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SOME PROPERTIES OF THE LINEAR POSITIVE OPERATORS (II)*

by

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1. This note is a sequel to the first note with the same title in this review, the contents of which are assumed to be known. One of our purposes in this paper is to generalize the results published in [4].

2. We recall that, in order to construct sequences of linear positive operators on $Q[0, a]$ with the aid of sequences of functions, V. A. BASKAKOV [1] and F. SCHURER [8] stated the following conditions. Let consider the sequence $\{\varphi_n\}$, $n \in \omega$, of real functions of the real variable x, each function having the following properties on the interval $[0, a]$, $a > 0$,

I. $\varphi_n(0) = 1$

II. φ_n is completely monotonic on $[0, a]$, that is φ_n has derivatives of all orders there and $(-1)^j \varphi_n^{(j)} \geqslant 0$, $j = 0, 1, 2, \ldots$.

III. there exists a positive integer m_n not depending on j, such that for $x \in [0, a]$ we have

$$-\varphi_n^{(j)}(x) = \Psi(n, x)\varphi_{m_n}^{(j-1)}(x)[1 + E_{j,n}(x)],\ j \in \omega,$$

where

$$\lim_{n\to\infty} E_{j,n} = 0, \quad x \in [0, a],\ j \in \omega,$$

$$\lim_{n\to\infty} \frac{\Psi(n, x)}{n} = 1 \text{ and } \lim_{n\to\infty} \frac{\Psi(m_n, x)}{n} = 1,\ x \in [0, a].$$

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The sequence $\{\varphi_n\}$, $n \in \omega$, generates the sequence of linear positive operators L_n, defined by the relation

$$(1) \qquad (L_n f)(x) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \varphi_n^{(\nu)}(x)\, x^\nu}{\nu!} f\left(\frac{\nu}{n}\right).$$

F. Schurer has shown in [8] that if $f \in Q[0, a]$ and $f(x) = O(x^2)$ when $|x| \to \infty$, then $\|L_n f - f\| \to 0$ where $\|f\| = \max\limits_{x \in [0, a]} |f(x)|$.

3. We denote by $\mathcal{A}_k^+$, $\mathcal{A}_k^0$ repsectively $\mathcal{A}_k^-$ the subsets of $Q[0, a]$ consisting of convex, polynomial respectively concave functions of k-order on an interval I of the real axis.

THEOREM 1. *If $f \in \mathcal{A}_k^+$, $\mathcal{A}_k^0$ resp. $\mathcal{A}_k^-$ on the interval $[0, \infty)$, then $L_n f \in \mathcal{A}_k^+$, $\mathcal{A}_k^0$ resp. $\mathcal{A}_k^-$ on the interval $[0, a]$, $k \in \omega$; in other words the operator L_n, defined by* (1), *preserves the convexity, the polynomiality respectively the concavity of any order on the interval* $[0, a]$.

Proof. It is known [6] that for $F \in C^{(k+1)}[0, a]$ the convexity, the polynomiality and the concavity conditions of k-order, on $[0, a]$, are the following

$$F^{(k+1)} > 0, \; F^{(k+1)} = 0 \text{ and } F^{(k+1)} < 0 \text{ on } [0, a].$$

Differentiating, therefore, the formula (1), we obtain

$$(L_n f)'(x) = \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu+1} \varphi_n^{(\nu+1)}(x)}{\nu!} \left[f\left(\frac{\nu+1}{n}\right) - f\left(\frac{\nu}{n}\right) \right] x^\nu =$$

$$= \frac{1}{n} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu+1} \varphi_n^{(\nu+1)}(x)}{\nu!} \left[\frac{\nu}{n}, \frac{\nu+1}{n}; f \right] x^\nu.$$

As a rule, by repeated differentiation, we give

(2)

$$(L_n f)^{(k+1)}(x) = \frac{(k+1)!}{n^{k+1}} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu+k+1} \varphi_n^{(\nu+k+1)}(x)}{\nu!} \left[\frac{\nu}{n}, \frac{\nu+1}{n}, \ldots, \frac{\nu+k+1}{n}; f \right] x^\nu.$$

Using the condition II in connection with (2) it follows that if $\left[\frac{\nu}{n}, \frac{\nu+1}{n}, \ldots, \frac{\nu+k+1}{n}; f\right] > 0, = 0$, resp. < 0 for n, $k \in \omega$, $\nu = 0, 1, \ldots$, namely if $f \in \mathcal{A}_k^+$, $\mathcal{A}_k^0$ resp. $\mathcal{A}_k^-$, then $(L_n f)^{(k+1)} > 0 = 0$, resp. < 0 on the interval $[0, a]$, and the theorem is proved.

Now it is easy to carry out a new form of the operator L_n, by using the ν-order divided differences.

THEOREM 2. *The operator L_n defined by* (1) *has the following form*

$$(3) \qquad (L_n f)(x) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \varphi_n^{(\nu)}(0)}{n^\nu} \left[0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{\nu}{n}; f\right] x^\nu.$$

Proof. Taking into account the relation (2) it is easily deduced that

$$(L_n f)^{(\nu)}(0) = \frac{(-1)^\nu \varphi_n^{(\nu)}(0)\, \nu!}{n^\nu} \left[0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{\nu}{n}; f\right].$$

which enables us to give the expression of the operator L_n in power series of x, that is the relation (3).

Corollary. *When f is a polynomial of s-degree, then $L_n f$ is itself a polynomial of degree $\leq s$.*

By using (3) this is obvious because the ν-order divided differences of f vanishes for $\nu > s$.

Now, let $V[f]$ be the variation of f defined as the number of changes of sign of the function as x varies across its domain. In [7] was introduced the concept of *variation diminishing operator*. According to I. J. SCHOENBERG such an operator has the following property

$$V[L_n f] \leq V[f].$$

THEOREM 3. *The operator L_n defined by* (1) *is variation diminishing.*

A proof of this result may be obtained by slightly modifying the proof of a somewhat weaker result given in [3].

4. Further, by means of many suitable choices of the sequences $\{\varphi_n\}$, $n \in \omega$, we give some well-known operators.

i) Let $\varphi_n(x) = (1 - x)^n$ where $x \in [0, 1]$; the operator generated by this sequence of functions is the Bernstein operator defined by

$$(B_n f)(x) = \sum_{\nu=0}^{n} \binom{n}{\nu} x^\nu (1-x)^{n-\nu} f\left(\frac{\nu}{n}\right).$$

Thirty years ago Prof. T. POPOVICIU [5] published the first result about the shape-preserving property by Bernstein operator. Also I. J. SCHOENBERG [7] proved that B_n is a variation diminishing operator.

ii) If $\varphi_n(x) = e^{-nx}$, $x \in [0, a]$, $0 < a < \infty$, the corresponding operator has the form

$$(S_n f)(x) = e^{-nx} \sum_{\nu=0}^{\infty} \frac{(nx)^\nu}{\nu!} f\left(\frac{\nu}{n}\right),$$

and we give the well-known Szász-Mirakyan operator [3—4]. Using the theorem 2 we remark that this operator has an interesting form, since in (3) we have

$$\frac{(-1)^{\nu}\varphi_n^{(\nu)}(0)}{n^{\nu}} = 1.$$

iii) For $\varphi_n(x) = (1+x)^{-n}$, $x \in [0, a]$, $0 < a < \infty$, we obtain the Baskakov operator defined by

$$(K_n f)(x) = \sum_{\nu=0}^{\infty} \binom{\nu+n-1}{\nu} \frac{x^{\nu}}{(1+x)^{n+\nu}} f\left(\frac{\nu}{n}\right).$$

For this operator the property of preserving convexity of any order is a new result.

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A PROPERTY OF THE S. N. BERNSTEIN OPERATOR

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1. Let f be a real function of real variable. We shall say that f is *starshaped* on an interval I, if

$$f(\alpha x) \leqslant \alpha f(x) \tag{1}$$

for every $\alpha \in [0, 1]$, $x \in I$.

Let us note with $\mathcal{F}[0, 1]$ the space of functions verifying the following properties

i) $f(0) = 0$

ii) f is nonnegative on $[0, 1]$

iii) $f \in C[0, 1]$.

Let $\mathcal{K}$, respectively $\mathcal{S}$ be the set of the functions $f \in \mathcal{F}[0, 1]$ which are non-concave of the first order, respectively starshaped on $[0, 1]$. These functions have been minutely studied in [1], [2] and [3]. In [1] and [2] the following inclusion relation was shown

$$\mathcal{K} \subset \mathcal{S}. \tag{2}$$

Likewise it is known that the reverse implication is not valid.

Let us consider the Bernstein operator defined on $C[0, 1]$ by the relation

$$(B_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) p_{nk}(x) \tag{3}$$

where

$$p_{nk}(x) = x^k (1-x)^{n-k}.$$