L'ensemble $W_1(x_0) = G_{j_0} \cap W(x_0)$ est un voisinage du point x_0 dans la topologie de X. De (3) et (4) on obtient, pour tout $x \in W_1(x_0)$,

$$(f(x_0), g_{j_0}(x)) \in U \circ U = \overset{2}{U},$$

et de (2) et de la dernière relation on trouve

$$(f(x_0), f(x)) \in \overset{2}{U} \circ \overset{-1}{U} = \overset{3}{U} \subseteq \overset{3}{U}_4 \subseteq U_1,$$

c'est-à-dire $f(x) \in V_1(f(x_0))$ pour tout $x \in W_1(x_0)$, donc l'application f est continue et la démonstration est achevée.

3. *Remarques.* Si l'espace topologique X est compact au sens de Borel-Lebesgue, alors de la suite généralisée $(G_j)_{j\in J}$ des ensembles ouverts G_j, la réunion desquels constitue un recouvrement de l'espace X, on peut extraire une suite finie $(G_{j_k})_{1<k<n}$ avec la même propriété. C'est pour cela que, dans ce cas, on peut remplacer dans la démonstration précédente de la condition suffisante les suites généralisées $(g_j)_{j\in J}$ et $(G_j)_{j\in J}$ par les suites finies $(g_{j_k})_{1<k<n}$, respectivement $(G_{j_k})_{1<k<n}$.

De même, lorsque l'espace topologique X possède une base dénombrable $\Gamma = (\Gamma_s)_{s\in\omega}$, pourt tout $x \in X$ et tout terme G_j de la suite $(G_j)_{j\in J}$ on trouve $\Gamma_s \in \Gamma$ tel que $x \in \Gamma_s \subseteq G_j$, et alors on peut remplacer les suites généralisées $(g_j)_{j\in J}$ et $(G_j)_{j\in J}$ par des suites habituelles correspondantes.

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HOMEOMORPHE PROJECTIONS OF k-INDEPENDENT SETS AND CHEBYSHEV SUBSPACES OF FINITE DIMENSIONAL CHEBYSHEV SPACES

by

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0. Known results, definitions and notations. We denote by $C(Q)$ the linear space of all continuous functions which map the compact set Q in R, the real numbers.

Definition 1. *The n-dimensional linear subspace F_n of the space $C(Q)$ is said to form a Chebyshev subspace of order $n-k$ $(1 \leq k \leq n)$ with respect to the function $f \in C(Q)$, if the set of elements of the best approximation of f in F has the dimension $\leq n-k$.*

The following theorem due to G. Š. RUBINŠTEĬN [3] is a generalisation of A. Haar's well known theorem:

The n-dimensional linear subspace F_n of the space $C(Q)$ is a Chebyshev subspace of order $n-k$ with respect to any function $f \in C(Q)$ if and only if each set of $n-k+1$ linearly independent functions of F has at most $k-1$ common zeros.

In our investigation concerning Chebyshev subspaces of finite dimensional Chebyshev spaces we have used the notion of the k-independent set introduced by K. BORSUK [2].

Definition 2. *The set A of the n-dimensional euclidean space E_n is said to be k-independent, if each set of k distinct vectors of A is linearly independent.*

The connection between Chebysev spaces of a given order and the notion of k-independent sets is by following theorem stated, similar to those of V.G. BOLTJANSKIĬ, S. S. RYŠKOV, JU. A. ŠAŠKIN ([1] Theorem 3)

THEOREM. *The n-dimensional subspace F_n of $C(Q)$ spanned by functions $\varphi_1, \dots \varphi_n$ is a Chebyshev space of order $n-k$ $(1 \leq k \leq n)$ if and only if the map $\Phi: Q \to E_n$ defined by the relation*

$$(*) \qquad \Phi(x) = (\varphi_1(x), \dots, \varphi_n(x))$$

is a homeomorphe imbedding of the compact set Q in a k-independent set of the n-dimensional euclidean space E_n.

An r-dimensional hyperplane of E_n passing through the origin is said to form an r-dimensional euclidean subspace of E_n and will be denoted with E_r.

As in the above mentioned theorem, in what follows, if $\varphi_1, \dots \varphi_n$ are n functions of $C(Q)$ which span an n-dimensional Chebyshev space of order $n-k$, then the map of the form (*) of Q into E_n will be denoted by Φ.

1. Homeomorphe projections of k-independent sets

THEOREM 1. *Let be A a compact set of the euclidean space E_n, which is k-independent. The set A can be topologically imbedded into an $k-s$-independent $(k \geq s+2)$ set A' of the $n-s-$dimensional subspace E_{n-s} of E_n by a projection, if and only if there are s vectors in E_n outside to A, with the property that each set of k vectors composed by these s vectors and any $k-s$ distinct vectors of A, spans a k-dimensional subspace of E_n.*

The necessity. Suppose that the projection of E_n into E_{n-s} is a homeomorphe map of the set A into the $k-s$-independent set A'. Let be $x \neq 0$ a point of the subspace E_s, the orthogonal complement of E_{n-s} in E_n. Then the set $A \cup \{x\}$ is a $k-s+1$-independent set in E_n. Really, if there would exist a $k-s$ -dimensional subspace E_{k-s} which would contain $k-s+1$ distinct points of $A \cup \{x\}$, then this subspace would contain the point x and then the projection of E_{k-s} into E_{n-s} would form a subspace of dimension $k-s-1$. But then the set A' would be intersected by a subspace of dimension $k-s-1$ of the space E_{n-s} in $k-s$ distinct points (the images by projection of the $k-s$ points of A which are common with E_{k-s}), which is in contradiction with our hypothesis, that A' is $k-s$-independent in E_{n-s}.

Let now the points $x_1, \dots, x_s$ form a basis of E_s. We shall show that these vectors have the property assumed by the Theorem 1. Suppose the contrary: there are $k-s$ distinct vectors $y_1 \dots y_{k-s}$ in A so that the vectors $x_1, \dots, x_s, y_1, \dots, y_{k-s}$ are linearly dependent, i.e. there are the reals a_i, $i=1, \dots, s$ and b_i, $i=1, \dots, k-s$ with the property: $\sum_{i=1}^{s} |a_i| + \sum_{i=1}^{k-s} |b_i| \neq 0$, for which

$$(1) \qquad \sum_{i=1}^{s} a_i x_i + \sum_{i=1}^{k-s} b_i y_i = 0.$$

From the linear independence of the vectors x_i, $i=1, \dots, s$, and the k-independence of A it follows that $\sum_{i=1}^{s} |a_i| \neq 0$ and $\sum_{i=1}^{k} |b_i| \neq 0$. But then $x = \sum_{i=1}^{k-s} a_i x_i$ is a non-zero vector in E_s, and by (1) it follows that the vectors $x, y_1, \dots, y_{k-s}$ are linearly dependent in contradiction with the result which has been established in the first part of our proof.

The sufficience. We suppose that the compact set A is k-independent in E_n and that the vectors $x_1, \dots, x_s$ have the property assumed in Theorem 1. Let be E_s the s-dimensional subspace spanned by the vectors $x_1, \dots, x_s$, and E_{n-s} the orthogonal complement of E_s in E_n. We shall show that the projection A' of A into the space E_{n-s} is a $k-s$-independent set in this space. Suppose the contrary: there is a $k-s-1$ - dimensional subspace E_{k-s-1} of E_{n-s}, which intersects the set A' in $k-s$ distinct vectors. All counter-images of these $k-s$ vectors are contained in the $k-1$ -dimensinoal subspace $E_{k-1} = E_{k-s-1} \oplus E_s$. If the $k-s$ distinct counter-images of the vectors of $A' \cap E_{k-s-1}$ are denoted by $y_1, \dots, y_{k-s}$, then obviously E_{k-1} will contain the vectors $x_1, \dots, x_s, y_1, \dots, y_{k-s}$ and hence it follows that these vectors are linearly dedendent, in contradiction with our assumption. The continuity of the projection of A into E_{n-s} is obvious. We prove that this map is one to one. If a vector $y \in A'$ would have two counter-images y' and y'', then if $y_1, \dots, y_{k-s-2}$ $(k-s-2 \geq 0)$ are counter-images of other $k-s-2$ distinct vectors of A', then the $k-1$-dimensional subspace $E_{k-1} = E_{k-s-1} \oplus E_s$, where E_{k-s-1} is the $k-s-1$-dimensional subspace of E_{n-s} spanned by y and those $k-s-2$ vectors whose counter-images are the vectors $y_1, \dots, y_{k-s-2}$, contain the vectors $y', y'', y_1, \dots, y_{k-s-2}, x_1, \dots, x_s$, and therefore these vectors must be linearly dependent, in contradiction with our assumption.

Observations. Firstly we observe that in Theorem 1 it is sufficient to consider the case $s \geq 1$, because if $s = 0$ then as a projection may be considered any continuous one-to-one linear map of E_n in itself and as it is easy to be seen such a transformation map any k-independent set of E_n in a k-independent set.

The condition $k \geq s+2$ of this theorem is an essential one. It has been applied in the first part of the proof of the necessity and in the proof of the one-to-ones of projection defined in the proof of sufficience. The essentiality of this condition may be seen in the following example. Let be $A \subset E_2$ the two-independent set obtained by the circumference of the circle $x^2 - 2x + y^2 = 0$ excluding an open arc of length which is smaller then the semicircle, containing the origin. This set may be projected with an one-to-one projection into none straight line.

In particular, if $A \cup \{x_1, \ldots, x_s\}$ is a compact k-independent set in E_n, then A is $k - s -$ independent projectable in E_{n-s}, the orthogonal complement of E_s, the s-dimensional space spanned by $x_1, \ldots, x_s$.

Lemma 1. *The compact two-independent set A of the n-dimensional euclidean space E_n may be topologically imbedded in a two-independent proper subset of S_{n-1}, the $n - 1$ sphere in E_n with its centre in the origin.*

Proof. Let S_{n-1} be an $n - 1$-sphere in E_n with its centre in the origin. Let be $x \in A$ and let be $(x)^+ = \{\lambda x \mid \lambda \geqq 0\}$. We consider the map $\psi(x) = S_{n-1} \cap (x)^+$ of A into S_{n-1}. It is easy to seen that this map is continuous. Moreover, it is one-to-one, because A is a two-independent set in E_n. According to the compacteness of A it follows that ψ is a homeomorphism. The set $\psi(A)$ is two-independent, because any one-dimensional subspace of E_n, which contains some points of $\psi(A)$, contains also the counter images of these points in A. It is a proper subset of S_{n-1}, because if $x \in A$ then from two-independence of A it follows that the point $\{\mu x \mid \mu \leq 0\} \cap S_{n-1}$ is outside to $\psi(A)$.

The Lemma 1, in conjunction with Theorem 1 gives the following analoge of K. Borsuk's well known imbedding theorem [2]:

THEOREM 2. *If A is a k-independent compact set in E_n $(k \geqq 2)$, and if $x_1, \ldots, x_{k-2}$ are $k-2$ vectors outside to A with the property that each set of k vectors which is composed by these vectors and any 2 distinct vectors of A, span a k-dimensional subspace of E_n, then the set A is topologically imbeddable into a proper subset of the $n - k + 1$-sphere S_{n-k+1}.*

THEOREM 3. *Let be A a compact two-independent set in the space E_n. Then there is an one-dimensional subspace E_1 with the property $E_1 \cap A = \emptyset$.*

Proof. Let be E_2 any two-dimensional subspace of E_n. From Lemma 1 it follows that $\psi(A)$ (where ψ is the map defined in the proof of Lemma 1) is two-independent. We consider the set $A' = \psi(A) \cap E_2$. Because A' is a two-independent set in E_2 it is a proper subset of the one-sphere $S_1 = S_{n-1} \cap E_2$. Because A' is compact it is a sum of closed arcs of S_1. Let be a an end-point of one of these arcs, and let be (a) the one-dimensional subspace which is spanned by a. From the two-independence of A' it follows that $-a \overline{\in} A'$. Because A' is a closed subset of S_1 it follows that there is a neighbourhood V of the point $-a$ in S_1 so that $V \cap A' = \emptyset$. But then the central homothety with respect to the origin and with the coefficient -1 map the neighbourhood V in U, which is a neighbourhood of the point a in S_1. Because a is an endpoint of an arc of A', its neighbourhood U contains a point b outside to A'. The one-dimensional subspace (b) spanned by the vector b intersects the set S_1 in b and in a point of V, and hence it follows that this subspace has no common point with A'. It also follows that the one-dimensional subspace $E_1 = (b)$ has the property stated by Theorem 3.

Definition 3. *The sequence of projections $\varkappa_1, \ldots, \varkappa_{k-2}$ is said to form an M-sequence of projections of the compact set A, which is k-independent in E_n if $\varkappa_i$ is a homeomorphe prjection of A into a $k-i$-independent set of E_{n-i} $(i = 1, \ldots, k - 2)$ and if $E_{n-i} \supset E_{n-i-1}$, $i = 1, \ldots, k - 3$.*

THEOREM 4. *The compact set A which is k-independent in E_n has an M-sequence of projections $\varkappa_1, \ldots, \varkappa_{k-2}$, if and only if there is a sequence of vectors $x_1, \ldots, x_{k-2}$ outside to A, with the property that each set of k vectors which is composed by the vectors $x_1, \ldots, x_i$ and $k - i$ vectors of A, span a k-dimensional subspace of E_n.*

The necessity. According to Theorem 1, for every i $(i = 1, ,, k - 2)$ there are the vectors $x_1^{(i)}, \ldots, x_i^{(i)}$ outside to A with the property that every k vectors composed by these i vectors and $k - i$ vectors of A are linearly independent. Furthermore, it is easy to seen that each basis of the space E_i, the space spanned by verctors $x_1^{(i)}, \ldots, x_i^{(i)}$ has too this property. From the proof of Theorem 1 it follows that the subspace E_i, $(i = 1, \ldots, k - 2)$ is the orthogonal complement of the space E_{n-i} $(i = 1, \ldots, k - 2)$ the space in which is projected A by the projection $\varkappa_i$. Because we have $E_{n-i} \supset E_{n-i-1}$ it followsthat $E_i \subset E_{i+1}$ $(i = 1, \ldots, k - 3)$. Let be $x_1, \ldots, x_i$ a basis of the space E_i and let be x_{i+1} a vector of E_{i+1} which is linearly independent of these vectors. Applying this constructive method for $i = 1, \ldots, k - 3$ we obtain the points $x_1, \ldots, x_{k-2}$ which have the property assumed by Theorem 4.

The sufficience follows immediately as a result of a repeated application of the sufficient part of the Theorem 1, remarking that according to the proof of this theorem it follows that $E_{n-i} \supset E_{n-i-1}$, $i = 1, \ldots, k - 3$.

In what follows we introduce a new kind of map of the k-independent sets.

Definition 4. *Let be A a compact set in the n-dimensional euclidean space E_n, which is k-independent. The projection β of E_n into E_{n-s} will be said to form a β-transformation of E_n into E_{n-s} with respect to the k-independent compact A, if β $(E_n) = E_{n-s}$, and if each $k - 1$-dimensional subspace E_{k-1} which intersects the set A in $s - (n - k)$ distinct points has in its orthogonal complement E_{n-k+1} in E_n an element, whose projection into E_{n-s} is different from zero.*

THEOREM 5. *The projection β forms a β-transformation of the n-dimensional euclidean space E_n into the $n - s$-dimensional subspace E_{n-s} with respect to the k-independent compact A in E_n, if and only if the projection of A into E_s, the orthogonal complement of E_{n-s} in E_n, is a homeomorphe map, and the image of A is an $s - (n - k)$-independent set in E_s $(s - (n - k) \geqslant 2)$.*

The necessity. We denote by E_s the orthogonal complement of the subspace E_{n-s} in which E_n may be projected with a β-transformation β with respect to the k-independent compact A in E_n. We denote by A' the projection of A into E_s. Suppose that A' is not $s-(n-k)$-independent in E_s, i.e. there is a subspace of the dimension $s-(n-k)-1$ $E_{s-(n-k)-1}$ in E_s, which intersects the set A' in $s-(n-k)$ distinct vectors (we remind that $s-(n-k) \geqq 2$). Let be E_{n-k+1} the orthogonal complement of $E_{s-(n-k)-1}$ in E_s and E_{k-1} the orthogonal complement of E_{n-k+1} in E_n. Then we have $E_{k-1} = E_{n-s} \oplus E_{s-(n-k)-1}$ and the counter-images of the $s-(n-k)$ distinct points in $E_{s-(n-k)-1} \cap A'$ are contained in E_{k-1}. It follows that E_{n-k+1} must have at least an element which is projected by β in a non-zero element of E_{n-s}. But this is impossible because $E_{n-k+1} \subset E_s$ and $\beta\ (E_s) = 0$. Therefore A' is $s-(n-k)$-independent in E_s.

With a similar argument it may be also proved that the projection of A in A' is an one-to-one map. Due to its continuity and because A is compact, it follows that this map is a homeomorphism.

The sufficience. We suppose that the projection of A into the s-dimensional subspace E_s is a homeomorphism, which map the set A in the $s-(n-k)$-independent set A' of the space E_s. Let β be the projection of E_n into E_{n-s}, the orthogonal complement of E_s in E_n. We shall show that this projection is a β-transformation of the space E_n in E_{n-s} with respect to the k-independent compact set A in E_n. Really, let be E_{n-k+1} an $n-k+1$-dimensional subspace of E_n, whose orthogonal complement E_{k-1} intersects the set A in $s-(n-k)$ distinct vectors. Then E_{n-k+1} contains at least a point which is outside to E_s. If we should have $E_{n-k+1} \subset E_s$, then, — if $E_{s-(n-k)-1}$ is the orthogonal complement of E_{n-k+1} in E_s — because $E_{k-1} = E_{n-s} \oplus E_{s-(n-k)-1}$, it would follow that the $s-(n-k)$ distinct points of the set $E_{k-1} \cap A$ would be contained in the projection of A into E_s, i.e. in the $s-(n-k)-1$-dimensional subspace $E_{s-(n-k)-1}$, in contradiction with the assumption that A' is $s-(n-k)$-independent homeomorphe image of A by the projection into E_s.

Observation. The essentiality of the condition $s-(n-k) \geqq 2$ follows from the observation after the Theorem 1.

2. Chebyshev subspaces of finite dimensional Chebyshev spaces

The notions and notations applied here were introduced in **0**.

THEOREM 6. *The n-dimensional Chebyshev space of order $n-k$ spanned by the functions $\varphi_1, \ldots, \varphi_n$ of the space $C(Q)$, has a Chebyshev subspace of dimension $n-s$ $(k \geqq s+2)$ and of the same order, if and only if the compact set $\Phi(Q)$ which is k-independent in E_n may be projected in a $k-s$-independent set of an $n-s$-dimensional subspace E_{n-s}, and if this projection of $\Phi(Q)$ in E_{n-s} is a homeomorphe map.*

The necessity. We suppose that F_n has an $n-s$-dimensional Chebyshev subspace F_{n-s} of the same order. Then there exists a linear inversable operator $A = \|a_{ij}\|$ of F_n onto itself so that the vector-function

$$(\psi_1(x), \ldots, \psi_n(x)) = (\varphi_1(x), \ldots, \varphi_n(x)) \cdot \|a_{ij}\|$$

has the property that $\psi_1(x), \ldots, \psi_{n-s}(x)$ is a basis of F_{n-s}. From the above equality it follows that

$$\Psi(Q) = \Phi(Q) \cdot \|a_{ij}\|$$

and from the inversability of $\|a_{ij}\|$ it follows that $\Psi(Q)$ is too a k-independent compact set of E_n. We consider the projection $\varkappa$ of E_n onto its first $n-s$ coordinates and will show that this projection is a homeomorphe map of $\Psi(Q)$ onto a $k-s$-independent subset of this subspace. Really, $\varkappa(\psi_1(x), \ldots, \psi_n(x)) = (\psi_1(x), \ldots, \psi_{n-s}(x), 0, \ldots, 0)$ and for any $k-s$ distinct points $x_1, \ldots, x_{k-s}$ of Q the rank of the matrix

$$\begin{Vmatrix} \psi_1(x_1) & \ldots & \psi_{n-s}(x_1) & 0 & \ldots & 0 \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \psi_1(x_{k-s}) & \ldots & \psi_{n-s}(x_{k-s}) & 0 & \ldots & 0 \end{Vmatrix}$$

will be $k-s$ since $\|\psi_i(x_j)\|_{\substack{i=1,\ldots,n-s \\ j=1,\ldots,k-s}}$ is of this rank by the hypothesis that $\psi_1, \ldots, \psi_{n-s}$ span an $n-s$-dimensional Chebyshev space F_{n-s} of order $n-k$. From Theorem 1 it follows that there are s vectors $y_1, \ldots, y_s$ with the property that the vectors $y_1, \ldots, y_s, y_{s+1}, \ldots, y_k$ are linearly independent for any $k-s$ distinct vectors $y_{s+1}, \ldots, y_k$ in $\Psi(Q)$. Now we consider the set of vectors $A^{-1}y_1, \ldots, A^{-1}y_s, A^{-1}y_{s+1}, \ldots, A^{-1}y_k$. Because A^{-1} is an inversable linear map of E_n onto itself it preserve the linear independence and the above vectors are linearly independent. This means that the vectors $y'_1 = A^{-1}y_1, \ldots y'_s = A^{-1}y_s$ have the property that the vectors $y'_1, \ldots, y'_s, y'_{s+1}, \ldots, y'_k$ are linearly independent for any $y'_{s+1}, \ldots, y'_k$ in $\Phi(Q)$, and we finish the proof of the necessity making use again of Theorem 1.

The sufficience. Suppose that $\Phi(Q)$ is projectable onto a $k-s$-independent set of E_{n-s}. By a coordinate transformation we map E_{n-s} in the subspace E'_{n-s} with the last s coordinates zeros. This coordinate transformation is a linear inversable application $A = \|a_{ij}\|$ of E_n onto itself and it means for the space F_n the change of the basis $\varphi_1(x), \ldots, \varphi_n(x)$ into a basis $\psi_1(x), \ldots, \psi_n(x)$ which is given by the equality

$$(\psi_1(x), \ldots, \psi_n(x)) = (\varphi_1(x), \ldots, \varphi_n(x)) \cdot \|a_{ij}\|.$$

The orthogonal projection of $\Phi(Q)$ into E_{n-s} become thus the orthogonal projection of $\Psi(Q)$ into the space E'_{n-s}, and because the image of $\Psi(Q)$

by this projection is a $k-s$-independent subset of E'_{n-s}, it follows that the matrix

$$\left\| \begin{matrix} \psi_1(x_1) & \dots & \psi_{n-s}(x_1) & 0 & \dots & 0 \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \psi_1(x_{k-s}) & \dots & \psi_{n-s}(x_{k-s}) & 0 & \dots & 0 \end{matrix} \right\|$$

is of rank $k-s$ for any $k-s$ distinct points $x_1, \dots, x_{k-s}$ of Q and so does the matrix $\|\psi_i(x_j)\|_{\substack{i=1,\dots,n-s \\ j=1,\dots,k-s}}$. It follows that $\psi_1(x), \dots, \psi_{n-s}(x)$ is a basis of a Chebyshev space of dimension $n-s$ and order $n-k$, which completes the proof of Theorem 6.

With help of Theorem 6 we obtain as consequences of those proved in **1.** the following theorems:

THEOREM 1'. *The n-dimensional Chebyshev space F_n of order $n-k$ $(1 \leq k \leq n)$ spanned by the functions $\varphi_1, \dots, \varphi_n$ of the space $C(Q)$ has an $n-s$-dimensional Chebyshev subspace $(k \geq s+2)$ of the same order, if and only if the real values $\varphi_i(x_j)$, $i = 1, \dots, n$, $j = 1, \dots, s$ may be defined so that the matrix*

$$\| \varphi_i(x_j) \|_{\substack{i=1,\dots,n \\ j=1,\dots,k}}$$

has rank k, for each set of $k-s$ distinct points $x_{s+1}, \dots, x_k$ in Q[1].

THEOREM 3'. *Each n-dimensional Chebyshev space of $C(Q)$, which is of order $n-2$ has a Chebyshev subspace of dimension $n-1$ and of the same order*[2].

THEOREM 4'. *The n-dimensional Chebyshev space F_n of order $n-k$ has Chebyshev subspaces F_i, $i = n-1, \dots, n-k+2$ of dimensions $n-1, \dots, n-k+2$ and with the properties $F_i \subset F_{i+1}$ $i = n-1, \dots, n-k+2$ which are of the same orders as F_n, if and only if the real numbers $\varphi_i(x_j)$, $i = 1, \dots, n$, $j = 1, \dots, k-2$ may be defined so that the matrices*

$$\left\| \begin{matrix} \varphi_1(x_1) & \dots & \varphi_n(x_1) \\ \vdots & & \vdots \\ \varphi_1(x_i) & \dots & \varphi_n(x_i) \\ \varphi_1(y_1) & \dots & \varphi_n(y_1) \\ \vdots & & \vdots \\ \varphi_1(y_{k-i}) & \dots & \varphi_n(y_{k-i}) \end{matrix} \right\| , \quad i = 1, \dots k-2$$

have rank k for any distinct points $y_1, \dots, y_{k-i}$ of Q.

[1] A particular case of this theorem was proved in [4] (Theorem 5).

[2] In [4] is given an example for 3-dimensional Chebyshev space of order 0 without 2-dimensional Chebyshev subspaces of order 0.

Definition 5. *The basis $\varphi_1, \dots, \varphi_n$ of the n-dimensional Chebyshev space F_n, which has the order $n-k$, is said to be a Markov basis, if the subspaces of F_n spanned by the functions $\varphi_1, \dots, \varphi_{n-k+i}$, $i = 1, \dots, k$, are Chebyshev subspaces of order $n-k$.*

Theorem 3' and Theorem 4' in conjuncture yield the

Consequence. The n-dimensional Chebyshev space F_n of the order $n-k$ has a Markov basis if and only if the conditions of Theorem 4' are fulfilled.

Definition 6. *The map B of the n-dimensional Chebyshev space F_n of order $n-k$ in the $n-s$-dimensional euclidean space E_{n-s} $(s+k > n+1)$ is said to from a B — transformation if B is onto, linear (additive and homogeneous) and if every $n-k+1$ lienarly independent functions of F_n which have at least $s-(n-k)$ common zeros span an $n-k+1$-dimensional subspace of F_n, which has at least an element φ so that $B\varphi \neq 0$.*

THEOREM 5'. *The linear operator B which map the n-dimensional Chebyshev space F_n of order $n-k$ onto the $n-s$ — dimensional euclidean space E_{n-s} $(s+k < n+1)$ is a B-transformation if and only if there is an s-dimensional subspace F_s of F_n which is a Chebyshev subpsace of the same order as F_n, and for which $B(F_s) = 0$.*

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