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THE GEOMETRY OF THE RELATIONS IN THE SET OF REAL NUMBERS

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1. Introduction

The purpose of this paper is the geometric study of the relations

$$M \subseteq R \times R$$

where R is the set of all real numbers. The algebraic theory of relations has been considered in some works (e.g. [2], [3], [4]).

Definition 1. *If T is a nonempty set, then a relation in T is a subset of the cartesian product $T \times T$:*

$$M \subseteq T \times T.$$

The empty relation ($\emptyset \subseteq T \times T$) is called the impossible relation and the total relation ($T \times T \subseteq T \times T$) the trivial relation. Let us denote by xMy the inclusion $(x, y) \in M$. The product of the relations M and N is defined by

$$M \cdot N = \{(x, y) \mid \exists\, z : xMz \wedge zNy\}. \tag{1.1}$$

Since the relations are the subsets of $T \times T$ we have

THEOREM 1. *The relations in T with the settheoretical operations is a Bool's algebra.*

THEOREM 2. *If M_α ($\alpha \in \mathcal{A}$) is a familly of relations in T and $M \subseteq T \times T$, then*

$$M\Big(\bigcup_{\alpha \in \mathcal{A}} M_\alpha\Big) = \bigcup_{\alpha \in \mathcal{A}} MM_\alpha \tag{1.2}$$

$$(1.3)\qquad (\bigcup_{\alpha\in\mathcal{A}} M_\alpha) M = \bigcup_{\alpha\in\mathcal{A}} M_\alpha M$$

$$(1.4)\qquad M(\bigcap_{\alpha\in\mathcal{A}} M_\alpha) \subseteq \bigcap_{\alpha\in\mathcal{A}} MM_\alpha$$

$$(1.5)\qquad (\bigcap_{\alpha\in\mathcal{A}} M_\alpha) M \subseteq \bigcap_{\alpha\in\mathcal{A}} M_\alpha M.$$

The proof of (1.2) and (1.3) is given in [2]. We prove (1.4). We see that

$$(x, y) \in M(\bigcap_{\alpha\in\mathcal{A}} M_\alpha) \Leftrightarrow \exists z: (x, z) \in M \wedge (z, y) \in \bigcap_{\alpha\in\mathcal{A}} M_\alpha \Leftrightarrow$$

$$\exists z: (x, z) \in M \wedge (z, y) \in M_\alpha \text{ for all } \alpha \in \mathcal{A} \Rightarrow$$

$$\exists z_\alpha(\alpha \in \mathcal{A}): (x, z_\alpha) \in M \wedge (z_\alpha, y) \in M_\alpha \Leftrightarrow$$

$$(x, y) \in MM_\alpha \text{ for all } \alpha \in \mathcal{A} \Leftrightarrow (x, y) \in \bigcap_{\alpha\in\mathcal{A}} MM_\alpha.$$

The proof of (1.5) is complet analogous.

Definition 2. *The algebraic structure of the relations in T with the settheoretical operations and the product is called the algebra of relations.*

In [1] we have given some results of the theory of relations is R

2. Preliminary definitions

Definition 3. *In the set R of real numbers we define the following particular relations:*

The point: $P_{a,b} = (a, b)$

where $a, b \in R$

The straight line: $D_{a,b,c} = \{(x, y) \mid ax + by + c = 0\}$

where $a, b, c \in R$

The interval: $I_{a,b,c,d} = \{(x, y) \mid x = (c - a)t + a, y = (d - b)t + b, 0 \leqslant t \leqslant 1\}$,

where $a, b, c, d \in R \wedge a \leqslant c$

The halfline: $H_{a,b,u} = \{(x, y) \mid x = a + t, y = b + t \cdot \operatorname{tg} u,\ t \geqslant 0\}$

where $a, b \in R$ and $u \in [0, 2\pi)$

The rectangle: $Q_{a,b,c,d} = \{(x, y) \mid a \leqslant x \leqslant c, b \leqslant y \leqslant d\}$

where $a \leqslant c$, $b \leqslant d \in R$.

These relations are sets. Therefore we have e.g. $P_{a,b} = I_{a,b,a,b} = Q_{a,b,a,b}$, $D_{0,0,0} = R \times R$, $D_{0,0,1} = \emptyset$, $H_{a,b,u} \cup H_{a,b,u+\pi} = D_{\operatorname{tg} u, -1, b - a \operatorname{tg} u}$ (where $u \in [0, \pi)$).

3. The formulae for the product of some particular relations

THEOREM 3. *If $M \subseteq R^2$, then*

$$(3.1)\qquad P_{a,b} \cdot M = \{(a, y) \mid (b, y) \in M\}$$

and

$$(3.2)\qquad M \cdot P_{a,b} = \{(x, b) \mid (x, a) \in M\}.$$

Proof. We have:

$$(x, y) \in P_{a,b} \cdot M \Leftrightarrow \exists z: (x, z) \in P_{a,b} \wedge (z, y) \in M \Leftrightarrow$$

$$\Leftrightarrow x = a \wedge z = b \wedge (b, y) \in M \Leftrightarrow (x, y) = (a, y) \text{ where } (b, y) \in M.$$

The proof of (3.2) is similar.

The formulae (1.3), (3.1) and (3.2) can suggest the consideration of the following formulae for the computation of the product of two relations

$$(3.3)\qquad M \cdot N = \bigcup_{(a,b)\in M} P_{a,b} \cdot N = \bigcup_{(a,b)\in N} MP_{a,b}.$$

THEOREM 4. *For a point and an interval we have*

$$(3.4)\qquad P_{a,b} \cdot I_{a',b',c',d'} = \begin{cases} P_{a, \frac{(d'-b')(b-a')}{c'-a'} + b'} \text{ if } c' \neq a' \wedge 0 < \frac{b-a'}{c'-a'} < 1 \\ I_{a,b',a,d'} \text{ if } c' = a' = b \\ \emptyset \text{ in the other cases} \end{cases}$$

and

$$(3.5)\qquad I_{a,b,c,d} \cdot P_{a',b'} = \begin{cases} P_{\frac{(b'-a)(a'-b)}{d-b} + a, b'}, \text{ if } d \neq b \wedge 0 \leqslant \frac{a'-b}{d-b} \leqslant 1 \\ I_{a,a',c,a'} \text{ if } d = b = a' \\ \emptyset \text{ in the other cases} \end{cases}$$

Proof. $P_{a,b}I_{a',b',c',d'} = \{(a, y) | (b, y) \in I_{a',b',c',d'}\} =$

$= \{(a, y) | b = (c' - a')t + a', y = (d' - b')t + b', \ 0 \leqslant t \leqslant 1\} =$

$$= \begin{cases} \left\{(a, y) | y = \dfrac{(d' - b')(b - a')}{c' - a'} + b'\right\} & \text{if } c' \neq a' \wedge 0 \leqslant \dfrac{b - a'}{c' - a'} \leqslant 1 \\ \emptyset & \text{if } c' \neq a' \wedge \dfrac{b - a'}{c' - a'} \notin [0, 1] \\ \{(a, y) | b = a', y = (d' - b')t + b', \ 0 \leqslant t \leqslant 1\} & \text{if } c' = a' \end{cases}$$

$$= \begin{cases} P_{a, \frac{(d' - b')(b - a')}{c' - a'} + b'} & \text{if } c' \neq a' \wedge 0 \leqslant \dfrac{b - a'}{c' - a'} \leqslant 1 \\ I_{a, b', a, d'} & \text{if } c' = a' = b \\ \emptyset & \text{in other cases.} \end{cases}$$

$$I_{a,b,c,d}P_{a',b'} = \{(x, b') | (x, a') \in I_{a,b,c,d}\} =$$

$= \{(x, b') | x = (b - a)t + a, a' = (d - b)t + b, 0 \leqslant t \leqslant 1\} =$

$$= \begin{cases} \left\{(x, b') | x = \dfrac{(b - a)(a' - b)}{d - b} + a, 0 \leqslant \dfrac{a' - b}{d - b} \leqslant 1\right\} & \text{if } d \neq b \\ \{(x, b') | x = (b - a)t + a, a' = b\} & \text{if } d = b \end{cases}$$

$$= \begin{cases} P_{\frac{(b - a)(a' - b)}{d - b} + a, b'} & \text{if } d \neq b, 0 \leqslant \dfrac{a' - b}{d - b} \leqslant 1 \\ I_{a, a', c, a'} & \text{if } d = b = a' \\ \emptyset & \text{in other cases.} \end{cases}$$

Thus the theorem is proved.

Corollary. *We have*

$$P_{a,b}P_{a',b'} = \begin{cases} P_{a,b'} & \text{if } a' = b \\ \emptyset & \text{if } a' \neq b. \end{cases} \tag{3.6}$$

The statement of corollary follows from (3.4):

$$P_{a,b}P_{a',b'} = P_{a,b}I_{a',b',a',b'} = \begin{cases} I_{a,b',a',b'} & \text{if } a' = b \\ \emptyset & \text{if } a' \neq b \end{cases} = \begin{cases} P_{a,b'} & \text{if } a' = b \\ \emptyset & \text{if } a' \neq b. \end{cases}$$

For the product of two intervals the following lemmas are proved

Lemma 1. *If* $c' \neq a' \wedge b \neq d \wedge [u, v] \neq \emptyset$

where

$$[u, v] = \left\{t \middle| \frac{(d - b)t + b - a'}{c' - a'} \in [0, 1]\right\} \cap [0, 1]$$

then

$$I_{a,b,c,d} \cdot I_{a',b',c',d'} = \tag{3.7}$$

$$= I_{(c - a)u + a, \frac{(d' - b')((d - b)u + b - a')}{c' - a'} + b', (c - a)v + a, \frac{(d' - b')((d - b)v + b - a')}{c' - a'} + b'}$$

Proof. From (3,4) and (1.3) it follows

$$I_{a,b,c,d} \cdot I_{a',b',c',d'} = \bigcup_{t \in [0, 1]} P_{(c - a)t + a, (d - b)t + b} \cdot I_{a',b',c',d'} =$$

$$= \bigcup_{t \in [0, 1] \wedge \frac{(d - b) + b - a'}{c' - a'} \in [0, 1]} P_{(c - a)t + a, \frac{(d' - b')((d - b)t + b - a')}{c' - a'}}$$

$$= \bigcup_{t \in [u, v]} P_{(c - a)t + a, \frac{(d' - b')((d - b)t + b - a')}{c' - a'} + b'} =$$

$$= I_{(c - a)u + a, \frac{(d' - b')((d - b)u + b - a')}{c' - a'} + b', (c - a)v + a, \frac{(d' - b')((d - b)v + b - a')}{c' - a'}}$$

and thus the lemma 1 is proved.

Lemma 2. *If for the intervals* $I_{a,b,c,d}$ *and* $I_{a',b',c',d'}$ *we have*

$$\frac{b - a'}{c' - a'} \in [0, 1] \wedge b = d \wedge c' \neq a'$$

then

$$I_{a,b,c,d}I_{a',b',c',d'} = I_{a,b,c,b}I_{a',b',c',d'} = \tag{3.8}$$

$$= I_{a, \frac{(d' - b')(b - a')}{c' - a'} + b', c, \frac{(d' - b')(b - a')}{c' - a'} + b'}$$

Proof. We see that

$$I_{a,b,c,b}I_{a',b',c',d'} = \bigcup_{t \in [0, 1]} P_{(c - a)t + a, b} \cdot I_{a',b',c',d'} =$$

$$= \bigcup_{t \in [0, 1]} P_{(c - a)t + a, \frac{(d' - b')(b - a')}{c' - a'} + b'} =$$

$$= I_{a, \frac{(d' - b')(b - a')}{c' - a'} + b', c, \frac{(d' - b')(b - a')}{c' - a'} + b'}$$

and the lemma is proved.

Lemma 3. *If for the intervales $I_{a,b,c,d}$ and $I_{a',b',c',d'}$*

$$a' = c' \wedge b \neq d \wedge \frac{a'-b}{d-b} \in [0, 1]$$

then

$$I_{a,b,c,d} I_{a',b',c',d'} = I_{a,b,c,d} I_{a',b',a',d'} = \tag{3.9}$$

$$= I_{\frac{(c-a)(a'-b)}{d-b}+a,\, b',\, \frac{(c-a)(a'-b)}{d-b}+a,\, d'}.$$

Proof. We have

$$I_{a,b,c,d} I_{a',b',a',d'} = \bigcup_{t\in[0,1]} P_{(c-a)t+a,\,(d-b)t+b}\, I_{a',b',a',d'} =$$

$$= \bigcup_{t\in[0,1]\wedge a'=(d-b)t+b} I_{(c-a)t+a,\,b',\,(c-a)t+a,\,d'} = I_{\frac{(c-a)(a'-b)}{d-b}+a,\, b',\, \frac{(c-a)(a'-b)}{d-b}+a,\, d'}$$

and the lemma is proved.

Lemma 4. *If $a' = c' = b = d$ then*

$$I_{a,b,c,d} I_{a',b',c',d'} = I_{a,b,c,b} I_{b,b',b,d'} = \tag{3.10}$$

$$= Q_{\min(a,c),\, \min(b',d'),\, \max(a,c),\, \max(b',d')}.$$

Proof. We have

$$I_{a,b,c,b} I_{b,b',b,d'} = \bigcup_{t\in[0,1]} P_{(c-a)t+a,\,b} I_{b,b',b,d'} =$$

$$= \bigcup_{t\in[0,1]} I_{(c-a)t+a,\,b',\,(c-a)t+a,\,d'} =$$

$$= \{P \mid P \in I_{u,b',u,d'} \text{ where } u \in [a, c] \wedge [c, a]\} =$$

$$= \{P \mid P \in I_{u,b',u,d}, \text{ where } u \in [\min(a, c), \max(a, c)]\} =$$

$$= Q_{\min(a,c),\, \min(b',d'),\, \max(a,c),\, \max(b',d')},$$

and the lemma 4 is proved.

From the lemmas 1–4 and the theorem 4 we can easy derive the following theorem:

THEOREM 5. *For the product of two intervals the following formula holds:*

$$I_{a,b,c,d} I_{a',b',c',d'} = \tag{3.11}$$

$$\begin{cases} I_{(c-a)u+a,\, q(u),\, (c-a)v+a,\, q(v)}, \\ \quad \text{if } (c'-a')(b-d) \neq 0, [u, v] \neq \emptyset \text{ and} \\ \quad [u, v] = \left\{t \mid \frac{(d-b)t+b-a'}{c'-a'} \in [0, 1]\right\} \cap [0, 1), \text{ where} \\ \quad q(x) = \frac{(d'-b')((d-b)x+b-a')}{c'-a'} + b' \\ I_{a,\, \frac{(d'-b')(b-a')}{c'-a'}+b',\, c,\, \frac{(d'-b')(b-a')}{c'-a'}+b'} \\ \quad \text{if } \frac{b-a'}{c'-a'} \in [0, 1] \wedge b = d \wedge c' \neq a' \\ I_{\frac{(c-a)(a'-b)}{d-b}+a,\, b',\, \frac{(c-a)(a'-b)}{d-b}+a,\, d'} \\ \quad \text{if } a' = c' \wedge b \neq d \wedge \frac{a'-b}{d-b} \in [0, 1] \\ Q_{\min(a,c),\, \min(b',d'),\, \max(a,c),\, \max(b',d')} \\ \quad \text{if } a' = c' = b = d \\ \emptyset \text{ otherwise} \end{cases}$$

THEOREM 6. *For an straight line and a point the product is given by the formulae*

$$P_{a,b} D_{a',b',c'} = \begin{cases} P_{a,\, -\frac{a'b+c'}{b'}} & \text{if } b' \neq 0 \\ D_{1,0,-a} & \text{if } b' = 0 \wedge a'b + c' = 0 \\ \emptyset & \text{otherwise} \end{cases} \tag{3.12}$$

and

$$D_{a,b,c} P_{a',b'} = \begin{cases} P_{-\frac{a'b+c}{a},\, b'} & \text{if } a' \neq 0 \\ D_{0,1,-b'} & \text{if } a' = 0 \wedge a'b + c = 0 \\ \emptyset & \text{otherwise} \end{cases} \tag{3.13}$$

Proof. From (3.1):

$$P_{a,b}D_{a',b',c'} = \{(a,y) \mid (b,y) \in D_{a',b',c'}\} = \{(a,y) \mid a'x + b'y + c' = 0\} =$$

$$= \begin{cases} \left\{\left(a, -\dfrac{a'b+c'}{b'}\right)\right\} & \text{if } b' = 0 \\ \{(a,y) \mid -\infty < y < +\infty\} & \text{if } b' = 0 \wedge a'b + c' = 0 \\ \emptyset & \text{otherwise} \end{cases} =$$

$$= \begin{cases} P_{a, -\frac{a'b+c}{b'}} & \text{if } b' \neq 0 \\ D_{1,0,-a} & \text{if } b' = 0 \;\; a'b + c' = 0 \\ \emptyset & \text{otherwise} \end{cases}$$

From (3.2):

$$D_{a,b,c}P_{a',b'} = \{(x,b') \mid (x,a') \in D_{a,b,c}\} = \{(x,b') \mid ax + ba' + c = 0\} =$$

$$= \begin{cases} \left\{\left(-\dfrac{a'b+c}{a}, b'\right)\right\} & \text{if } a \neq 0 \\ \{(x,b') \mid -\infty < x < +\infty\} & \text{if } a = 0 \wedge a'b + c = 0 \end{cases} =$$

$$= \begin{cases} P_{-\frac{a'b+c}{a}, b'} & \text{if } a \neq 0 \\ D_{0,1,-b'} & \text{if } a = 0 \wedge a'b + c = 0 \\ \emptyset & \text{otherwise} \end{cases}$$

THEOREM 7. *For two straight lines the product is given by the formula*

$$(3.14) \qquad D_{a,b,c} \cdot D_{a',b',c'} =$$

$$= \begin{cases} D_{aa', -bb', a'c-bc'} & \textit{if } b \neq 0 \vee a' \neq 0 \\ P_{-\frac{c}{a}, -\frac{c'}{b'}} & \textit{if } a \neq 0 \wedge b' \neq 0 \wedge b = a' = 0 \\ D_{0,b',c'} & \textit{if } a = b = c = a' = 0 \wedge b' \neq 0 \\ D_{a,0,c} & \textit{if } b = a' = b' = c' = 0 \wedge a \neq 0 \\ \emptyset & \textit{if } (b = a' = 0) \wedge ((a = 0 \wedge c \neq 0) \vee \\ & \quad \vee (b' = 0 \wedge c' \neq 0)) \\ R \times R & \textit{if } a = a' = b = b' = c = c' = 0. \end{cases}$$

Proof. We have the equivalences

$$(x,y) \in D_{a,b,c}D_{a',b',c'} \Leftrightarrow \exists z : ax + bz + c = 0 \wedge a'z + b'y + c' = 0$$

if $b \neq 0 \vee a' \neq 0$, then

$$(x,y) \in D_{a,b,c}D_{a',b',c'} \Leftrightarrow \begin{cases} \exists z : z = -\dfrac{ax+c}{b} = -\dfrac{b'y+c'}{a'} & \textit{if } a'b \neq 0 \\ \exists z : z = -\dfrac{ax+c}{b} b'y + c' = 0 & \text{if } a' = 0 \wedge b \neq 0 \\ \exists z : ax + c = 0 \wedge z = -\dfrac{b'y+c'}{a'} & \text{if } a' \neq 0 \wedge b = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} (x,y) \in D_{aa', -bb', a'c-bc'} & \text{if } a'b \neq 0 \\ (x,y) \in D_{0,0,0} = R \times R & \text{if } (a' = b' = c' = 0 \wedge b \neq 0) \vee \\ & \quad \vee (a = c = b = 0 \wedge a' \neq 0) \\ (x,y) \in D_{0,0,1} = \emptyset & \text{if } (a' = b' = 0 \wedge b \neq 0 \wedge c' \neq 0) \vee \\ & \quad \vee (b = a = 0 \wedge a' \neq 0 \wedge c \neq 0) \\ (x,y) \in D_{0,b',c'} & \text{if } a' = 0 \wedge b \neq 0 \wedge b' \neq 0 \\ (x,y) \in D_{a,0,c} & \text{if } a' \neq 0 \wedge b = 0 \wedge a \neq 0. \end{cases} \Leftrightarrow$$

$$\Leftrightarrow (x,y) \in D_{aa', -bb', a'c-bc'} \text{ if } b \neq 0 \vee a' \neq 0.$$

If $b = a' = 0$ then $(x,y) \in D_{a,b,c}D_{a',b',c'} \Leftrightarrow (x,y) \in D_{a,0,c}D_{0,b',c'} \Leftrightarrow$

$$\Leftrightarrow \exists z : (x,z) \in D_{a,0,c} \wedge (z,y) \in D_{0,b',c'} \Leftrightarrow \exists z : ax + c = 0 \wedge b'y + c' = 0 \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} (x,y) \in \emptyset & \text{if } (a = 0 \wedge c \neq 0) \vee (b' = 0 \wedge c' \neq 0) \\ D_{a,0,c} & \text{if } a \neq 0 \wedge b' = c' = 0 \\ D_{0,b',c} & \text{if } a = c \wedge b' \neq 0 \\ P_{-\frac{c}{a}, -\frac{c'}{b'}} & \text{if } a \neq 0 \wedge b' \neq 0 \\ R \times R & \textit{if } a = c = b' = c' = a' = b' \end{cases}$$

and thus the theorem is proved.

By definition the relations $D_{a,b,c}$ (for $a, b, c \in R$) are straight lines, it follows that $D_{0,0,0} = R \times R$ and $D_{0,0,1} = \emptyset$ are straight lines (in generalized interpretation).

THEOREM 8. *For two rectangles the product is given by*

$$(3.15) \qquad Q_{a,b,c,d} \cdot Q_{a',b',c',d'} = \begin{cases} Q_{a,b',c,d'} & \textit{if } d \geqslant a' \wedge b \leqslant c' \\ \emptyset & \textit{if } d < a' \vee b > c' \end{cases}$$

Proof. We have

$$Q_{a,b,c,d}Q_{a',b',c',d'} = \bigcup_{u\in[a,c]\wedge v\in[b',d']} I_{u,b,u,d}\cdot I_{a',v,c',v} =$$

$$= \begin{cases} \bigcup\limits_{u\in[a,c],\, v\in[b',d']} P_{u,v} & \text{if } [a',c'] \cap [b,d] \neq \emptyset \\ \emptyset & \text{if } [a',c'] \cap [b,d] = \emptyset \end{cases} =$$

$$= \begin{cases} Q_{a,b',c,d'} & \text{if } [a',c'] \cap [b,d] \neq \emptyset \\ \emptyset & \text{if } [a',c'] \cap [b,d] = \emptyset \end{cases} =$$

$$= \begin{cases} Q_{a,b',c,d'} & \text{if } d \geqslant a' \wedge b \leqslant c' \\ \emptyset & \text{if } d < a' \vee b > c' \end{cases}$$

and the theorem is proved.

THEOREM 9. *The product of two squares have the area* 0 *or* $\sqrt{AA'}$ *where* A *and* A' *are respectively the areas of the two squares.*

Proof. If the squares are $Q_{a,b,a+k,b+k}$ and $Q_{a',b',a'+k',b'+k'}$ and

$$[a', a'+k'] \cap [b, b+k] \neq 0$$

then

$$Q_{a,b,a+k,b+k}\, Q_{a',b',a'+k',b'+k'} = Q_{a,b,a+k,b'+k'}.$$

We denote by αQ the area of square Q and we have in this case for areas

$$\alpha Q_{a,b,a+k,b+k} = k^2, \quad \alpha Q_{a',b',a'+k',b'+k'} = k'^2, \quad \alpha Q_{a,b,a+k,b+k'} = kk'.$$

If $[a', a'+k'] \cap [b, b+k] = \emptyset$ then the area of product is 0, as it was to be proved.

THEOREM 10. *If the rectangles* Q *and* Q' *are similar rectangles with the areas* A *and* A' *then the area of product is*

$$\alpha QQ' = \begin{cases} 0 & \text{if } QQ' = \emptyset \\ \sqrt{AA'} & \text{if } QQ' \neq \emptyset. \end{cases} \tag{3.16}$$

Proof. Let $Q = Q_{a,b,a+s,b+t}$, $Q' = Q_{c,d,c+sk,d+tk}$ be the two rectangles, thus $A = st$ and $A' = stk^2$. If $QQ' \neq \emptyset$ then $QQ' = Q_{a,d,a+s,d+kt}$ and $\alpha QQ' = skt = \sqrt{AA'}$, as it was to be proved.

The interval $I_{a,b;u,T}$ is defined by

$$I_{a,b;u,T} = \{(x,y) \mid x = a + t\cos u,\ y = b + t\sin u,\ 0 \leqslant t \leqslant T\} = \\ = I_{a,b,a+T\cos u,b+T\sin u} = I_{a+T\cos u,\, b+T\sin u;\, u+\pi,\, T} \tag{3.17}$$

and therefore

$$I_{a,b,c,d} = I_{a,b;\,\operatorname{arctg}\frac{d-b}{c-a},\ \sqrt{(c-a)^2+(d-b)^2}}. \tag{3.18}$$

For $T \to \infty$ we have

$$H_{a,b,u} = I_{a,b;u,\infty}. \tag{3.19}$$

THEOREM 11. *The product of a point and of an interval is*

$$P_{a,b} I_{a',b';u',T'} = \begin{cases} P_{a,\, b'+(b-a')\operatorname{tg} u'} & \text{if } \cos u' \neq 0 \wedge \dfrac{b-a'}{\cos u'} \in [0, T'] \\ I_{a,b';u',T'} & \text{if } \cos u' = 0 \wedge b = a' \\ \emptyset & \text{otherwise} \end{cases} \tag{3.20}$$

and

$$I_{a,b;u,T} P_{a',b'} = \begin{cases} P_{a+(a'-b)\operatorname{ctg} u,\, b'} & \text{if } \sin u \neq 0 \wedge \dfrac{a'-b}{\sin u} \in [0, T] \\ I_{a,b';u,T} & \text{if } \sin u = 0 \wedge a' = b \\ \emptyset & \text{otherwise.} \end{cases} \tag{3.21}$$

Proof. The equivalences $(x,y) \in P_{a,b}I_{a',b';u',T} \Leftrightarrow x = a \wedge (b,y) \in \in I_{a',b';u',T} \Leftrightarrow x = a \wedge b = a' + t\cos u',\ y = b' + t\sin u' \Leftrightarrow$

$$\Leftrightarrow \begin{cases} x = a \wedge y = b' + (b-a')\operatorname{tg} u' \wedge \cos u' \neq 0 \wedge \dfrac{b-a'}{\cos u'} \in [0, T] \\ x = a \wedge y = b' + t\sin u' \wedge b = a' \\ (x,y) \in \emptyset \text{ otherwise} \end{cases}$$

and $(x,y) \in I_{a,b;u,T} P_{a',b'} \Leftrightarrow y = b \wedge x = a + t\sin u \wedge a' = b + t\cos u \Leftrightarrow$

$$\Leftrightarrow \begin{cases} x = a + (a'-b)\operatorname{ctg} u \wedge y = b \wedge \dfrac{a'-b}{\cos u} \in [0, T] \wedge \cos u \neq 0 \\ x = a + t\sin u \wedge y = b \wedge \cos u = 0 \wedge a' = b \\ (x,y) \in \emptyset \text{ otherwise} \end{cases}$$

proved the theorem.

Corollary. *The product of a point and of a half-line is given by the formulae*

$$(3.22)\quad P_{a,b}H_{a',b',u'} = \begin{cases} P_{a,b'+(b-a')\mathrm{tg}\,u'} & \text{if } \cos u' \neq 0 \text{ and } \dfrac{b-a'}{\cos u'} \geqslant 0 \\ H_{a,b',u'} & \text{if } \cos u' = 0 \text{ and } b = a' \\ \varnothing & \text{otherwise} \end{cases}$$

and

$$(3.23)\quad H_{a,b,u}P_{a',b'} = \begin{cases} P_{a+(a'-b)\mathrm{ctg}\,u,\,b'} & \text{if } \sin u \neq 0 \text{ and } \dfrac{a'-b}{\sin u} \geqslant 0 \\ H_{a,b',u} & \text{if } \sin u = 0 \text{ and } a' - b = 0 \\ \varnothing & \text{otherwise} \end{cases}$$

Lemma 5. *If* $\sin u \cos u' \neq 0$ *and* $[v, w] \neq \varnothing$ *where*

$$[u, w] = \left\{ t \,\middle|\, \frac{\sin u}{\cos u'} t + \frac{b-a'}{T\cos u'} \in [0,1] \right\} \cap [0, 1]$$

then

$$(3.24)\quad I_{a,b;u,T}I_{a',b';u',T'} = I_{Tv\cos u+a,\,(Tv\sin u+b-a')\mathrm{tg}\,u'+b',\,Tw\cos u+a,\,(Tw\sin u+b-a')\mathrm{tg}\,u'+b'}$$

Proof. From Lemma 1 and formula (3.13) we have

$$I_{a,b;u,T}I_{a',b';u',T'} = I_{a,b,a+T\cos u,\,b+T\sin u}\, I_{a',b',a'+T'\cos u',\,b'+T'\sin u'} =$$

$$= {}_{Tv\cos u+a,\,(Tv\sin u+b-a')\mathrm{tg}\,u'+b',\,Tw\cos u+a\;(Tw\sin u+b-a')\mathrm{tg}\,u'+b'}.$$

Lemma 6. *If*

$$\cos u' = b - a' = 0$$

then

$$I_{a,b;u,T} \cdot I_{a',b';u',T'} = Q_{a,b',a+T\cos u,\,b'+T}.$$

Proof.

$$I_{a,b;u,T}I_{a',b';u',T'} = \bigcup_{P_{u,w}\in I_{a,b;u,T}} P_{u,w}I_{a',b';u',T'} = \bigcup_{P_{u,w}\in I_{a,b;u,T}} I_{u,b';u',T'} =$$

$$= \bigcup_{t\in[0,T]} I_{a+t\cos u,\,b';u',T'} = Q_{a,b',a+t\cos\; u'\,b'+T}.$$

The interval J is defined by

$$(3.25)\quad I_{a-S\cos u,\,b-S\sin u;\,u,\,S+T} = J_{a,b;u,S,T}$$

where $S, T \geqslant 0$. In particular in this way can be obtained the following formulae

$$(3.26)\quad J_{a,b;u,0,T} = I_{a,b;u,T}$$

$$(3.27)\quad J_{a,b;u,\infty,\infty} = D_{\sin u,\,-\cos u,\,b\cos u - a\sin u}$$

$$(3.28)\quad J_{a,b;u,0,\infty} = H_{a,b,u}$$

$$(3.29)\quad J_{a,b;u,0,0} = P_{a,b}$$

$$(3.30)\quad J_{a,b;u,S,T} = J_{a,b;u+\pi,T,S}$$

$$(3.31)\quad D_{a,b,c} = \begin{cases} J_{\frac{c}{a},0;\,-\mathrm{arctg}\frac{a}{b},\,\infty,\infty} & \text{if } a \neq 0 \wedge b \neq 0 \\ J_{0,b;0,\infty,\infty} & \text{if } a = 0 \end{cases}$$

4. The radical from relations

Definition 4. *If* $A \subseteq R \times R$ *and* $\mathcal{A}$ *is a familly of relations*, then

$$(4.1)\quad \sqrt[n]{A} = \{M \mid M^n = A \wedge M \in \mathcal{A}\}$$

is called the n-radical of A by $\mathcal{A}$ *(where* $M^n = \underbrace{M \cdot M \ldots M}_{n}$*).*

For the complex numbers $\alpha = a + a_1 i$, $\beta = b + b_1 i$ and $\gamma = c + c_1 i$ by definition

$$(4.2)\quad D_{\alpha,\beta,\gamma}$$

is called *complex straight-line* and it is a complex relation.

If $\beta \neq 0 \vee \alpha' \neq 0$ then by definition (see 3.14)

$$(4.3)\quad D_{\alpha,\beta,\gamma}D_{\alpha',\beta',\gamma'} = D_{\alpha\alpha',\,-\beta\beta',\,\alpha'\gamma-\beta\gamma'}.$$

The line

$$(4.4)\quad i_R = D_{1,i,0}$$

is called the complex unit-line. From (4.3) we have

$$(4.5)\quad i_R^2 = D_{1,i,0} \cdot D_{1,i,0} = D_{1,1,0}.$$

The necessary and sufficient condition that

$$D_{\alpha,\beta,\gamma}D_{\alpha',\beta',\gamma'} = \text{real relation}$$

is given by

$$(4.6)\qquad \begin{cases} aa_1' + a'a_1 = 0 \\ bb_1' + b'b_1 = 0 \\ a_1'c + a'c_1 = bc_1' + b_1c' \end{cases} \quad \text{or} \quad \begin{cases} aa' - a_1a_1' = 0 \\ c_1c_1' - cc' = 0 \\ a'c + b_1c_1' = a_1'c_1 + bc' \end{cases}$$

THEOREM 12. *If $a, b, c \in R$, $a > 0$, $b < 0$ and $a^2 \neq b^2$ then*

$$(4.7)\qquad \sqrt{D_{a,b,c}} = \left\{ D_{\sqrt{a}, \sqrt{-b}, \frac{c}{\sqrt{a}-\sqrt{-b}}},\ D_{\sqrt{a}, -\sqrt{-b}, \frac{c}{\sqrt{a}+\sqrt{-b}}} \right\}.$$

Proof. From (3.14)

$$D^2_{u,v,w} = D_{u^2, -v^2, uw - vw}.$$

From

$$D_{u^2, -v^2, uw - vw} = D_{a,b,c}$$

it follows

$$u = \pm\sqrt{a},\ v = \pm\sqrt{-b},\ w = \frac{c}{u - v}.$$

As for $k \neq 0$ we have $D_{a,b,c} = D_{ak,bk,ck}$ it follows

$$D^2_{u,v,w} = D_{a,b,c} \Leftrightarrow D_{u,v,w} \in \left\{ D_{\sqrt{a}, \sqrt{-b}, \frac{c}{\sqrt{a}-\sqrt{-b}}},\ D_{\sqrt{a}, -\sqrt{-b}, \frac{c}{\sqrt{a}+\sqrt{-b}}} \right\}$$

and the theorem is proved.

THEOREM 13. *If D, D' and D'' are real lines and*

$$D^2 = D',\ D''^2 = D'$$

then the lines D, D' and D'' will intersect in one point of the first bisector $D_{1,-1,0}$.

The proof of the theorem is given in.[1]

THEOREM 14. *If $D_{a,b,c}$ is a real line and $a > 0$, $b > 0$, then*

$$\sqrt{D_{a,b,c}} = \left\{ D_{\sqrt{a}, \sqrt{b}, \frac{c\sqrt{a}}{a+b}}\, i_R D_{1,1,\frac{c}{a+b}},\ D_{\sqrt{a}, -\sqrt{b}, \frac{c\sqrt{a}}{a+b}}\, i_R D_{1,1,\frac{c}{a+b}} \right\}.$$

Proof. From (4.3) $D_{x,y,z} i_R D_{1,1,u} =$

$$= D_{x,y,z} D_{1,i,0} D_{1,1,u} = D_{x,-yi,z} D_{1,1,u} = D_{x,yi,z+yui}.$$

For $a > 0$, $b > 0$

$$\sqrt{D_{a,b,c}} = \left\{ D_{\sqrt{a}, i\sqrt{b}, \frac{c\sqrt{a}}{a+b} + \frac{c\sqrt{b}}{a+b}\cdot i},\ D_{\sqrt{a}, -i\sqrt{b}, \frac{c\sqrt{a}}{a+b} - \frac{c\sqrt{b}}{a+b} i} \right\}.$$

Then

$$x = \sqrt{a},\ y = \sqrt{b},\ z = \frac{c\sqrt{a}}{a+b},\ yu = \frac{c\sqrt{b}}{a+b},\ u = \frac{c}{a+b}$$

and

$$x = \sqrt{a},\ y = -\sqrt{b},\ z = \frac{c\sqrt{a}}{a+b},\ yu = \frac{-c\sqrt{b}}{a+b},\ u = \frac{c}{a+b},$$

as it was to be proved.

5. Some equations connecting relations

In the above section, we derived (for $a > 0$, $b < 0$ and $a^2 \neq b^2$) that the line-roots of the equation $D^2_{u,v,w} = D_{a,b,c}$ are

$$D_{\sqrt{a}, \sqrt{-b}, \frac{c}{\sqrt{a}-\sqrt{-b}}} \text{ and } D_{a, -\sqrt{-b}, \frac{c}{\sqrt{a}-\sqrt{-b}}}.$$

In this section we consider the set of some equations with relations.

THEOREM 15. *The solution of the equation*

$$(5.1)\qquad MN = \emptyset$$

is given by

$$(5.2)\qquad M = \{(x, y) \mid y \in A\} \cap C,\ N = \{(x, y) \mid x \in B\} \cap D$$

where A and B are sets of real numbers, C and D are relations and

$$A \cap B = \emptyset.$$

We denote this solution by

$$(5.3)\qquad S_{A,B,C,D}.$$

Proof. Using (1.2) we obtain $MN = \bigcup_{P \in M, Q \in N} P \cdot Q$. If $P \in \{(x, y) \mid y \in A\} \cap C$ and $Q \in \{(x, y) \mid x \in B\} \cap D$ then $P = P_{u,y}$ and $Q = P_{x,v}$ where $u, v \in R$ and $x \in A$, $y \in B$, therefore $x \neq y$ and $P \cdot Q = \emptyset$. When $MN = \emptyset$, then

$$A = \{y \mid \exists x : (x, y) \in M\}$$

$$B = \{x \mid \exists y : (x, y) \in N\}$$

$$C = M \text{ and } D = M.$$

This will complete the proof of our theorem.

THEOREM 16. *The solution of the equation*

$$M^2 = \emptyset \tag{5.4}$$

is given by

$$M = \{(x, y) \mid y \in A\} \cap \{(x, y) \mid x \in B\} \cap C \tag{5.5}$$

where $A, B \subseteq R$, $A \cap B = \emptyset$ *and* $C \subseteq R \times R$.

This theorem follows from theorem 15.

THEOREM. 17. *If* $A = \{P_{a_1, b_1}, P_{a_2, b_2}, \ldots, P_{a_n, b_n}\}$ *is a set of n points then a particular solution of the equation*

$$MN = A \tag{5.6}$$

is

$$M = \{P_{a_1, x_1}, \ldots, P_{a_n, x_n}\}$$
$$N = \{P_{x_1, b_1}, \ldots, P_{x_n, b_n}\} \tag{5.7}$$

where $x_j \neq x_k$ $(j, k = 1, 2, \ldots, n;\ j \neq k)$.

Proof. This follows from the representation $MN = \bigcup_{P \in M, N \in Q} PQ = A$ and therefore $M = \{P_{a_j, x_j} \mid j = 1, \ldots, n\}$, $N = \{P_{x_j, b_j} \mid j = 1, \ldots, n\}$.

In this case

$$MN = \Big(\bigcup_{j=1,\ldots,n} P_{a_j, x_j} P_{x_j, b_j}\Big) \cup \Big(\bigcup_{\substack{j, k = 1, \ldots, n \\ j \neq k}} P_{a_j, x_j} P_{x_k, b_k}\Big) = A \cup \Big(\bigcup_{\substack{j, k = 1, \ldots, n \\ j \neq k}} P_{a_j, x_j} P_{x_k, b_k}\Big)$$

and $MN = A$ when for $j \neq k$ we have

$$P_{a_j, x_j} P_{x_k, b_k} = \emptyset$$

which is valid when $x_j \neq x_k$ $(j \neq k)$.

Definition 5. *The solution* $M = M'$, $N = N'$ *of the equation*

$$MN = A$$

is called minimal if for $B, C \subseteq R^2$ *and* $B \cup C \neq \emptyset$ *we have*

$$(M' - A)(N' - C) \neq A.$$

In assumption of this definition the solutions (5.7) of equation (5.6) are minimal solutions.

THEOREM 18. *If the equation* $MN = A$ *has the minimal solutions*

$$M'_\alpha,\ N'_\alpha \quad (\alpha \in \mathfrak{A}) \tag{5.9}$$

then the general solution of the above equation is given by

$$M' = M'_\alpha \cup U_\alpha,\ N' = N'_\alpha \cup V_\alpha \tag{5.9}$$

where

$$M'_\alpha V_\alpha \cup U_\alpha N'_\alpha \cup U_\alpha V_\alpha \subseteq A. \tag{5.10}$$

Proof. Let M'_α, N'_α be the minimal solution of the equation $MN = A$. We consider the equation

$$(M'_\alpha \cup U_\alpha)(N'_\alpha \cup V_\alpha) = MN.$$

We have

$$(M'_\alpha \cup U\ )(N'_\alpha \cup V_\alpha) = M'_\alpha N'_\alpha \cup M'_\alpha V_\alpha \cup U_\alpha N'_\alpha \cup U_\alpha V_\alpha =$$
$$= A \cup M_\alpha V_\alpha \cap U_\alpha N'_\alpha \cup U_\alpha V_\alpha.$$

6. Some special problems

We denote by $S_{a, b}$ the square $Q_{a, b, a+1, b+1}$. From (3.15) we have

$$S_{a, b} S_{c, d} = \begin{cases} S_{a, d} & \text{if } c - 1 \leqslant b \leqslant c + 1 \\ \emptyset & \text{otherwise.} \end{cases} \tag{6.1}$$

Therefore for the equation

$$MN = \bigcup_{j=1}^{k} S_{a_j, b_j} \tag{6.2}$$

we have the minimal solution of the form

$$M = \bigcup_{j=1}^{k'} S_{a'_j, x_j},\ N = \bigcup_{j=1}^{k''} S_{y_j, b'_j} \tag{6.3}$$

where $a'_1 \ldots, a'_{k'}$ are the distinct numbers $a_1, \ldots, a_k$ and $b'_1, \ldots, b'_{k''}$ are the distinct numbers $b_1, \ldots, b_k$ and $x_j \neq y_k$ for $j \neq k$.

E x e m p l e. For the relation

$$A = (Q_{4,2,7,5} - Q_{5,3,6,4}) \cup I_{5,3,7,3} \cup I_{7,3,7,5} \cup I_{7,5,5,5} \cup I_{5,5,5,3} =$$
$$= S_{4,2} \cup S_{5,2} \cup S_{6,2} \cup S_{7,2} \cup S_{4,3} \cup S_{7,3} \cup S_{4,4} \cup S_{7,4} \cup$$
$$S_{4,5} \cup S_{5,5} \cup S_{6,5} S_{7,5}$$

according to (6.3) we have the representation

$$A = MN$$

where

$$M = S_{4,9} \cup S_{5,9} \cup S_{6,9} \cup S_{6,10} \cup S_{7,11}$$

and

$$N = S_{9,2} \cup S_{10,2} \cup S_{11,2} \cup S_{12,2} \cup S_{9,3} \cup S_{12,3} \cup S_{9,4} \cup S_{12,4} \cup$$
$$\cup S_{9,5} \cup S_{10,5} \cup S_{11,5} \cup S_{12,5}$$

This method can be applied to obtain approximative solutions of the equation in relation $MN = A$, when we determine two finite sistems of squares $\mathcal{S}$ and $\mathcal{S}'$ where

$$\mathcal{S} \subseteq A \subseteq \mathcal{S}'.$$

An analogous method can be applied for the approximative solutions of the implication

$$MN \subseteq A$$

(see theorem 18) where are unknown the relations M and N.

7. The product of relations and the composite functions

If $y = f(x)$ and $y = g(x)$ are increasing functions defined respectively for $x \in [a, b]$ and $x \in [f(a), f(b)]$ then we define

$$f \circ g$$

by

$$(f \circ g)(x) = g(f(x)).$$

If we consider the relations

$$F = \{(x, y) \mid y = f(x)\} \text{ and } G = \{(x, y) \mid y = g(x)\}$$

then the product of these is

(7.1)
$$FG = \{(x, y) \mid \exists z : (x, z) \in F \wedge (z, y) \in G\} =$$
$$= \{(x, y) \mid z = f(x) \wedge y = g(z)\} = \{(x, y) \mid y = g(f(z))\}.$$

In the case of two lines ($bb' \neq \emptyset$)

$$D_{a,b,c} = F = \left\{(x, y) \mid y = f(x) = -\frac{ax + c}{b}\right\}, \ D_{a',b',c'} =$$
$$G = \left\{(x, y) \mid y = g(x) = -\frac{a'x + c'}{b'}\right\}$$

and

$$FG = \{(x, y) \mid y = g(f(x))\} = \left\{(x, y) \mid y = g(f(x)) = \frac{aa'x + a'c - bc'}{bb'}\right\} =$$
$$= \{(x, y) \mid aa'x - bb'y + a'c - bc'\} = D_{aa', -bb', a'c - bc'}.$$

Therefore the theory of relations is an untrivial generalisation of the theory of functions.

We conclude by the observations that the theory of relations can be generalized for the cartesian product

$$\underbrace{R \times R \times \ldots \times R}_{k} = R^{(k)}.$$

The product of relations in this case is inducted by the composite of functions. E.g. if $k = 3$ and

$$z = f(x, y), \ z = g(x, y), \ z = h(x, y)$$
$$F = \{(x, y, z) \mid z = f(x, y)\}, \ G = \{(x, y, z) \mid z = g(x, y)\}, \ H =$$
$$= \{(x, y, z \mid z = h(x, y)\}$$

and we define the composite

(7.2) $$z = f(g(x, y), \ h(x, y))$$

then the inducted product is

(7.2′) $FHG = \{(x, y, z) \mid \exists u, v : u = g(x, y) \wedge v = h(x, y) \wedge z = f(u, v)\}.$

If

(7.3) $$z = f(h(x, y), g(y, x))$$

then the inducted product is

(7.3′) $FHG = \{(x, y, z) \mid u, v : u = g(y, x) \wedge v = h(x, y) \wedge z = f(g, h)\}.$

The theory of the relations in $R^{(3)}$ with the products of the forms (7.2′), (7.3′) is the object of the so called „grudd-theory" (теория грудд). The geometry of these relations is analogous with the present considerations.

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ZUR CHARAKTERISIERUNG VON MINIMALLÖSUNGEN IN NORMIERTEN LINEAREN RÄUMEN

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1. Es sei X ein normierter linearer Raum über dem Körper der reellen oder komplexen Zahlen und Y eine nichtleere Teilmenge von X. Ein Element y_0 aus Y ist eine Minimallösung bezüglich Y für ein vorgegebenes Element x_0 aus X, wenn y_0 der Gleichung

$$\|x_0 - y_0\| = \inf_{y \in Y} \|x_0 - y\|$$

genügt. Bekanntlich (vgl. I. SINGER [8, Kap. I,1.1, 1.8]) können die Minimallösungen im Falle eines linearen Teilraumes Y von X folgendermassen charakterisiert werden:

Ein Element y_0 des linearen Teilraumes Y von X ist genau dann eine Minimallösung für das Element x_0 aus X, wenn eines der beiden folgenden Kriterien erfüllt ist:

(1): Es gibt ein Funktional $f \in M[x_0 - y_0]$, so dass

$$\operatorname{Re} f(y - y_0) \leq 0 \qquad \text{für alle } y \in Y$$

gilt.

(2): Jedes Element y aus Y genügt der Ungleichung

$$\min_{f \in \mathcal{E}(M[x_0 - y_0])} \operatorname{Re} f(y - y_0) \leq 0.$$

Dabei ist

$$M[x_0 - y_0] = \{f \mid f \in X^*, \|f\| \leq 1, f(x_0 - y_0) = \|x_0 - y_0\|\}$$

und $\mathcal{E}(M[x_0 - y_0])$ bezeichnet die Menge der Extremalpunkte von $M[x_0 - y_0]$.