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Dedicated to professor
TIBERIU POPOVICIU
on his 60th birthday

ON THE CONVERGENCE OF SOME SEQUENCES OF MEANS

by

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1. Let

(1) $$x_1^0,\ x_2^0,\ \ldots\ x_m^0$$

be a system of real numbers, and

(2) $$s_1,\ s_2,\ s_3,\ \ldots$$

a sequence, the terms of which are elements of the set of indices $M = \{1, 2, \ldots, m\}$. Consider the sequence

(3) $$a_0,\ a_1,\ a_2,\ \ldots$$

the terms of which are constructed as follows:

(4) $$a_k = \frac{1}{m}\sum_{i=1}^{m} x_i^k, \quad k = 0, 1, 2, \ldots$$

where

(5) $$x_i^k = \begin{cases} a^{k-1} & \text{for} \quad i = s_k \\ x_i^{k-1} & \text{for} \quad i \neq s_k \end{cases}$$

THEOREM 1. *Whichever should be the sequence* (2), *the sequence* (3) *is convergent.*

Before demonstrate the theorem we point out a simple property of the means a_k. Note

$$X_k = \min_i x_i^k, \quad X^k = \max_i x_i^k. \tag{6}$$

We determine the low limites of the differences $a_k - X_k$ and $X^k - a_k$. We have

$$a_k - X_k = \frac{1}{m}\sum_{i=1}^{m} x_i^k - X_k = \frac{1}{m}\sum_{i=1}^{m}(x_i^k - X_k)$$

and according to (6) we obtain

$$a_k - X_k \geqslant \frac{1}{m}(X^k - X_k)$$

and similarly

$$X^k - a_k \geqslant \frac{1}{m}(X^k - X_k). \tag{7}$$

Proof of theorem 1. We note with M_1 that subset of M the elements of which appear for infinit times in the sequence (2). Particularly M_1 may coincide with M. Without restricting the generality we assume from the beginning that in the sequence (2) there are only elements of M_1 because the elimination of a finite number of terms from the sequence (3) does not affect its convergence. A *cycle* of the sequence (2) is any section of this sequence which contains all the elements of M_1, the last term of the section being contained only once. Whichever should be a term of the (2), there exists a *cycle* which begins with it.

According to the assumption above the points of the system (1) whose low indices belong to the set $M - M_1$, will not be substituted with elements of the sequence (3). Each of these points may be substituted in (1) with their arithmetical mean ξ without the sequence (3) being affected.

The inclusions

$$[X_0, X^0] \supseteq [X_1, X^1] \supseteq \ldots \supseteq [X_n, X^n] \supseteq \ldots \tag{8}$$

result from (4), (5), (6) and (7).

The limit of the sequence (8) is a point. In order to prove this assertion we assume the contrary, that is, the sequence (8) has the interval $[X_*, X^*]$ as limit; the length of this interval differing from zero. This means that for any $\varepsilon > 0$ there exists an N such as for any $n > N$, we have $X_* - X_n < \varepsilon$ and $X^n - X^* < \varepsilon$. In the same time for any n and p we have

$$X_{n+p} < X_* \text{ and } X^* < X^{n+p} \tag{9}$$

According to the inequalities (7), a_n and the others that follow belong to the interval $[Y_n, Y^n]$ where

$$Y_n = X_n + \frac{1}{m}(X^n - X_n)$$

$$Y^n = X^n - \frac{1}{m}(X^n - X_n).$$

For $n > N$ we have

$$Y_n > X_* + \frac{1}{m}(X^* - X_*) - \frac{m-1}{m}\varepsilon$$

$$Y^n < X^* - \frac{1}{m}(X^* - X_*) + \frac{m-1}{m}\varepsilon$$

and for a sufficiently small ε, the interval $[Y_n, Y^n]$ is inside the interval $[X_*, X^*]$. The sequence (2) contains a cycle which begins with s_n. Let this be $s_n, s_{n+1}, \ldots, s_{n+p}$. All the points of the system $\{x_i^n\}$, $i = 1, 2, \ldots, m$, except that which is equal to ξ, will be substituted with elements of the section $a_n, \ldots a_{n+p}$, belonging to the interval $[Y_n, Y^n]$. This contradicts the inequalities (9) hence the limit of the sequence (8) cannot be but a point. As $a_n \in [X_n, X^n]$ results that the sequence (3) is convergent.

2. Let f be a continuous and strict monotonous function defined on an interval which contains the points (1) and let F be its inverse. We consider the sequence

$$b_0, b_1, b_2, \ldots \tag{10}$$

constructed as follows:

$$b_k = F\left[\frac{1}{m}\sum_{i=1}^{m} f(x_i^k)\right], \quad k = 0, 1, 2, \ldots$$

where

$$x_i^k = \begin{cases} b_{k-1} & \text{for } i = s_k \\ x_i^{k-1} & \text{for } i \neq s_k \end{cases} \qquad k = 1, 2, \ldots$$

THEOREM 2. *Whichever should be the sequence* (2), *the sequence* (10) *is convergent.*

Proof: We apply the theorem 1 to the following system of numbers

$$f(x_1^0),\ f(x_2^0),\ \ldots,\ f(x_m^0)$$

The sequence (3) will be replaced by

$$f(b_0),\ f(b_1),\ f(b_2),\ \ldots \tag{11}$$

constructed as follows:

$$f(b_k) = \frac{1}{m}\sum_{i=1}^{m} f(x_i^k), \quad k = 0, 1, 2, \ldots$$

Where

$$f(x_i^k) = \begin{cases} f(b_{k-1}) & \text{for } i = s_k \\ f(x_i^{k-1}) & \text{for } i \neq s_k \end{cases} \quad k = 1, 2, 3, \ldots$$

Hence, the sequence (11) is convergent. The convergence of sequence (10) results now from the restrictions imposed to the function f.

The theorem 1 and 2 admit more generalizations. They are valid if we consider instead of the system (1) of numbers, a system of n-dimensional, m vectors or m real functions, defined and bounded on an I interval, making the corresponding modification in (3), (4) and (5).

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Hommage à M. TIBERIU POPOVICIU
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PRODUIT CONTINU DANS LES ALGÈBRES DE BANACH

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On sait que la résolution des systèmes différentiels linéaires dans certains espaces normés fait intervenir systématiquement des fonctions qui généralisent l'exponentielle. C'est ainsi que dans E_n la solution générale de l'équation

$$\frac{dX}{dt} = u(t)X + v(t)$$

s'exprime à l'aide du matrizant $\Delta(u, v)$. [1]

Dans ce mémoire nous développons, à la suite de Mac Nerney [2] une notion plus générale applicable dans les espaces de Banach usuels : les produits continus. Nous indiquerons les propriétés fondamentales de ces produits continus et leurs applications dans la résolution de certains systèmes différentiels des espaces de Banach notamment les systèmes différentiels avec des conditions d'interface.

I.

Dans ce qui suit on désigne par Δ un intervalle $[a, b]$ de la droite réelle ; par $\mathscr{X}$ un expace de Banach quelconque ; par $\mathscr{A}$ une algèbre de Banach à unité e, telle que $\|e\| = 1$. (Il pourra s'agir dans les applications de l'algèbre des endomorphismes de $\mathscr{X}$).

Une subdivision σ de Δ sera définie par la donnée d'une suite finie de points de Δ : $\sigma = \{a = t_0 < t_1 \ldots < t_n < t_{n+1} = b\}$; on notera $\delta_i = [t_i, t_{i+1}]$.

On appellera module de la subdivision le nombre :

$$|\sigma| = \max\ |t_{i+1} - t_i|$$