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TWO GENERALIZATIONS OF THE MEYER-KÖNIG AND ZELLER OPERATOR

by

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1. Introduction

Various generalizations of the Bernstein operator, defined on $C[0,1]$ by the relation

$$(B_n f)(x) = \sum_{\nu=0}^{n} b_{n\nu}(x) f\left(\frac{\nu}{n}\right)$$

where

$$b_{n\nu}(x) = \binom{n}{\nu} x^{\nu}(1-x)^{n-\nu}, \quad \nu = 0, 1, \ldots, n,$$

have recently been found.

It is well known the fact that both the B_n operator and its other generalizations are built by starting from the identity

$$1 = \sum_{\nu=0}^{n} b_{n\nu}(x); \tag{1}$$

In recent times, E. W. CHENEY and A. SHARMA [1] have obtained a Bernstein — type operator begining with a generalization of the identity (1).

The purpose of this paper is to generalize, in the same way, the Meyer-König and Zeller operator [4–5], which is a Bernstein power series defined on $C[0, 1]$, as

$$(M_n f)(x) = \sum_{\nu=0}^{\infty} m_{n\nu}(x) f\left(\frac{\nu}{n+\nu}\right)$$

where

$$m_{n\nu}(x) = (1-x)^{n+1}\binom{n+\nu}{\nu}x^{\nu}, \quad \nu = 0, 1, \ldots,$$

this operator having its origin in the identity

(2) $$1 = \sum_{\nu=0}^{\infty} m_{n\nu}(x), \quad |x| < 1.$$

For non-negative α let us note

$$l_{n\nu}(x, \alpha) = \frac{n+1}{n+\alpha\nu+1}\binom{n+\alpha\nu+\nu}{\nu}x^{\nu}(1-x)^{n+\alpha\nu+1}$$

$$g_{n\nu}(x, \alpha) = \binom{n+\alpha\nu+\alpha+\nu}{\nu}[1-(\alpha+1)x]\,x^{\nu}(1-x)^{n+\alpha\nu+\alpha}$$

$$n = 1, 2, \ldots, \quad \nu = 0, 1, \ldots.$$

We shall consider further down the following generalizations of the identity (2) (see [6])

(3) $$1 = \sum_{\nu=0}^{\infty} l_{n\nu}(x, \alpha)$$

$$, \quad |x| < \frac{1}{\alpha+1}.$$

(3′) $$1 = \sum_{\nu=0}^{\infty} g_{n\nu}(x, \alpha)$$

Starting from these relations and by conveniently choosing the interpolation knots, we shall define on $C\left[0, \frac{1}{\alpha+1}\right]$ the following linear and positive operators

(4) $$(L_n f)(x, \alpha) = \sum_{\nu=0}^{\infty} l_{n\nu}(x, \alpha)\, f\left(\frac{\nu}{n+\alpha\nu+\nu}\right)$$

(4′) $$(G_n f)(x, \alpha) = \sum_{\nu=0}^{\infty} g_{n\nu}(x, \alpha)\, f\left(\frac{\nu}{n+\alpha\nu+\alpha+\nu}\right).$$

It is clear that $(L_n f)(x, 0) = (G_n f)(x, 0) = (M_n f)(x)$.

We recall the following result

THEOREM A. *If $K_n : C[a^*, b^*] \to C[a, b]$, $n = 1, 2, \ldots,$ is a sequence of linear positive operators so that*

$$\lim_{n\to\infty} \| K_n e_j - e_j \| = 0, \quad e_j(x) = x^j, \quad j = 0, 1, 2,$$

then

$$\lim_{n\to\infty} \| K_n f - f \| = 0,$$

whatever $f \in C[a^*, b^*]$ is, ($\|\cdot\|$ is the uniform norm).

In a particular case this result was established for the first time by Prof. T. POPOVICIU [8] and in a general framework, by H. BOHMAN [2] and P. P. KOROVKIN [3]. Recently, A. JAKIMOVSKI and M. S. RAMANUJAN [2] have brought essential contributions.

In this paper, by using the above result, the convergency of the $\{L_n f\}_{n=1}^{\infty}$ and $\{G_n f\}_{n=1}^{\infty}$ sequences to the function f is proved.

In the same time the approximation degrees furnished by the above mentioned operators are studied, and in conclusion it is shown that L_n and G_n are variation diminishing operators according to I. J. SCHOENBERG [9].

2. Convergence properties

THEOREM 1. *If $L_n : C\left[0, \frac{1}{\alpha+1}\right] \to C[0, \lambda_\alpha]$, $\lambda_\alpha < \frac{1}{\alpha+1}$, $n = 1, 2, \ldots,$ is the operators sequence given by (4), then*

$$\lim_{n\to\infty} \| L_n f - f \| = 0,$$

for every $f \in C\left[0, \frac{1}{\alpha+1}\right]$.

Proof. According to the theorem A it is sufficient to show that the following three relations holds

$$\lim_{n\to\infty} \| L_n e_j - e_j \| = 0, \quad e_0(x) = 1, \quad e_1(x) = x, \quad e_2(x) = x^2.$$

From (3) and (4) it is easy to obtain the relations

(5) $$(L_n e_0)(x, \alpha) = e_0(x)$$

(6) $$(L_n e_1)(x, \alpha) = \sum_{\nu=0}^{\infty} l_{n\nu}(x, \alpha)\frac{\nu}{n+\alpha\nu+\nu} =$$

$$= \frac{n+1}{n+\alpha+1} e_1(x) \sum_{\nu=0}^{\infty} l_{n+\alpha,\nu}(x, \alpha) =$$

$$= \frac{n+1}{n+\alpha+1} e_1(x).$$

Further

$$(L_n e_2)(x, \alpha) = \sum_{\nu=0}^{\infty} l_{n\nu}(x, \alpha) \frac{\nu^2}{(n + \alpha\nu + \nu)^2} <$$

$$< \frac{n+1}{n+2\alpha+1} e_2(x) \sum_{\nu=0}^{\infty} l_{n+2\alpha,\nu}(x, \alpha) \frac{\nu+2}{\nu+1} = \frac{n+1}{n+2\alpha+1} e_2(x) + O_n(x, \alpha),$$

where

$$O_n(x, \alpha) = \frac{(n+1)\, x(1-x)^{n+\alpha+1}}{(n+\alpha+1)^2} + \frac{(n+1)\, x(1-x)^{1-\alpha}}{(n+2\alpha)(n+2\alpha+1)}.$$

Similarly, by obtaining an inferior delimitation we can write

$$\left[\frac{n+1}{n+2\alpha+1} e_2(x) + O_n(x, \alpha)\right]\left(1 - \frac{1}{n}\right) < (L_n e_2)(x, \alpha) < \frac{n+1}{n+2\alpha+1} e_2(x) + O_n(x, \alpha). \tag{7}$$

As $\{O_n\}_{n=1}^{\infty}$ converges uniformly to zero, from (5) – (7) and by using theorem A, the proof is complete.

THEOREM 2. *If* $G_n : C\left[0, \frac{1}{\alpha+1}\right] \to C[0, \mu_\alpha]$, $\mu_\alpha < \frac{1}{\alpha+1}$, $n = 1, 2, \ldots$, *is the operators sequence given by* (4'), *then*

$$\lim_{n\to\infty} \| G_n f - f \| = 0,$$

for every $f \in C\left[0, \frac{1}{\alpha+1}\right]$.

Proof. Proceeding as before, we find

$$(G_n e_0)(x, \alpha) = e_0(x) \tag{8}$$

$$(G_n e_1)(x, \alpha) = e_1(x). \tag{9}$$

Now

$$(G_n e_2)(x, \alpha) = \frac{g_{n1}(x, \alpha)}{(n+2\alpha+1)^2} + e_2(x) \sum_{\nu=0}^{\infty} g_{n+2\alpha,\nu}(x, \alpha)\, C_{n\alpha\nu}$$

where

$$C_{n\alpha\nu} = \left(1 + \frac{1}{\nu+1}\right)\left(1 - \frac{1}{n+\alpha\nu+3\alpha+\nu+2}\right).$$

As

$$\left(1 + \frac{1}{\nu+1}\right)\left(1 - \frac{1}{n}\right) < C_{n\alpha\nu} < 1 + \frac{1}{\nu+1}$$

we conclude

$$[e_2(x) + Q_n(x, \alpha)]\left(1 - \frac{1}{n}\right) < (G_n e_2)(x, \alpha) < e_2(x) + Q_n(x, \alpha), \tag{10}$$

and

$$Q_n(x, \alpha) = \frac{g_{n1}(x, \alpha)}{(n+2\alpha+1)^2} + \frac{x[1-(\alpha+1)x]}{n+2\alpha}.$$

Consequently, by means the fact that $\{Q_n\}_{n=1}^{\infty}$ converges uniformly to zero, from (8) – (10) it follows that $\|G_n e_j - e_j\| \to 0$ for $n \to \infty$, $j = 0, 1, 2,$ and again the theorem A completes the proof.

3. Approximation degree

THEOREM 3. *If* $f \in C\left[0, \frac{1}{\alpha+1}\right]$ *and* $L_n : C\left[0, \frac{1}{\alpha+1}\right] \to C[0, \lambda_\alpha]$, $\lambda_\alpha < \frac{1}{\alpha+1}$, *is the operator defined by* (4), *then*

$$\| L_n f - f \| < \frac{3}{2}\, \omega_f(N_\alpha^{-1/2})$$

where $N_\alpha = n + \alpha + 1$, *and* ω_f *is the modulus of continuity attached to the function* f.

Proof. Let $K_n : C[a^*, b^*] \to C[a, b]$ be a linear and positive operator so that $K_n e_0 = e_0$ and let $\delta > 0$. In this case it is known the fact that the inequality

$$\|K_n f - f\| < \omega(\delta)\left[1 + \frac{1}{\delta^2}\|K_n e\|\right] \tag{11}$$

takes place, where we put $e(t) = (t - x)^2$.
Taking into account the relations (5)–(7) we are led to the delimitation

$$(L_n e)(x, \alpha) = (L_n e_2)(x, \alpha) - 2x(L_n e_1)(x, \alpha) + x^2(L_n e_0)(x, \alpha) <$$

$$< \left[\frac{n+1}{n+2\alpha+1} - \frac{2(n+1)}{n+\alpha+1} + 1\right] e_2(x) + O_n(x, \alpha)$$

or

$$(L_n e)(x, \alpha) < \frac{1}{n+\alpha+1}\left[x(1-x)^{n+\alpha+1} + x(1-x)^{1-\alpha} + \alpha x^2\right] \leqslant \frac{1}{2(n+\alpha+1)}$$

for $x \in [0, \lambda_\alpha]$, $\lambda_\alpha < \frac{1}{\alpha+1}$, and in this way

$$\|L_n e\| \leqslant \frac{1}{2(n+\alpha+1)} = \frac{1}{2N_\alpha}. \tag{12}$$

Now, for $\delta = N_\alpha^{-1/2}$ and from (11) combined with (12) we have the desired inequality.

THEOREM 4. *If* $f \in C\left[0, \frac{1}{\alpha+1}\right]$ *and* $G_n : C\left[0, \frac{1}{\alpha+1}\right] \to C[0, \mu_\alpha]$, $\mu_\alpha < \frac{1}{\alpha+1}$, *is the operator defined by* (4'), *then*

$$\|G_n f - f\| < \frac{3}{2}\,\omega_f(M_\alpha^{-1/2})$$

where $M_\alpha = n + 2\alpha$, *and* ω_f *is the modulus of continuity attached to the function* f.

Proof. Similar to the operator L_n, from (8) — (10) we find

$$(G_n e)(x, \alpha) < \frac{x(1-x)[1+(1-x)^{n+2\alpha}]}{n+2\alpha}.$$

Therefore

$$\|G_n e\| < \frac{1}{2M_\alpha}$$

and by substituting δ in (11) by $M_\alpha^{-1/2}$ the theorem is proved.

Remark 1. The theorems 3 and 4 are analogous to that obtained by T. POPOVICIU [7] in the case of the B_n operator and to one obtained by M. MÜLLER [5] when $\alpha = 0$, that is for the M_n operator.

Remark. 2. The approximation degree for the L_n operator respectively for the G_n one can not be improved. More precisely, $N_\alpha^{-1/2}$ respectively $M_\alpha^{-1/2}$ can not be substituted in the above inequalities by other function decreasing more rapidly to zero.

We shall prove this statement for only the L_n operator. For G_n the proof is similar. We consider the probability density as being defined on $\left[0, \frac{1}{\alpha+1}\right]$ by

$$p_n(\nu) = p_n(\nu, x, \alpha) = l_{n,\nu-n-1}(x, \alpha).$$

In this case the mean value and the variance corresponding to this probability density are given by the relations

$$q_n = q_n(x, \alpha) = \sum_{\nu=n+1}^{\infty} \nu l_{n,\nu-n-1}(x, \alpha) = \frac{(n+1)(1-\alpha x)}{1-(\alpha+1)x}$$

and

$$v_n^2 = v_n^2(x, \alpha) = \frac{(n+1)\,x(1-x)}{[1-(\alpha+1)\,x]^3}.$$

By using the Chebischev inequality, it is easy to show that

$$\sum_{|t-x|\geqslant N_\alpha^{-1/2}} (L_n e_0)(x, \alpha) \cong C, \quad C \in (0,1]. \tag{13}$$

Let us consider the function h defined on $\left[0, \frac{1}{\alpha+1}\right]$ as

$$h(t) = |t - x_0|^\beta, \quad \beta \in (0,1], \quad x_0 \in \left(0, \frac{1}{\alpha+1}\right).$$

Since $h \in \mathrm{Lip}\,\beta$ it follows that $\omega_h(\delta) \leqslant \delta^\beta$.

On the other hand, taking into account theorem 3 and also (13), for this function the

$$CN_\alpha^{-\beta/2} \leqslant |h(x_0) - (L_n h)(x_0, \alpha)| < \frac{3}{2} N_\alpha^{-\beta/2}$$

inequalities are found, and in this way our affirmation results.

4. Variation diminishing properties

Let $V_D[g]$ be the variation of the function g defined as the number of the function sign changes when x varies along its domain D. Let $\mathcal{F}$ be a set of real functions defined on a subset D of the real axis. We consider an operator K transforming any $f \in \mathcal{F}$ in a Kf function defined on a subset Ω of the real axis. According to I. J. Schoenberg [9] we say that the K operator is a *variation diminishing operator* iff

$$V_\Omega[Kf] \leqslant V_D[f] \text{ for each } f \in \mathcal{F}.$$

THEOREM 5. *The operator* L_n, *defined by* (4), *is a variation diminishing operator, in other words*

$$V_{[0,\lambda_\alpha]}[L_n f] \leqslant V_{\left[0, \frac{1}{\alpha+1}\right]}[f].$$

Proof. Let us note

$$\theta_{n\nu}(x, \alpha) = (1 - x)^{n+1} \frac{n+1}{n + \alpha\nu + 1} \binom{n + \alpha\nu + \nu}{\nu}$$

and by

$$V[\{a_k\},\ k = 1, 2, \ldots]$$

we shall mean the number of sign changes of the sequence $a_1, a_2, \ldots$. Because

$$\theta_{n\nu}(x, \alpha) > 0,\ x \in \left[0, \frac{1}{\alpha + 1}\right]$$

making use of a known result we can write

$$V_{[0, \lambda_\alpha]}[L_n f] \leqslant V\left[\left\{f\left(\frac{\nu}{n + \alpha\nu + \nu}\right)\right\},\ \nu = 0, 1, \ldots\right] \leqslant V_{\left[0, \frac{1}{\alpha+1}\right]}[f]$$

and this proves the affirmation of the theorem.

Similarly we can establish

THEOREM 6. *The operator G_n, defined by (4′), is a variation diminishing operator.*

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À M. le Prof. TIBERIU POPOVICIU
pour son 60-ème anniversaire

SOLUTIONS PRESQUE-PÉRIODIQUES DE CERTAINES ÉQUATIONS PARABOLIQUES

par

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à Jassy

Dans cette Note, nous allons considérer l'équation parabolique non-linéaire

(1) $$u_{xx} = u_t + f(x, t, u),$$

où $f(x, t, u)$ est une fonction continue dans l'ensemble

(Δ_1) $$-\infty < x < +\infty,\ 0 \leqslant t \leqslant T,\ |u| \leqslant A,$$

y satisfaisant à la condition de Lipschitz par rapport à u

(2) $$|f(x, t, u) - f(x, t, v)| \leqslant L|u - v|.$$

On suppose que T, A et L sont des nombres positifs.

Soit $u(x, t)$ une solution de l'équation (1), définie dans

(Δ) $$-\infty < x < +\infty,\ 0 \leqslant t \leqslant T,$$

satisfaisant à la condition initiale

(3) $$u(x, 0) = u_0(x), \quad -\infty < x < +\infty.$$

Le problème suivant se pose d'une façon naturelle : en admettant que $f(x, t, u)$ et $u_0(x)$ sont des fonctions presque-périodiques, peut-on affirmer que la solution $u(x, t)$ est aussi presque-périodique ?

Il s'agit, évidemment, de la presque-périodicité par rapport à x, c'est-à-dire, par rapport à la variable spatiale. Nous soulignons cette particularité du problème envisagé, parce que les résultats connus sur la presque-