

AN ALGEBRA OF THE OPTIMAL PATH IN NETWORKS

by

VASILE PETEANU

Cluj

The problems concerning the determination of the optimal paths in networks have been very much studied lately. There were elaborated various algorithms for the determination of these paths.

In [2] the author makes an algebrical study which holds for many problems of optimal paths. We notice that the optimal path exists in all cases studied in [2]. If we enlarge the class of the studied problems, for instance if in the case of the minimal path problem in a graph we admit the existence of some arcs which have negative lengths, the problem of the optimal path may not have solution. A study on the compatibility was made in [4] and [3] for the problem of the maximal path. In this paper we extend this study to a more general case.

Let $\mathcal{S}$ be a set of elements which has the following properties: An internal law of composition is defined in $\mathcal{S}$ which is termed „addition” and written $\oplus$. The addition is associative

$$(1) \quad a \oplus (b \oplus c) = (a \oplus b) \oplus c,$$

it is commutative

$$(2) \quad a \oplus b = b \oplus a;$$

there exists an element θ , termed the neuter element, characterized by

$$(3) \quad a \oplus \theta = \theta \oplus a = a$$

for all the a in $\mathcal{S}$, and at least one and only one equality of the following two is satisfied,

$$(4) \quad a \oplus b = a, \quad a \oplus b = b.$$

It is also defined in $\mathcal{S}$ a second law of composition which is termed „multiplication” and written $\circ$. This law of composition assigns to each pair a, b of elements of $\mathcal{S}$ the product $a \circ b$ which is again an element of $\mathcal{S}$. The multiplication is associative

$$(5) \quad a \circ (b \circ c) = (a \circ b) \circ c,$$

it is commutative

$$(6) \quad a \circ b = b \circ a$$

there exists a unit element denoted by e , which is characterized by

$$(7) \quad a \circ e = e \circ a = a.$$

The multiplication is distributive relating to the addition

$$(8) \quad a \circ (b \oplus c) = (a \circ b) \oplus (a \circ c)$$

and moreover

$$(9) \quad a \oplus e = a, \quad a \neq e, \quad b \neq \theta \Rightarrow a \circ b \neq b$$

$$(10) \quad a \circ \theta = \theta \circ a = \theta.$$

The set $\mathcal{S}$ which has the properties mentioned above can be ordered. We define $a > b$ if and only if $a \oplus b = a$. Evidently

$$(11) \quad a > b, \quad b > c \Rightarrow a > c$$

$$(12) \quad a > b, \quad b > a \Leftrightarrow a = b$$

$$a > \theta, \text{ for } a \in \mathcal{S}.$$

From (4) it results that whichever would be a, b of $\mathcal{S}$, we have either $a > b$ or $b > a$.

Next properties follow directly:

$$P_1: \quad a > b, \quad c > d \Rightarrow a \oplus c > b \oplus d$$

$$P_2: \quad a > b, \quad c > d \Rightarrow a \circ c > b \circ d$$

We denote

$$\bigoplus_{i=1}^n a_i = a_1 \oplus a_2 \oplus \dots \oplus a_n.$$

Let A, B, C , be the matrices whose elements belong to the set $\mathcal{S}$. The sum of two matrices A and B both having the dimension $m \times n$ will

be a matrix C , of the same dimensions. Its elements are $c_{ij} = a_{ij} \oplus b_{ij}$. The following properties are obvious:

$$P_3: \quad A \oplus (B \oplus C) = (A \oplus B) \oplus C$$

$$P_4: \quad A \oplus B = B \oplus A$$

$$P_5: \quad A \oplus \Theta = \Theta \oplus A = A, \quad \text{where } \Theta \text{ is the matrix with all elements equal to } \theta.$$

The product of two matrices is defined like this:

$$A \circ B = C$$

where

$$c_{ij} = \bigoplus_{k=1}^n a_{ik} \circ b_{kj}.$$

These three matrices have the dimensions $m \times n$, respectively $n \times p$ and $m \times p$.

$$P_6: \quad A \circ (B \circ C) = (A \circ B) \circ C$$

Proof.

$$\begin{aligned} \bigoplus_k a_{ik} \circ \left(\bigoplus_s b_{ks} \circ c_{sj} \right) &= \bigoplus_k \bigoplus_s a_{ik} \circ b_{ks} \circ c_{sj} = \\ &= \bigoplus_s \bigoplus_k a_{ik} \circ b_{ks} \circ c_{sj} = \bigoplus_s \left(\bigoplus_k a_{ik} \circ b_{ks} \right) \circ c_{sj} \end{aligned}$$

$$P_7: \quad A \circ (B \oplus C) = (A \circ B) \oplus (A \circ C)$$

Proof.

$$\bigoplus_k a_{ik} \circ (b_{kj} \oplus c_{kj}) = \left(\bigoplus_k a_{ik} \circ b_{kj} \right) \oplus \left(\bigoplus_k a_{ik} \circ c_{kj} \right).$$

The set of matrices of the same dimensions is partially ordered. We have $A > B$ if and only if $A \oplus B = A$.

It follows directly that

$$P_8: \quad A > B, \quad C > D \Rightarrow A \oplus C > B \oplus D$$

$$P_9: \quad A > B, \quad C > D \Rightarrow A \circ C > B \circ D.$$

In the set of $n \times n$ -dimensional matrices, the matrix E will be called the unit. matrix. Its elements e_{ij} are defined like this:

$$e_{ij} = \begin{cases} \theta & \text{for } i \neq j \\ e & \text{for } i = j. \end{cases}$$

The matrix E has the following property:

$$P_{10}: \quad A \circ E = E \circ A = A$$

where A is an arbitrary $n \times n$ matrix over E .

Proof.

$$\sum_{k=1}^n a_{ik} \circ e_{kj} = \sum_{k=1}^n e_{ik} \circ a_{kj} = a_{ij}.$$

For the square matrices we define

$$A^1 = A, \quad A^p = A^{p-1} \circ A.$$

Evidently

$$A^p \circ A^q = A^{p+q}.$$

Further on we denote by a_{ij}^p the element of the line i and the column j of a matrix A^p .

We denote by $\mathcal{A}$ the set of $n \times n$ - dimensional matrices which has the property that $a_{ii} = e$, $i = 1, 2, \dots, n$. For any integer and positive s we have

$$P_{11}: \quad A \in \mathcal{A} \Rightarrow \sum_{p=1}^s A^p = A^s.$$

Proof.

$$A \in \mathcal{A} \Rightarrow E < A \Rightarrow A < A^2 < \dots < A^s \Rightarrow \sum_{p=1}^s A^p = A^s.$$

Let us consider now the matricial equation

$$(13) \quad X = X \circ A$$

in which $A \in \mathcal{A}$ is a given matrix and $X = \{x_1, \dots, x_n\}$, $x_i \in \mathcal{S}$, $x_i \neq \theta$ $i = 1, 2, \dots, n$ is an n -dimensional line vector which is going to be determined. We have

P_{12} : The equation (13) is equivalent to the inequality

$$(14) \quad X > X \circ A.$$

Proof. It is sufficient to show that (13) follows from (14). In order to prove this, we notice that $A > E \Rightarrow X \circ A > X$ for any X , in accordance with P_9 ; it follows $X > X \circ A > X \Rightarrow X = X \circ A$

P_{13} : The equation (13) is equivalent to the equation

$$(15) \quad X = X \circ A^p$$

whichever should be the natural number p .

Proof. (15) follows from (13).

$$X = X \circ A \Rightarrow X \circ A = X \circ A^2 = \dots = X \circ A^p$$

(13) follows from (15)

$$A^p > A \Rightarrow X \circ A^p > X \circ A \Rightarrow X > X \circ A \Rightarrow X = X \circ A.$$

The next property is clear

P_{14} : The equation (13) is equivalent to the inequality

$$X > X \circ A^p$$

whichever should be the natural number p .

THEOREM 1. *The necessary and sufficient condition for the compatibility of the equation (13) is*

$$(16) \quad A^{n-1} = A^n.$$

Proof. The condition is necessary. We suppose that the equation (13) admits the solution X_0 . According to P_{14} we have

$$(17) \quad X_0 > X_0 \circ A^p \quad p = 1, 2, \dots$$

and hence according to (9)

$$a_{ii} < e \quad i = 1, 2, \dots, n.$$

On the other hand

$$A \in \mathcal{A} \Rightarrow A > E \Rightarrow A^p > E \Rightarrow a_{ii}^p > e$$

hence according to (11)

$$(18) \quad a_{ii}^p = e, \quad i = 1, 2, \dots, n; \quad p = 1, 2, \dots$$

For $p = p_1 + p_2$ we have

$$A^p = A^{p_1} \circ A^{p_2}$$

that is

$$a_{ij}^p = \sum_{k=1}^n a_{ik}^{p_1} \circ a_{kj}^{p_2}.$$

From the former equality it results on the one hand that for any index k we have

$$(19) \quad a_{ij}^p > a_{ik}^{p_1} \circ a_{kj}^{p_2}.$$

and on the other hand, according to (5) there exists an index s such that

$$(20) \quad a_{ij}^p = a_{is}^{p_1} \circ a_{sj}^{p_2}.$$

It results from (19) that whichever would be the natural number p and the indices $k_1, k_2, \dots, k_{p-1}$,

$$(21) \quad a_{ij}^p > a_{ik_1} \circ a_{k_1 k_2} \circ \dots \circ a_{k_{p-1} j}$$

and it results from (20) that the indices $s_1, s_2, \dots, s_{n-1}$ exists such that

$$(22) \quad a_{ij}^n = a_{is_1} \circ a_{s_1 s_2} \circ \dots \circ a_{s_{n-1} j}.$$

In the right side of the equality (22) there exist $n+1$ indices belonging to the set $\{1, 2, \dots, n\}$. At least two indices coincide. Let s_q and s_r be two of the indices which coincide ($q < r$). The element a_{ij}^n can be represented like this:

$$a_{ij}^n = \alpha_{is_q} \circ \beta_{s_q s_r} \circ \gamma_{s_r j}$$

where

$$\alpha_{is_q} = a_{is_1} \circ \dots \circ a_{s_{q-1} s_q}$$

$$\beta_{s_q s_r} = a_{s_q s_{q+1}} \circ \dots \circ a_{s_{r-1} s_r}$$

$$\gamma_{s_r j} = a_{s_r s_{r+1}} \circ \dots \circ a_{s_{n-1} j}.$$

According to (21)

$$a_{is_q}^q > \alpha_{is_q}, \quad a_{s_q s_r}^{r-q} > \beta_{s_q s_r}, \quad a_{s_r j}^{n-r} > \gamma_{s_r j}$$

hence

$$a_{ij}^n < a_{is_q}^q \circ a_{s_q s_r}^{r-q} \circ a_{s_r j}^{n-r}$$

But $s_q = s_r$, and according to (18) we have $a_{s_q s_r}^{r-q} = e$

hence

$$a_{ij}^n < a_{is_q}^q \circ a_{s_r j}^{n-r}.$$

According to (19) we have

$$a_{is_q}^q \circ a_{s_r j}^{n-r} < a_{ij}^{n+q-r}$$

hence

$$a_{ij}^n < a_{ij}^{n+q-r}.$$

As $n > n+q-r$ it results

$$(23) \quad a_{ij}^n < \bigoplus_{p=1}^{n-1} a_{ij}^p$$

hence

$$(24) \quad A^n < \bigoplus_{p=1}^{n-1} A^p = A^{n-1}$$

the last equality being P_{11} .

On the other hand $A > E \Rightarrow A^n > A^{n-1}$ and in view of (24)

$$A^n = A^{n-1}$$

which proves the necessity of the condition (16).

The condition is sufficient. Let X_1 be an arbitrary n -dimensional vector. If $A^{n-1} = A^n$ then

$$(25) \quad X = X_1 \circ A^{n-1}$$

verifies the equation (13). Really we have

$$X_1 \circ A^{n-1} = X_1 \circ A^n = (X_1 \circ A^{n-1}) \circ A$$

which proves that the condition (16) is sufficient.

THEOREM 2. If $A \in \mathcal{A}$, $B \in \mathcal{A}$, $A > B$ and $A^{n-1} = A^n$ then also $B^{n-1} = B^n$.

Proof. According to the theorem 1, $A^{n-1} = A^n$ implies the existence of a vector X_0 for which $X_0 = X_0 \circ A$. On the other hand $A > B \Rightarrow X_0 \circ A > X_0 \circ B \Rightarrow X_0 > X_0 \circ B$ and according to P_{13} , $X_0 = X_0 \circ B$ hence $B^{n-1} = B^n$.

COROLLARY. If $B \in \mathcal{A}$ and $C \in \mathcal{A}$ and $(B \oplus C)^{n-1} = (B \oplus C)^n$ then also $B^{n-1} = B^n$ and $C^{n-1} = C^n$.

Proof. Evidently $B \oplus C \in \mathcal{A}$ and $B \oplus C > B$, $B \oplus C > C$.

Let G be a directed network with n nodes. We assign to each arc (i, j) a real number a_{ij} called "the length" of the arc. Hence the matrix $A = (a_{ij})$ is the distance matrix. If we define the addition $a \oplus b = \max(a, b)$ and the multiplication $a \circ b = a + b$ then the matrix A^{n-1} is the matrix of the maximal paths between the nodes. But if we define $a \oplus b =$

$= \min(a, b)$ and $a \circ b = a + b$ then A^{n-1} is the matrix of the minimal paths. These two cases together with other possible ones are contained in the following table.

N	δ	$\oplus$	θ	$\circ$	e	$>$
1	$[-\infty, +\infty)$	max	$-\infty$	+	0	$\geq$
2	$(-\infty, +\infty]$	min	$+\infty$	+	0	$\leq$
3	$[0, +\infty)$	max	0	$\times$	1	$\geq$
4	$(0, +\infty]$	min	$+\infty$	$\times$	1	$\leq$
5	$[-\infty, +\infty]$	max	$-\infty$	min	$+\infty$	$\geq$
6	$[-\infty, +\infty]$	min	$+\infty$	max	$-\infty$	$\leq$
7	$[-\infty, +\infty)$	max	$-\infty$	max	$-\infty$	$\geq$
8	$(-\infty, +\infty]$	min	$+\infty$	min	$+\infty$	$\leq$

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SOME OBSERVATIONS CONCERNING THE APPLICATION OF THE ELECTRONIC COMPUTERS IN ORDER TO SOLVE NONARITHMETICAL PROBLEMS

by

TEODOR RUS

Cluj

1. — Introduction

The elaboration of a theory using electronic computers in order to solve mathematical problems of nonarithmetical character requires firstly a certain generalisation of the computing notion itself, and, in the same time, a new computing theory answering the purpose. The idea of such a theory was clearly exposed by J. Mc.Carty [22]. But this idea is founded on the usual computing notion, namely on the numerical calculation.

For the electronic computer could become an effective instrument in solving some scientific problems related to the research especially in mathematics or in other fields strongly connected with it, it is necessary to understand the computing notion in a larger sense. This understanding must have in view the fact that the nature of the magnitudes taking part in the computing process can be different from the numerical one, and that the operations performed in the computing process can be different from the common numerical ones.

Some of the desiderata of such a theory are formulated within the different specification of the algorithm notion. But in view using effectively these desiderata in the machanization of that part of the mathematical research process that refers to the effectuation of a certain computing process, having abstract magnitudes, given rules, different from those commonly used the actual computers, or from the numerical ones, it is necessary to reconsider the mathematical theory itself, in the new spirit of the electronic computer. But this would lead to serious difficulties.