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SOME PROPERTIES OF THE LINEAR POSITIVE OPERATORS (I)*

by

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1. In this paper we give new properties of the Szász-Mirakyan operator and of the operator introduced by Meyer-König and Zeller. Thus we establish some shape-preserving properties and we derive an other form of the Szász-Mirakyan operator.

2. For the formulation of a number of theorems concerning operators werecall the following definitions.

Definition 1. A real function f is called *convex, non-concave, polynomial, non-convex* respectively *concave*, of *k-order*, on $[a, b]$, iff

$$[x_1, x_2, \ldots, x_{k+2}; f] > 0, \geqslant 0, = 0, \leqslant 0 \text{ resp. } < 0,$$

for any system of $k + 2$ knots from $[a, b]$.

In the above definition by $[x_1, x_2, \ldots, x_{k+2}; f]$ we mean the $k+1$- *order divided difference* of the function f on the *knots* $x_1, x_2, \ldots, x_{k+2}$.

Let $\mathcal{F}$ denote the set of all real functions f, defined on an interval $[a, b]$ of the real axis. Let L be an operator which maps each $f \in \mathcal{F}$ into a subset $\mathcal{F}_1 \subseteq \mathcal{F}$ of real functions Lf which are defined on an interval $[\alpha, \beta]$ of the real axis.

Definition 2. An operator $L : \mathcal{F} \to \mathcal{F}_1$ *preserves the convexity of k-order* (on the interval $[\alpha, \beta]$) if for any convex function f of k-order (on the interval $[a, b]$) the function Lf is also convex of k-order (on the interval $[\alpha, \beta]$).

* This work was communicated at the I.C.M. — 1966 — Moskow.

If in this definition we shall replace respectively the term of convex function by non-concave function, polynomial function, non-convex function, concave function of k-order, then we say that the operator L preserves the non-concavity, the polynomiality, the non-convexity respectively the concavity of k-order.

We specify that from now on the word operator means linear positive operator in the usually sense from numerical analysis.

3. The set of functions which are defined and bounded on $[0, +\infty)$ continuous on the interval $[0, a]$, $a > 0$, and continuous to the right on $x = a$, is denoted by $Q[0, a]$.

Let S_n the Szász-Mirakyan operator be defined by

$$(S_n f)(x) = e^{-nx}\sum_{\nu=0}^{\infty}\frac{(nx)^{\nu}}{\nu!}f\left(\frac{\nu}{n}\right). \tag{1}$$

It is known [9] that if $f \in Q[0, a]$, $a > 0$, and if $\|f\| = \max\limits_{x\in[0,a]}|f(x)|$, then $\|S_n f - f\| \to 0$. In the same time in [1] and [3] the monotony of the sequence of operators S_n on the class of the convex functions was studied and the „simple form" of the remainder is given.

We also investigate the Meyer-König and Zeller operator which is defined on $C[0, 1]$ by the relation

$$(M_n f)(x) = (1-x)^{n+1}\sum_{\nu=0}^{\infty}\binom{\nu+n}{\nu}f\left(\frac{\nu}{\nu+n}\right)x^{\nu}. \tag{2}$$

With regard to this operator we notice that $\|M_n f - f\| \to 0$, where $\|f\| = \max\limits_{x\in[0,a]}|f(x)|$, $a < 1$, (see [4]).

THEOREM 1. *If for a function $f \in Q[0, a]$, of real t variable, we denote $D_k^f(t) = [t, t+1/n, \ldots, t+k/n; f]$, then the k-derivative of $S_n f$ has the following form*

$$(S_n f)^{(k)} = k!\, S_n D_k^f. \tag{3}$$

Proof. Differentiating the relation (1), we have

$$(S_n f)'(x) = ne^{-nx}\sum_{\nu=0}^{\infty}\frac{(nx)^{\nu}}{\nu!}\left[f\left(\frac{\nu}{n}+\frac{1}{n}\right) - f\left(\frac{\nu}{n}\right)\right] =$$

$$= e^{-nx}\sum_{\nu=0}^{\infty}\frac{(nx)^{\nu}}{\nu!}\left[\frac{\nu}{n}, \frac{\nu}{n}+\frac{1}{n}; f\right] = (S_n D_1^f)(x).$$

Further we obtain

$$(S_n f)''(x) = ne^{-nx}\sum_{\nu=0}^{\infty}\frac{(nx)^{\nu}}{\nu!}\left\{\left[\frac{\nu}{n}+\frac{1}{n}, \frac{\nu}{n}+\frac{2}{n}; f\right] - \left[\frac{\nu}{n}, \frac{\nu}{n}+\frac{1}{n}; f\right]\right\} =$$

$$= 2!\, e^{-nx}\sum_{\nu=0}^{\infty}\frac{(nx)^{\nu}}{\nu!}\left[\frac{\nu}{n}, \frac{\nu}{n}+\frac{1}{n}, \frac{\nu}{n}+\frac{2}{n}; f\right] =$$

$$= 2!\,(S_n D_2^f)(x).$$

As a rule we suppose that for $k = s$ the derivative of the s^{th} order satisfies the equality

$$(S_n f)^{(s)} = s!\, S_n D_s^f. \tag{4}$$

Making use of (4) we can write

$$(S_n f)^{(s+1)}(x) = s!\, ne^{-nx}\sum_{\nu=0}^{\infty}\frac{(nx)^{\nu}}{\nu!}\left\{\left[\frac{\nu}{n}+\frac{1}{n}, \frac{\nu}{n}+\frac{2}{n}, \ldots, \frac{\nu}{n}+\frac{s+1}{n}; f\right] -\right.$$

$$\left. - \left[\frac{\nu}{n}, \frac{\nu}{n}+\frac{1}{n}, \ldots, \frac{\nu}{n}+\frac{s}{n}; f\right]\right\} =$$

$$= (s+1)!\, e^{-nx}\sum_{\nu=0}^{\infty}\frac{(nx)^{\nu}}{\nu!}\left[\frac{\nu}{n}, \frac{\nu}{n}+\frac{1}{n}, \ldots, \frac{\nu}{n}+\frac{s+1}{n}; f\right] =$$

$$= (s+1)!\,(S_n D_{s+1}^f)(x),$$

and the theorem is proved.

THEOREM 2. *The Szász-Mirakyan operator has the following form*

$$(S_n f)(x) = \sum_{\nu=0}^{\infty}\left[0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{\nu}{n}; f\right]x^{\nu}. \tag{5}$$

Proof. Method I. By developing the function e^{-nx} in power series we have

$$e^{-nx} = \sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{n^{\nu}}{\nu!}x^{\nu}$$

and the Szász-Mirakyan operator defined by (1) can be written in the following manner

$$(S_n f)(x) = \sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{n^{\nu}}{\nu!}x^{\nu}\sum_{\nu=0}^{\infty}\frac{n^{\nu}}{\nu!}f\left(\frac{\nu}{n}\right)x^{\nu} =$$

$$= \sum_{\nu=0}^{\infty}x^{\nu}\sum_{j=0}^{\nu}(-1)^{\nu-j}\frac{n^{\nu}}{(\nu-j)!\,j!}f\left(\frac{j}{n}\right) = \sum_{\nu=0}^{\infty}D_{\nu}^f(0)\,x^{\nu},$$

and we give (5).

Method II. From (4)

$$(S_n f)^{\nu}(0) = \nu! \left[0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{\nu}{n}; f\right],$$

is easily deduced and it gives the expression of the Szász-Mirakyan operator in power series of x, namely

$$(S_n f)(x) = \sum_{\nu=0}^{\infty} \frac{(S_n f)^{(\nu)}(0)}{\nu!} x^{\nu} = \sum_{\nu=0}^{\infty} \left[0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{\nu}{n}; f\right] x^{\nu}.$$

This theorem enables us to establish the following properties which are similar to those of the Bernstein operator defined on $C[0, 1]$ by the relation

$$(B_n f)(x) = \sum_{\nu=0}^{n} \binom{n}{\nu} f\left(\frac{\nu}{n}\right) x^{\nu}(1-x)^{n-\nu}, \tag{6}$$

(see G. G. LORENTZ [2] and T. POPOVICIU [5]).

Corollary I. *When f is a polynomial of s-degree then $S_n f$ is itself a polynomial of the degree $\leqslant s$.*

This statement results taking into account that for a polynomial f of the s-degree, we have

$$\left[0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{\nu}{n}; f\right] = 0 \text{ for } \nu > s.$$

Corollary II. *If $f^{(\nu)}(0)$, $\nu \in \omega$, exists, then*

$$Sf = \lim_{n \to \infty} S_n f$$

is the Maclaurin's expansion of the f function.

It is known [6] that

$$\lim_{n \to \infty} \left[0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{\nu}{n}; f\right] = \frac{f^{(\nu)}(0)}{\nu!}$$

Therefore

$$(Sf)(x) = \sum_{\nu=0}^{\infty} \frac{f^{(\nu)}(0)}{\nu!} x^{\nu},$$

and we may say that the Szász-Mirakyan operator is a „quasi'' Maclaurin operator.

THEOREM 3. *The Szász-Mirakyan operator preserves the convexity of any order.*

Proof. It is known that if $g \in C^{(k+1)}[a, b]$, that is $g^{(k+1)}$ exists and it is continuous on $[a, b]$, then we say that g is a convex function of k-order on $[a, b]$ iff $g^{(k+1)} > 0$ on this interval. From (3) it is readily verified that for $k = -1, 0, 1, \ldots$ we can write

$$\operatorname{sgn} D^f_{k+1} \to \operatorname{sgn} (S_n f)^{(k+1)}$$

and in accordance with the definition 2 we conclude that the Szász-Mirakyan operator preserves the convexity of any order. On account of (3) we also observe that S_n likewise preserves the non-concavity, the polynomiality, the non-convexity and the concavity of any order.

In the case of the Bernstein operator, Prof. T. Popoviciu [5] proved in 1934 that this operator, defined by (6), preserves the convexity of any order.

THEOREM 4. *The Meyer-König and Zeller operator defined by* (2) *preserves the convexity of $-1, 0, 1, 1$ & 2 order of a function.*

Proof. The convexity of -1 order, that is the positivity, is immediately obtained.

From (2) we get

$$(M_n f)'(x) = (n+1)(1-x)^n \left\{ \sum_{\nu=0}^{\infty} \binom{\nu+n+1}{\nu} f\left(\frac{\nu+1}{\nu+n+1}\right) x^{\nu} - \right.$$

$$\left. - \sum_{\nu=1}^{\infty} \binom{\nu+n}{\nu-1} f\left(\frac{\nu}{\nu+n}\right) x^{\nu} - \sum_{\nu=0}^{\infty} \binom{\nu+n}{\nu} f\left(\frac{\nu}{\nu+n}\right) x^{\nu} \right\}.$$

Since

$$\sum_{\nu=0}^{\infty} \binom{\nu+n}{\nu-1} f\left(\frac{\nu}{\nu+n}\right) x^{\nu} + \sum_{\nu=1}^{\infty} \binom{\nu+n}{\nu} f\left(\frac{\nu}{\nu+n}\right) x^{\nu} =$$

$$= \sum_{\nu=0}^{\infty} \binom{\nu+n+1}{\nu} f\left(\frac{\nu}{\nu+n}\right) x^{\nu},$$

we have

$$\begin{aligned}(M_n f)'(x) &= (n+1)(1-x)^n \sum_{\nu=0}^{\infty} \binom{\nu+n+1}{\nu} \left[f\left(\frac{\nu+1}{\nu+n+1}\right) - f\left(\frac{\nu}{\nu+n}\right)\right] x^{\nu} = \\ &= (1-x)^n \sum_{\nu=0}^{\infty} \binom{\nu+n-1}{\nu} \left[\frac{\nu}{\nu+n}, \frac{\nu+1}{\nu+n+1}; f\right] x^{\nu},\end{aligned} \tag{7}$$

and thus we conclude that this operator preserves the convexity of 0-order.

By differentiating the formula (7), we obtain

$$(M_nf)''(x) = (1-x)^{n-1}\sum_{\nu=0}^{\infty}\binom{\nu+n}{\nu+1}\left[\frac{\nu+1}{\nu+n+1}, \frac{\nu+2}{\nu+n+2}; f\right](\nu+1)x^{\nu} -$$

$$-(1-x)^{n-1}\sum_{\nu=1}^{\infty}\binom{\nu+n-1}{\nu}\left[\frac{\nu}{\nu+n}, \frac{\nu+1}{\nu+n+1}; f\right]\nu x^{\nu} -$$

$$-(1-x)^{n-1}\sum_{\nu=0}^{\infty}\binom{\nu+n-1}{\nu}\left[\frac{\nu}{\nu+n}, \frac{\nu+1}{\nu+n+1}; f\right]nx^{\nu} =$$

$$=(1-x)^{n-1}\sum_{\nu=0}^{\infty}\binom{\nu+n-1}{1}n\left\{\left[\frac{\nu+1}{\nu+n+1}, \frac{\nu+2}{\nu+n+2}; f\right]-\left[\frac{\nu}{\nu+n}, \frac{\nu+1}{\nu+n+1}; f\right]\right\}x^{\nu}=$$

$$=(1-x)^{n-1}\sum_{\nu=0}^{\infty}\binom{\upsilon+n-1}{\nu}\frac{2n}{\nu+n+2}\left[\frac{\nu}{\nu+n}, \frac{\nu+1}{\nu+n+1}, \frac{\nu+2}{\nu+n+2}; f\right]x^{\nu}$$

and thus the operator M_n preserves the convexity of 1-order. Further we get

$$(M_nf)'''(x) = (1-x)^{n-2}\sum_{\nu=0}^{\infty}\binom{\nu+n-1}{\nu}2nH^f_{n\nu}x^{\nu},$$

where

$$H^f_{n\nu} = \frac{\nu+n}{\nu+n+3}\left[\frac{\nu+1}{\nu+n+1}, \frac{\nu+2}{\nu+n+2}, \frac{\nu+3}{\nu+n+3}; f\right] -$$

$$-\frac{\nu+n-1}{\nu+n+2}\left[\frac{\nu}{\nu+n}, \frac{\nu+1}{\nu+n+1}, \frac{\nu+2}{\nu+n+2}; f\right].$$

In the following we assume that f is a convex function of 1-order on $[0, 1]$. In this case, as

$$\frac{\nu+n}{\nu+n+3} > \frac{\nu+n-1}{\nu+n+2}$$

we conclude that

$$H^f_{n\nu} > \frac{\nu+n-1}{\nu+n+2}\left\{\left[\frac{\nu+1}{\nu+n+1}, \frac{\nu+2}{\nu+n+2}, \frac{\nu+3}{\nu+n+3}; f\right] -\right.$$

$$\left.-\left[\frac{\nu}{\nu+n}, \frac{\nu+1}{\nu+n+1}, \frac{\nu+2}{\nu+n+2}; f\right]\right\} =$$

$$= \frac{3n(\nu+n-1)}{(\nu+n+3)(\nu+n+2)(\nu+n)}\left[\frac{\nu}{\nu+n}, \frac{\nu+1}{\nu+n+1}, \frac{\nu+2}{\nu+n+2}, \frac{\nu+3}{\nu+n+3}; f\right]$$

and consequently, if f is convex of 1 and 2-order on $[0, 1]$, then $(M_nf)'' > 0$ and also $(M_nf)''' > 0$ on $[0, a]$, $a < 1$.

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