefore it is identical to zero. If the variables $x_{l+2}, \dots, x_n$ are not all distinct then this function is zero because it is a determinant with two identical rows.

2. The B-transformations of the Chebyshev systems. In this point we investigate the transformations B of the n-dimensional Chebyshev space F_n in the $n-s$-dimensional Chebyshev space F_{n-s} which satisfays the conditions 1., 2. for $i = 0$ and 3. for $k = 1$[1]. The operator B having these properties will be said to form a B-transformation of the Chebyshev space F_n in the Chebyshev space F_{n-s}.

Operators $B : F_n \to F_{n-s}$ having the properties 1. and 2. may be easyly defined as follows:

a) $B\left(\sum_{i=1}^{n} c_i \varphi_i\right) = \sum_{i=1}^{n} c_i B\varphi_i;$

b) $B\varphi_i = \psi_i,\ i = 1, \dots, n - s,$

$B\varphi_i = 0,\ i = n - s + 1, \dots, n,$

where $\varphi_1, \dots, \varphi_n$ is a basis of the space F_n, $\psi_1, \dots, \psi_{n-s}$ is a basis of the space F_{n-s}. Conversly, each operator B with the properties 1. and 2. may be thus defined. The question is, which are the conditions necessary

[1] In followings, where reffering to conditions 1, 2 and 3 we understand alwais these cases.

to be imposed on operator B satisfying the conditions 1. and 2., and on space F_n, so to fulfil condition 3.

For the caracterisation of these conditions we introduce the notion of the L-basis.

D e f i n i t i o n 4. *The basis $\varphi_1, \dots, \varphi_n$ of the n-dimensional Cebyshev space F_n is said to be an L-basis with respect to the points $x_1, \dots, x_n$, if the following conditions hold:*

$$\varphi_i(x_j) = \delta_i^j,\ i, j = 1, \dots, n,$$

where δ_i^j is the symbol of Kronecker.

THEOREM 2. *Any function $\varphi \in F_n$, $\varphi \neq 0$, having at least s distinct zeros will be transformed by the operator B which has the properties 1. and 2. in a non zero element of F_{n-s}, if and only if for every L-basis $\varphi_1, \dots, \varphi_n$ of F_n, every $n - s$ different functions of the set $B\varphi_1, \dots, B\varphi_n$ of the space F_{n-s} are linearly independent.*

Proof of the necessity. Suppose that the operator $B : F_n \to F_{n-s}$ has the properties 1., 2. and 3., and that there is an L-basis $\varphi_1, \dots, \varphi_n$ with respect to a set of distinct points $x_1, \dots, x_n$ with the property that there are $n - s$ different functions of the set $B\varphi_1, \dots, B\varphi_n$, which are linearly dependent. Let be these functions the functions $B\varphi_1, \dots, B\varphi_{n-s}$. Then there are the real numbers a_i, $i = 1, \dots, n - s$ with the property that $\sum_{i=1}^{n-s} |a_i| \neq 0$ and

$$\sum_{i=1}^{n-s} a_i B\varphi_i = 0.$$

The function $\varphi = \sum_{i=1}^{n-s} a_i \varphi_i$ is not zero because of the condition imposed on coefficients a_i, $i = 1, \dots, n - s$. This function vanishes in s distinct points $x_{n-s+1}, \dots, x_n$. Therefore there is a function $\varphi \in F_n$, $\varphi \neq 0$, vanishing in s distinct points, so that $B\varphi = 0$, in contradiction with the property 3. of the operator B.

Proof of the sufficience. Suppose that the operator $B : F_n \to F_{n-s}$ having the properties 1. and 2. transforms each L-basis $\varphi_1, \dots, \varphi_n$ in a set of functions $B\varphi_1, \dots, B\varphi_n$ with the property that every $n - s$ different functions of this set are linearly independent. Let be $\varphi \in F_n$, $\varphi \neq 0$ a function vanishing in s distinct points $x_1, \dots, x_s$. Let be $x_{s+1}, \dots, x_n$ other $n - s$ distinct points, and let be $\varphi_1, \dots, \varphi_n$ the L-basis of the space F_n with respect to the points $x_1, \dots, x_n$. Then φ may be written as

$$\varphi = \sum_{i=s+1}^{n} a_i \varphi_i,$$

where obviously $\sum_{i=s+1}^{n} |a_i| \neq 0$. If $B\varphi$ would be $= 0$, then according to the above representation it would follow that

$$\sum_{i=s+1}^{n} a_i B\varphi_i = 0$$

i.e. the functions $B\varphi_{s+1}, \ldots, B\varphi_n$ are linearly dependent which contradicts our assumption.

THEOREM 3. *For every operator $B : F_n \to F_{n-s}$ having the properties 1. and 2., and for every $s-1$ distinct points $x_1, \ldots, x_{s-1}$ there is a function $\psi \in F_n$, $\psi \neq 0$ which vanishes in the points $x_1, \ldots, x_{s-1}$ and has the property that $B\psi = 0$. If the operator B has also the property 3., then this function is the single except a multiplicative constant.*

Proof. We add to the points $x_1, \ldots, x_{s-1}$ the other distinct points $x_s, \ldots, x_n$. Let be $\varphi_1, \ldots, \varphi_n$ the L-basis with respect to these points. Each function $\varphi \in F_n$ which vanishes in the points $x_1, \ldots, x_{s-1}$ may be written as

$$\varphi = \sum_{i=1}^{n} a_i \varphi_i.$$

Because the functions $B\varphi_s, \ldots, B\varphi_n$ are elements of the $n-s$-dimensional space F_{n-s}, they must be linearly dependent, i. e. there are the numbers c_i, $i = s, \ldots, n$, $\sum_{i=s}^{n} |c_i| \neq 0$, so that

$$\sum_{i=s}^{n} c_i B\varphi_i = 0,$$

but this means that the not identic to zero function $\psi = \sum_{i=s}^{n} c_i \varphi_i$, which vanishes in $s-1$ distinct points $x_1, \ldots, x_{s-1}$, has the property that $B\psi = 0$.

We suppose now that the operator B has also the property 3. If the function $\eta \in F_n$, $\eta \neq 0$ which vanishes in the points $x_1, \ldots, x_{s-1}$, is linearly independent of ψ, and has the property that the operator B vanishes too on it, then if x_0 is a point $x_0 \neq x_i$, $i = 1, \ldots, s-1$ and we have $\eta(x_0) = k\psi(x_0)$ (it is possible because the functions $\eta(x_0) \neq 0$, $\psi(x_0) \neq 0$), it follows that the function $\eta - k\psi \in F_n$ is not identic to zero, has s distinct zeros in $x_1, \ldots, x_{s-1}$, x_0 and has the property that $B(\eta - k\psi) = 0$ in contradiction with the property 3. of the operator B.

THEOREM 4. *The operator $B : F_n \to F_{n-s}$ having the properties 1. and 2. has also the property 3. if and only if the space F_n has an s-dimensional Chebyshev subspace F_s for which $B(F_s) = 0$.*

Proof of the sufficience. Suppose that the operator B having the properties 1. and 2. vanishes on an s-dimensional Chebyshev subspace F_s of the space F_s. We shall show that in this case the operator B has also the property 3. From Theorem 3. it follows that it is sufficient to show that each $n-s$ different functions of the set $B\varphi_1, \ldots, B\varphi_n$ are linearly independent for any L-basis $\varphi_1, \ldots, \varphi_n$ of the space F_n. We suppose the contrary: there is an L-basis $\varphi_1, \ldots, \varphi_n$ of the space F_n with respect to the points $x_1, \ldots, x_n$ so that $n-s$ of the functions $B\varphi_1, \ldots, B\varphi_n$ are linearly dependent. Let be these functions $B\varphi_1, \ldots, B\varphi_{n-s}$ and we suppose that may be written

$$B\varphi_1 = \sum_{i=2}^{n-s} c_i B\varphi_i. \tag{18}$$

The set of functions $B\varphi_2, \ldots, B\varphi_{n-s}$ may be completed with functions of the set $B\varphi_{n-s+1}, \ldots, B\varphi_n$ to a set which contains a basis of the space F_{n-s}. We may suppose that the functions which have non zero coefficients in the representation (18) are linearly independent. We may suppose that the functions $B\varphi_2, \ldots, B\varphi_{n-s}$, $B\varphi_{n-s+1}$ form a basis of the F_{n-s}, and we also suppose that the first $n-s-1$ of these functions are those which appear in the representation (18). Let be the function ψ the not identical to zero function of the space F_s, which vanishes in the points $x_{n-s+2}, \ldots, x_n$. We have the representation

$$\psi = \sum_{i=1}^{n-s+1} a_i \varphi_i.$$

Applying the operator B to this function which according to the hypothesis vanishes on F_s and then vanishes also on ψ, we obtain

$$\sum_{i=1}^{n-s+1} a_i B\varphi_i = 0,$$

from which, applying the reprezentation (18) we obtain

$$(a_2 + a_1 c_2) B\varphi_2 + (a_3 + a_1 c_3) B\varphi_3 + \ldots + (a_{n-s} + a_1 c_{n-s}) B\varphi_{n-s} + \\ + a_{n-s+1} B\varphi_{n-s+1} = 0.$$

But the functions $B\varphi_2, \ldots, B\varphi_{n-s} B\varphi_{n-s+1}$ are linearly independent and then must to be zero all coefficients in the above linear form. It follows that $a_{n-s+1} = 0$, and then

$$\psi = \sum_{i=1}^{n-s} a_i \varphi_i,$$

i.e. the function $\psi \in F_s$ vanishes in s distinct points $x_{n-s+1}, \ldots, x_n$, in contradiction with our hypothesis that $\psi \neq 0$, and F_s is Chebyshev space of dimension s.

Proof of the necessity. From the linearity of the operator B it follows that the set $F_s = \{\varphi | \varphi \in F_n, \ B\varphi = 0\}$ form a linear subspace of F_n. We show that this subspace has the dimension s. Let be $x_1, \ldots, x_s$ s distinct points in $[a, b.]$ From Theorem 3. we may construct the functions η_i with the properties

$$\eta_i(x_j) = \delta_i^j, \ i, j = 1, \ldots, s, \ B\eta_i = 0, \ i = 1, \ldots, s. \tag{19}$$

The functions η_i, $i = 1, \ldots, s$ are obviously linearly independent, and therefore $\dim F_s \geqq s$. If F_s would be of dimension $> s$, then would also follow that $\dim F_{n-s} < n - s$, in contradiction whith the hypothesis, and then $\dim F_s = s$. From property 3. of the operator B it follows that every function of F_s may have at most $s - 1$ distinct zeros and therefore F_s is a Chebyshev space.

Another proof of Theorem 4. We give also a very short proof of Theorem 4. applying a simple result of theory of finite dimensional vector spaces. If we denote by F_s the set of functions of F_n on which the linear operator B vanishes, it follows by the linearity of B, that F_s is a linear subspace of F_n and that $\dim F_n = \dim F_s + \dim F_{n-s}$, where $F_{n-s} = B(F_n)$. If the dimension of F_{n-s} is equal with $n - s$, then it follows that $\dim F_s = s$.

The sufficience. Because F_s is Chebyshev space its every non zero element has at most $s - 1$ distinct zeros. If $\varphi \in F_n$, $\varphi \neq 0$, and has s distinct zeros, then $\varphi \notin F_s$ and then $B\varphi \neq 0$.

The necessity. The dimension of linear subspace F_s of F_n, on which vanishes the operator B is s. According to property 3. of the operator B, it follows that every non-zero element of this subspace, has at most $s-1$ distinct zeros, i.e. it is a Chebyshev subpace.

Observation. In proving theorems 2, 3, 4 it has not been applied the condition that F_{n-s} is a Chebyshev space. It may be nobody $n - s$-dimensional linear space.

Conclusion. From Theorem 4 it follows that for B-interpolation the Chebyshev spaces with Chebyshev subspaces of each dimension are very important. Such Chebyshev spaces are those which have a Markov basis. In what follows we shall show that no every Chebyshev space has thus a basis.

3. The problem of existence of an $n-1$-dimensional Chebyshev subspace of the n-dimensional Chebyshev space. In the previous point we have seen the importance of Chebyshev spaces with a Markov basis of out point of viw of B-interpolation. If we consider the space $C(S)$, where S is a compact Hausdorff space, then if S is the circumference of the circle then $C(S)$ contains only Chebyshev spaces with odd dimension [2], i.e. in this case there is no Chebyshev space with Markov basis of dimension > 1.

In what follows we shall investigate the problem of existence of an $n - 1$-dimensional Chebyshev subspace of the n-dimensional Chebyshev space F_n of the space $C(S)$. The main result of the present point is the

THEOREM 5. *The n-dimensional Chebyshev space F_n of the space $C(S)$ has an $n-1$-dimensional Chebyshev subspace F_{n-1} if and only if the domain of definition S of functions of F_n may be extended to a set $S \cup \{\alpha\}$, where α is a point not in S, so that the property of Chebyshev space of F_n to maintain on the set $S \cup \{\alpha\}$.*

The sufficience of condition of Theorem 5 is well known, we shall prove it here only for completeness.

Before proving our theorem, we present a simple lemma.

L e m m a 3.

$$\left| \sum_{i=1}^{n} a_{ji} b_{ik} \right|_{j, k = 1, \ldots, n-1} = \sum_{i=1}^{n} \begin{vmatrix} a_{11} & \ldots & a_{1, i-1} & a_{1, i+1} & \ldots & a_{1n} \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ a_{n-1, 1} & \ldots & a_{n-1, i-1} & a_{n-1, i+1} & \ldots & a_{n-1, n} \end{vmatrix} \times \tag{19}$$

$$\times \begin{vmatrix} b_{11} & \ldots & b_{1, n-1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ b_{i-1, 1} & \ldots & b_{i-1, n-1} \\ b_{i+1, 1} & \ldots & b_{i+1, n-1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ b_{n1} & \ldots & b_{n, n-1} \end{vmatrix}.$$

This result is well known in linear algebra.

The proof of necessity of Theorem 5. Suppose that the n-dimesional Chebyshev space F_n contains an $n-1$-dimensional Chebyshev subspace F_{n-1}. Let the functions $\psi_1, \ldots \psi_{n-1}$ form a basis of the space F_{n-1}. They may be represented in the form

$$\psi_j = \sum_{i=1}^{n} a_{ji} \varphi_i, \quad j = 1, \ldots, n-1, \tag{20}$$

where the functions $\varphi_1, \ldots, \varphi_n$ form a basis of the space F_n, and a_{ji}, $j = 1, \ldots, n-1$, $i = 1, \ldots, n$ are reals. Because F_{n-1} is a Chebyshev space, it follows that for any $n-1$ distinct points $x_1, \ldots, x_{n-1}$, $x_i \in S$, $i = 1, \ldots, n-1$, we have

$$|\psi_j(x_k)|_{j, k = 1, \ldots, n-1} \neq 0, \tag{21}$$

from which applying the representation (20) we obtain

$$\left| \sum_{i=1}^{n} a_{ji} \varphi_i(x_k) \right|_{j, k = 1, \ldots, n-1} \neq 0.$$

From (19) it follows that

$$\left|\sum_{i=1}^{n} a_{ji}\varphi_i(x_k)\right|_{j,k=1,\ldots,n-1} =$$

$$= \sum_{i=1}^{n} \begin{vmatrix} a_{11} & \ldots a_{1,i-1} & a_{1,i+1} & \ldots a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n-1,1} & \ldots a_{n-1,i-1} & a_{n-1,i+1} & \ldots a_{n-1,n} \end{vmatrix} \cdot \begin{vmatrix} \varphi_1(x_1) & \ldots \varphi_1(x_{n-1}) \\ \vdots & \vdots \\ \varphi_{i-1}(x_1) & \ldots \varphi_{i-1}(x_{n-1}) \\ \varphi_{i+1}(x_1) & \ldots \varphi_{i+1}(x_{n-1}) \\ \vdots & \vdots \\ \varphi_n(x_1) & \ldots \varphi_n(x_{n-1}) \end{vmatrix}.$$

We introduce the notations

$$A_i = (-1)^{i-1} \begin{vmatrix} a_{11} & \ldots a_{1,i-1} & a_{1,i+1} & \ldots a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n-1,1} & \ldots a_{n-1,i-1} & a_{n-1,i+1} & \ldots a_{n-1,n} \end{vmatrix}$$

then the condition (21) may be written as

$$\text{(22)} \qquad \begin{vmatrix} A_1 & \varphi_1(x_1) & \ldots \varphi_1(x_{n-1}) \\ A_2 & \varphi_2(x_1) & \ldots \varphi_2(x_{n-1}) \\ \vdots & \vdots & \vdots \\ A_n & \varphi_n(x_1) & \ldots \varphi_n(x_{n-1}) \end{vmatrix} \neq 0.$$

Let be now α a point not in S. We define the functions φ_i, $i=1, \ldots, n$ on the point α setting

$$\varphi_i(\alpha) = A_i, \quad i = 1, \ldots, n.$$

Since the condition (22) is fulfilled for any distinct points $x_1, \ldots, x_{n-1}$ in S, and since the functions $\varphi_1, \ldots, \varphi_n$ form a Chebyshev system on S, it follows that these functions form a Chebyshev system too on the set $S \cup \{\alpha\}$.

Proof of sufficience of the condition of Theorem 5. We suppose that the functions $\varphi_1, \ldots, \varphi_n$ form a Chebyshev system on the set $S \cup \{\alpha\}$, where α is a point not in the set S. It follows that for any $n-1$ distinct points $x_1, \ldots, x_{n-1}$ of S we have

$$\text{(23)} \qquad \begin{vmatrix} \varphi_1(\alpha) & \varphi_1(x_1) & \ldots \varphi_1(x_{n-1}) \\ \vdots & \vdots & \vdots \\ \varphi_n(\alpha) & \varphi_n(x_1) & \ldots \varphi_n(x_{n-1}) \end{vmatrix} \neq 0.$$

We may supose that $\varphi_1(\alpha) \neq 0$, and then may determine the reals k_j so that

$$\varphi_j(\alpha) - k_j\varphi_1(\alpha) = 0, \; j = 2, \ldots, n.$$

Then the condition (23) may be written as

$$|\varphi_{j+1}(x_k) - k_{j+1}\varphi_1(x_k)|_{j,k=1,\ldots,n-1} \neq 0.$$

But this means that the functions

$$\psi_i = \varphi_{i+1} - k_{i+1}\varphi_1, \; i = 1, \ldots, n-1,$$

form an $n-1$-parameter Chebyshev system on the set S, which complete our proof.

In what follows we shall give an exemple of three-parameter Chebyshev system which is defined on the closed interval of the real axis, and whose domain of definition may be not extended with a point, so that the property of Chebyshev system remains valid. An exemple of three-parameter Chebyshev system, which is defined on a closed interval and may be not extended to an open interval which contains this closed interval has been given by V. I. VOLKOV [3].

We define the functions $\varphi_i(t)$, $i = 1, 2, 3$ as follows

$$\varphi_1(t) = \begin{cases} \sin t \cos t, & 0 \leqq t < \frac{\pi}{2}, \\ \cos t, & \frac{\pi}{2} \leqq t \leqq \pi; \end{cases}$$

$$\varphi_2(t) = \begin{cases} \sin^2 t, & 0 \leqq t < \frac{\pi}{2}, \\ \sin t, & \frac{\pi}{2} \leqq t \leqq \pi; \end{cases}$$

$$\varphi_3(t) = 1, \qquad 0 \leqq t \leqq \pi.$$

Firstly we shall show that the functions φ_i, $i = 1, 2, 3$ form a three-parameter Chebyshev system on the interval $[0, \pi]$. Suppose the contrary: there are three points t_1, t_2, t_3 so that

$$\text{(24)} \qquad |\varphi_i(t_j)|_{i,j=1,2,3} = 0.$$

We define the transformation $\Phi: [0, \pi] \to E_3$ as follows

$$\Phi(t) = (\varphi_1(t), \varphi_2(t), \varphi_3(t)).$$

From (24) it follows that the points

$$\text{(25)} \qquad (\varphi_1(t_j), \varphi_2(t_j), \varphi_3(t_j)), \; j = 1, 2, 3$$

are in a plane H which pass through the origin. The relations

$$x = \varphi_1(t),$$
$$y = \varphi_2(t),$$
$$z = \varphi_3(t),$$

define a curve in the space E_3. This curve is completly in the plane $z = 1$. The plane H, which contains the vectors (25), intersect the plane $z = 1$ in the straight line (Δ), which contains the points (25), i.e. the straight line (Δ) interesect the curve with equations

$$(26)\qquad \begin{aligned} x &= \varphi_1(t), \\ y &= \varphi_2(t) \end{aligned} \qquad 0 \leqq t \leqq \pi$$

in three distinct points. In the polar coordinates the equation of the curve (26) is

$$\rho = \begin{cases} \sin t, & 0 \leqq t < \dfrac{\pi}{2}, \\ 1, & \dfrac{\pi}{2} \leqq t \leqq \pi, \end{cases}$$

i. e. the curve is in the first square a semi-circle with its centre in $x = 0$, $y = \frac{1}{2}$, and with radius $\frac{1}{2}$, in the second square it is the quarter of circle with its centre in origin and with radius 1. As it may be easily observed, every straight line interesect this curve in at most two points in contradiction with our conclusion. It follows that the equality (24) is not true, and therefore the functions $\varphi_1, \varphi_2, \varphi_3$ form a three-parameter Chebyshev system on $[0, \pi]$.

We shall show that the domain of definition of the above defined three-parameter Chebyshev system may be not extended i.e. the functions φ_i, $i = 1, 2, 3$ may be not defined on a point α not in $[0, \pi]$ so that these functions form a three-parameter Chebyshev system on $[0, \pi] \cup \{\alpha\}$. Really, let a, b and c be three real numbers, and let be

$$\varphi_1(\alpha) = a, \qquad \varphi_2(\alpha) = b, \qquad \varphi_3(\alpha) = c.$$

We distinguish two cases:

i) $a = 0$, $b = 0$, or $a \neq 0$, b is of same sign as a, or equal with zero.

ii) $a = 0$, $b \neq 0$, or a and b are of oposite signs.

i) The case $a = 0$, $b = 0$ may be excluded, because in this case the vectors $(\varphi_1(0), \varphi_2(0), \varphi_3(0))$ and (a, b, c) are linearly dependent and then

$\varphi_1, \varphi_2, \varphi_3$ may be not form a Chebyshev system on $[0, \pi] \cup \{\alpha\}$. Suppose that $a > 0$, $b \geqq 0$, if $a < 0$, $b \leqq 0$ we consider instead of a and b the numbers $-a$ and $-b$.

Let be

$$(27)\qquad \begin{aligned} t_1 &= 0, \\ t_2 &= \arccos \frac{a}{\sqrt{a^2 + b^2}}, \\ t_3 &= \alpha. \end{aligned}$$

In this case $0 \leqq t_2 < \frac{\pi}{2}$, and

$$|\varphi_i(t_j)|_{i, j = 1, 2, 3} = \begin{vmatrix} 0 & \sin t_2 \cos t_2 & a \\ 0 & \sin^2 t_2 & b \\ 1 & 1 & c \end{vmatrix} = \begin{vmatrix} \sin t_2 \cos t_2 & a \\ \sin^2 t_2 & b \end{vmatrix} =$$

$$= \sin t_2 \begin{vmatrix} \dfrac{a}{\sqrt{a^2 + b^2}} & a \\ \dfrac{b}{\sqrt{a^2 + b^2}} & b \end{vmatrix} = 0,$$

and therefore in this case the above functions may be not form a Chebyshev system on $[0, \pi] \cup \{\alpha\}$.

ii) Suppose $a \leqq 0$, $b > 0$; if we have $a \geqq 0$, $b < 0$ instead of these numbers we consider $-a$, $-b$. Let be the points t_1, t_2, t_3 defined as in (27). In this case $\frac{\pi}{2} \leqq t_2 \leqq \pi$ and

$$|\varphi_i(t_j)|_{i, j = 1, 2, 3} = \begin{vmatrix} 0 & \cos t_1 & a \\ 0 & \sin t_2 & b \\ 1 & 1 & c \end{vmatrix} = \begin{vmatrix} \cos t_2 & a \\ \sin t_2 & b \end{vmatrix} =$$

$$= \begin{vmatrix} \dfrac{a}{\sqrt{a^2 + b^2}} & a \\ \dfrac{b}{\sqrt{a^2 + b^2}} & b \end{vmatrix} = 0.$$

In conclusion for any values of φ_1, φ_2 and φ_3 in the point α there are two points $t_1, t_2 \in [0, \pi]$ so that the determinant

$$|\varphi_i(t_j)|_{i, j = 1, 2, 3}$$

vanishes for $t_3 = \alpha$. Therefore the domain of definition of the Chebyshev system $\varphi_1, \varphi_2, \varphi_3$ may be not extended.

From this exemple applying the Theorem 5 it follows that the above three-dimensional Chebyshev space has none two-dimensional Chebyshev subspace, i. e. it has not a Markov basis.

4. The densitiy of Chebyshev spaces which have Markov basis. The following lemma may be obtained as a consequence of Theorem 5. We give a constructive proof for it by means of notion of the L-basis.

Lemma 4. *If the functions $\varphi_1, \ldots, \varphi_n$ form an n-parameter Chebyshev system on the set S, and if $x_1, \ldots, x_{n-1}$ are $n-1$ distinct points in S then the functions $\varphi_1 \ldots, \varphi_n$ spans a Chebyshev space on* on $S - \{x_1, \ldots, x_{n-1}\}$, *which has a Markov basis.*

Proof. Let $\psi_1, \ldots, \psi_n$ be an L-basis with respect to the points $x_1, \ldots, x_{n-1}, x_n$ where $x_n \in S - \{x_1 \ldots, x_{n-1}\}$. Then the functions $\psi_2, \ldots, \psi_n$ form an $n-1$ parameter Chebyshev system on $S - \{x_1\}$, the functions $\psi_3, \ldots, \psi_n$ form an $n-2$ parameter Chebyshev system on $S - \{x_1, x_2\}$ and so on, the function ψ_n form an one parameter Chebyshev system on $S - \{x_1, \ldots, x_{n-1}\}$. It follows that the functions $\psi_1, \ldots, \psi_n$ form a Markov basis of the n-dimensional Chebyshev space spanned by functions $\varphi_1, \ldots, \varphi_n$ defined on $S - \{x_1, \ldots, x_{n-1}\}$. We show that the functions $\psi_i \ldots, \psi_n$ form an $n - i + 1$ parameter Chebyshev system on $S - \{x_1, \ldots, x_{n-1}\}$. Really, if it would exist a linear combination of these functions, having $n - i + 1$ distinct zeros on $S - \{x_1, \ldots, x_{i-1}\}$ then this linear combination would have n distinct zeros in S in contradiction with the fact that the functions $\psi_1, \ldots, \psi_n$ form an n-parameter Chebyshev system on S.

THEOREM 6. *For every n-parameter Chebyshev system of continuous functions $\varphi_1(x), \ldots, \varphi_n(x)$ defined on the closed interval $[a, b]$, and for every positive ε there exists an n parameter Chebyshev system $\psi_1(x), \ldots, \psi_n(x)$ which has a Markov basis and for which*

$$|\varphi_i(x) - \psi_i(x)| < \varepsilon, \quad i = 1, \ldots, n.$$

Proof. Because the functions $\varphi_1, \ldots, \varphi_n$ are uniform continuous on the interval $[a, b]$ there is a positive δ so that

$$|\varphi_i(x') - \varphi_i(x'')| < \varepsilon, \qquad \text{if } |x' - x''| < \delta.$$

Let be a'in (a, b) and so that $a' - a < \delta$. The set $[a, a')$, contains $n-1$ distinct points and therefore it follows from Lemma 4 that the functions $\varphi_1, \ldots, \varphi_n$ form a Chebyshev system on $[a, b]$, which has a Markov basis. By the transformation

$$x = \frac{b - a'}{b - a} y + \frac{a' - a}{b - a} b$$

the Chebyshev system $\varphi_1, \ldots, \varphi_n$ defined on $[a', b]$ will be turned in the Chebyshev system

$$\psi_i(y) = \varphi_i\left(\frac{b - a'}{b - a} y + \frac{a' - a}{b - a} b\right), \quad i = 1, \ldots, n$$

which is defined on $[a, b]$ and has a Markov basis (Lemma 4). We replace y in x and show that

$$|\psi_i(x) - \varphi_i(x)| < \varepsilon, \quad i = 1, \ldots, n.$$

Really, let be $x \in [a, b]$. Then we have

$$|\psi_i(x) - \varphi_i(x)| = \left|\varphi_i\left(\frac{b - a'}{b - a} x + \frac{a - a'}{b - a} b\right) + \varphi_i(x)\right| < \varepsilon, \; i = 1, \ldots, n$$

because

$$\left|\frac{b - a'}{b - a} x + \frac{a' - a}{b - a} b - x\right| < \delta$$

which complete the proof of Theorem 6.

REFERENCES

[1] Birkhoff G. D., *General mean value and remainder theorems with applications to mechanical differentiation and quadrature.* Trans. Amer. Math. Soc., **7**, 107—136 (1906)

[2] Sieklucki K., *Topological properties of sets admitting the Tschebycheff systems.* Bull. de l'Acad. Polonaise des sciences, Serie des sci. math. astr. et phys. **6**, 603—606 (1958).

[3] Волков В. И., *Некоторые свойства систем Чебышева.* Учен. зап. Калининского гос. пед. института им. М. И. Калинина, **26**, 41—48 (1958).

Received 21. IX. 1966.