MATHEMATICA VOL. 11 (34), 2, 1969, pp. 189–194

THE SOLUTION OF AN INTERVAL EQUATION

by

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The equation

(1) $$f(A_1, \ldots, A_m, X) = g(B_1, \ldots, B_n, X)$$

where $A_1, \ldots, A_m, B_1, \ldots, B_n$ are the given intervals, is called interval equation. The solution of this equation is a system of intervals

$$X_u \quad (u \in U).$$

The purpose of this paper is to solve the equation

(2) $$AX + B = C.$$

We denote $A = [a_1, a_2]$, $B = [b_1, b_2]$, $C = [c_1, c_2]$ and $X = [x_1, x_2]$.

§ 1.

The solution of the equation (2) in the case $a_1 < a_2 < 0$

In this case for a given X we have

$$AX = \begin{cases} [a_2x_2, a_1x_1] & \text{if } x_2 < 0 \\ [a_1x_2, a_1x_1] & \text{if } x_1 < 0 < x_2 \\ [a_1x_2, a_2x_1] & \text{if } x_1 > 0. \end{cases}$$

1.1 We consider first the equation

(1.1) $$[a_2x_2 + b_1, a_1x_1 + b_2] = [c_1, c_2]$$

which has the solution

(1.2) $$X = \left[\frac{c_2 - b_2}{a_1}, \frac{c_1 - b_1}{a_2}\right].$$

We have in this case the conditions

$$(1.3)\quad \begin{aligned} & a_1 < a_2 < 0 \\ & b_1 < b_2 \\ & c_1 < c_2 \\ & \frac{c_2 - b_2}{a_1} < \frac{c_1 - b_1}{a_2} \\ & \frac{c_1 - b_1}{a_2} < 0. \end{aligned}$$

These conditions are equivalents with the system of conditions

$$(1.4)\quad \begin{aligned} & a_1 < a_2 < 0 \\ & b_1 < b_2 \\ & c_1 > b_1 \\ & c_2 > c_1 \\ & c_2 > b_2 + \frac{a_1(c_1 - b_1)}{a_2}. \end{aligned}$$

We see that $b_1 > \frac{a_2 b_2 - a_1 b_1}{a_2 - a_1}$ implies $c_2 > b_2 + \frac{a_1(c_1 - b_1)}{a_2}$ and therefore the system of conditions (1.4) is equivalent with

$$(1.5)\quad \begin{aligned} & a_1 < a_2 < 0 \\ & b_1 < b_2 \\ & c_1 > b_1 \\ & c_2 > b_2 + \frac{a_1(c_1 - b_1)}{a_2}. \end{aligned}$$

Hence we have the following assertion:

If for the intervals A, B and C we have (1.5) *then the equation* (2) *has the solution* (1.2).

1.2 For the equation

$$(1.6)\quad [a_1 x_2 + b_1,\ a_1 x_1 + b_2] = [c_1,\ c_2]$$

we have the solution

$$(1.7)\quad X = \left[\frac{c_2 - b_2}{a_1},\ \frac{c_1 - b_1}{a_1}\right]$$

and the system of conditions for this case is

$$(1.8)\quad \begin{aligned} & a_1 < a_2 < 0\,. \\ & b_1 < b_2 \\ & c_1 < c_2 \\ & \frac{c_2 - b_2}{a_1} < 0 \\ & \frac{c_1 - b_1}{a_1} > 0. \end{aligned}$$

From $c_1 < c_2$, $\frac{c_2 - b_2}{a_1} < 0$, $\frac{c_1 - b_1}{a_1} > 0$ we see that $c_1 < b_1$, $c_2 > b_2$ and the system of conditions in this case is

$$(1.9)\quad \begin{aligned} & a_1 < a_2 < 0 \\ & b_1 < b_2 \\ & c_1 < b_1 \\ & c_2 > b_2. \end{aligned}$$

We have the following assertion:

If for the intervals A, B, and C we have (1.9) *then the equation* (2) *has the solution* (1.7).

1.3 For the equation

$$(1.10)\quad [a_1 x_2 + b_1,\ a_2 x_1 + b_2] = [c_1,\ c_2]$$

we have the solution

$$(1.11)\quad X = \left[\frac{c_2 - b_2}{a_2},\ \frac{c_1 - b_1}{a_1}\right]$$

and the system of conditions

$$(1.12)\quad \begin{aligned} & a_1 < a_2 < 0 \\ & b_1 < b_2 \\ & c_1 < c_2 \\ & \frac{c_2 - b_2}{a_2} > 0 \\ & \frac{c_1 - b_1}{a_1} > \frac{c_2 - b_2}{a_2}. \end{aligned}$$

For c_1 we have the conditions

$$c_1 < c_2, \; c_1 < \frac{a_1(c_2 - b_2)}{a_1}$$

and the relation

$$b_2 < \frac{a_2 b_1 - a_1 b_2}{a_2 - a_1}$$

implies the system of conditions

$$a_2 < 0, \; a_1 < a_2, \; b_1 < b_2, \; c_2 < b_2 \tag{1.13}$$

$$c_1 < b_1 + \frac{a_1(c_2 - b_2)}{a_2}.$$

In this case the solution of (2) is (1.11).

Observation. The case $0 < a_1 < a_2$ is similar with the case $a_1 < a_2 < 0$.

§ 2.

The case $a_1 < 0 < a_2$

The discussion of this case is similar with the cases of §1. We consider the following two assertions:

2.1. For the system of conditions

$$\begin{aligned} & a_1 < 0 < a_2 \\ & b_1 < b_2 \\ & c_1 < b_1 \\ & c_2 = b_2 + \frac{a_2(c_1 - b_1)}{a_1} \end{aligned} \tag{2.1}$$

the solution of equation (2) has the form

$$X = \left[x, \frac{c_1 - b_1}{a_1}\right] \tag{2.2}$$

where

$$\frac{c_1 - b_1}{a_2} < x < 0 \text{ if } a_1 < 0 \text{ and } a_2 > -a_1 \tag{2.3}$$

and

$$\frac{c_2 - b_2}{a_1} < x < 0 \text{ if } a_2 > 0 \text{ and } a_1 < -a_2. \tag{2.3'}$$

2.2. For the system of conditions

$$\begin{aligned} & a_2 > 0 \\ & a_1 < -a_2 \\ & b_1 < b_2 \\ & c_1 < b_1 \\ & b_2 + \frac{a_2(c_1 - b_1)}{a_1} < c_2 < b_2 + \frac{a_1(c_1 - b_1)}{a_2} \end{aligned} \tag{2.4}$$

the equation (2) has the solution

$$X = \left[\frac{c_2 - b_2}{a_1}, \frac{c_1 - b_1}{a_1}\right]. \tag{2.5}$$

§ 3.

A parametric reprezentation of the solution of equation (2)

Let a, b, c, d, e, f, are real positive numbers and

$$0 < s < 1.$$

With these parameters we have in the case (1.1), the representation

$$A = [-a - b, -a], \quad B = [c - d, c - d + e]$$

$$C = \left[c - d + f, \; c - d + f + e + \frac{bf}{a} + g\right]$$

$$X = \left[-\frac{f}{a} - \frac{g}{a + b}, \; -\frac{f}{a}\right].$$

In the case (1.2), we have

$$A = [-a - b, \; -a] \quad B = [c - d, \; c - d + e]$$

$$C = [c - d - f, \; c - d + e + g]$$

$$X = \left[-\frac{g}{a + b}, \quad \frac{f}{a + b}\right].$$

In the case (1.3), we have

$$A = [-a - b, \; -a], \quad B = [c - d, \; c - d + e]$$

$$C = \left[c - d - \frac{(a + b) f}{a} - g, \; c - d + e - f\right]$$

$$X = \left[-\frac{f(a + b)}{a} - g, \; -f\right].$$

In the case 2.1 we have

$$A=[-a,\ a+b],\quad B=[c-d,\quad c-d+e]$$

$$C=\left[c-d-f,\quad c-d+e+\frac{(a+b)f}{a}\right]$$

$$X=\left[-\frac{fs}{a+b},\ \frac{f}{a}\right]$$

or

$$A=[-a-b,\ a],\quad B=[c-d,\ c-d+e]$$

$$C=\left[c-d-f,\ c-d+e+\frac{af}{a+b}\right]$$

$$X=\left[-\frac{afs}{(a+b)^2},\ \frac{f}{a+b}\right].$$

In the case 2.2 we have

$$A=[-a-b,\ a]\quad B=[c-d,\ c-d+e]$$

$$C=\left[c-d-f,\ c-d+e+\frac{a^2+(2a+b)bs}{a(a+b)}\,f\right]$$

$$X=\left[-\frac{a^2+b(2a+b)s}{a(a+b)^2}f,\ \frac{f}{a+b}\right].$$

I should like to thank A. NEY for his comments about the manuscript.

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Received 12. VII. 1969.

MATHEMATICA VOL. 11 (34), 2, 1969, pp. 195–220

FIXPUNKTSÄTZE IN DER APPROXIMATIONSTHEORIE

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Zusammenfassung. In der folgenden Arbeit berichten wir über die Anwendung verschiedener Fixpunktsätze bei dem folgenden Approximationsproblem: Man bestimme zu einem gegebenen Element f eines normierten linearen Raumes R und zu einer gegebenen Teilmenge $V \subset R$ ein Element v_0 aus V in der Art, dass $||f-v_0|| \leq ||f-v||$ für alle $v \in V$ gilt. Jedes derartige Element v_0 nennt man eine Minimallösung für f bezüglich V. Wir bezeichnen mit $P_V(f)$ die Menge der Minimallösungen für f bezüglich V. In dieser Arbeit wenden wir Fixpunktsätze an, um Fragen der Existenz und Invarianz von Minimallösungen zu untersuchen. Wir behandeln ferner Fragen der Charakterisierung von Sonnen in normierten linearen Räumen. Als Anwendung ergeben sich Bedingungen für die Notwendigkeit des verallgemeinerten Kolmogoroffschen Kriteriums und Kriterien für die Konvexität von Mengen in flach konvexen Räumen.

1. Vorbereitungen

Es seien X und Y Hausdorff-Räume und es sei $\mathfrak{A}_0(Y)$ die Menge der nichtleeren abgeschlossenen Teilmengen von Y, also die Menge

$$\mathfrak{A}_0(Y):=\{A\subset Y: A\neq\emptyset \wedge A=\bar{A}\}.$$

Wir betrachten Abbildungen der Art

$$\Lambda: X\to\mathfrak{A}_0(Y).$$

Λ ist also eine Abbildung, die jedem Punkt $x\in X$ eine nichtleere abgeschlossene Teilmenge von Y zuordnet. Für derartige Abbildungen gibt es verschiedene Stetigkeitsbegriffe (vgl. HAHN [10]).