

ABOUT THE EXTENSION OF THE DOMAIN OF DEFINITION OF THE CHEBYSHEV SYSTEMS DEFINED ON INTERVALS OF THE REAL AXIS

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Denote by $\mathcal{C}$ the linear space of the real, continuous functions defined on $\mathbf{R}$, the real axis.

Definition 1. The set of elements $f_1, \dots, f_n$ in $\mathcal{C}$ is said to form a Chebyshev system of order r , ($0 \leq r \leq n-1$), (denote CS_r) on the subset $I \subset \mathbf{R}$, if the restrictions to I of these functions are linearly independent functions, and if each set of $r+1$ linearly independent elements of the linear space spanned by $f_1, \dots, f_n$ have at most $n-r-1$ distinct common zeros in I .

Definition 2. The subset A of the n -dimensional, real Euclidean space $\mathbf{R}^n$ is said to be k -vectorial-independent, if any k distinct vectors of A are linearly independent.

By a similar reasoning as in the paper [1], Theorem 3., may be easily proved the following proposition:

The elements $f_1, \dots, f_n$ of $\mathcal{C}$ form a CS_r , ($0 \leq r \leq n-1$) on I , if and only if the mapping $\Phi: I \rightarrow \mathbf{R}^n$ defined by

$$(1) \quad \Phi(x) = (f_1(x) \dots, f_n(x))$$

is one to one, and $\Phi(I)$ is an $(n-r)$ -vectorial-independent set in $\mathbf{R}^n$.

In a previous paper [3] we have considered the connection between some extension property of the domain of definition of the CS_r -s, defined on a compact Hausdorff space, and the existence of the Chebyshev subspaces of the same order r (see the terminology in [3]) of the space spanned

by these CS_r -s. In the present note we consider some extension problems of the domain of definition of the CS_r -s defined on the subintervals of the real axis, which are of different nature. However, the method in our note is closed to that in [3].

THEOREM 1. Suppose that the elements $f_1, \dots, f_n$, $n \geq 3$, of $\mathcal{C}$ form a CS_0 on $[a, b]$, and that this interval is maximal in the sense that the above considered system of functions has not the property CS_0 on $[a, b]$. Then $f_1, \dots, f_n$ form no CS_r on $[a, b]$ for any $r = 1, \dots, n-2$.

Proof. Denote by Φ the mapping of $[a, b]$ into $\mathbf{R}^n$ defined by (1). From the our above proposition it follows that the set $\Phi([a, b])$ is n -vectorial-independent in $\mathbf{R}^n$. From the same proposition it follows that Theorem 1. is equivalent with the assertion: The set $\Phi([a, b])$ is not a 2-vectorial-independent set in $\mathbf{R}^n$. Because the system of functions $f_1, \dots, f_n$ has not the property CS_0 on $[a, b]$, it follows that there exists a nonzero linear combination

$$(2) \quad \sum_{i=1}^n c_i f_i$$

with the property that it has n distinct zeros, say $x_1, \dots, x_n$ in $[a, b]$. This means that the subspace $\mathbf{R}^{n-1}$ of $\mathbf{R}^n$ having the vector $c = (c_1, \dots, c_n)$ as normal vector, intersects the arc $\Phi([a, b])$ in the points $\Phi(x_1), \dots, \Phi(x_n)$. Because $f_1, \dots, f_n$ form a CS_0 on $[a, b]$, it follows that a point x_i must coincide with b . If the points $x_1, \dots, x_n$ are arranged increasingly, then $x_n = b$. From the one-to-ones of the mapping Φ on $[a, b]$ it follows that the points $\Phi(x_1), \dots, \Phi(x_{n-1})$ are distinct. According to the condition $n \geq 3$, at least a point $\Phi(x_i)$ is in the interior of the arc $\Phi([a, b])$ and corresponds to a point in (a, b) . Because the nonzero linear combination (2) has $n-1$ zeros in $[a, b]$, it follows from the classical theory of the Chebyshev system that all the zeros in (a, b) must be simple. Geometrically this means, that the subspace $\mathbf{R}^{n-1}$ is pierced by $\Phi([a, b])$ in all the points corresponding to zeros of (2) in (a, b) . Consider now the set of vectors

$$(3) \quad \Phi(x_1), \dots, \Phi(x_{i-1}), \Phi(x_{i+1}), \dots, \Phi(x_{n-1}),$$

where $x_i \in (a, b)$. Suppose that $\Phi(x_n) = \Phi(b)$ forms with (3) a linearly independent set of vectors, and such this set of vectors spans the subspace $\mathbf{R}^{n-1}$. Let be now x_0 a point in (a, b) , $x_0 \neq x_i$, $i = 1, \dots, n-1$. Then the set of vectors (3) and $\Phi(x_0)$ span a subspace $\mathbf{R}_1^{n-1}$ of $\mathbf{R}^n$ according to the n -vectorial-independency of $\Phi([a, b])$. Letting now $x_0 \rightarrow b$, the subspace $\mathbf{R}_1^{n-1}$ may be made arbitrary closed to $\mathbf{R}^{n-1}$. From the fact that $\Phi([a, b])$ pierces $\mathbf{R}^{n-1}$ in $\Phi(x_i)$, it follows that for $\Phi(x_0)$ sufficiently closed to $\Phi(b)$, $\mathbf{R}_1^{n-1}$ will be pierced by $\Phi([a, b])$ at a point $\Phi(x')$ closed to $\Phi(x_i)$, and distinct from $\Phi(x_j)$, $j = 1, \dots, n-1$. But then the points (3), $\Phi(x_0)$ and $\Phi(x')$ of $\Phi([a, b])$ are contained in $\mathbf{R}_1^{n-1}$, in contradiction with the n -vectorial-independency of this set. From this contradiction it follows that $\Phi(b)$ is

linearly dependent on each set of vectors of the form (3), for $i = 2, 3, \dots, n-1$. But the representation of $\Phi(b)$ in the form of linear combination of the vectors $\Phi(x_1), \dots, \Phi(x_{n-1})$ is unique and therefore

$$\Phi(b) = k\Phi(x_1),$$

which proves that $\Phi([a, b])$ is not a 2-vectorial-independent set. Implicitly was proved that $x_1 = a$.

THEOREM 2. Let $f_1, \dots, f_n$ form a CS_0 on (a, b) . Denote by Φ the mapping of $\mathbf{R}$ into $\mathbf{R}^n$ defined by (1). If $\Phi(b) \neq 0$ (or $\Phi(a) \neq 0$), then the above functions form a CS_0 on (a, b) (respective on $[a, b]$).

Proof. Suppose $\Phi(b) \neq 0$ and that $f_1, \dots, f_n$ do not form a CS_0 on (a, b) . It follows that the set $\Phi((a, b))$ is not an n -vectorial-independent subset of $\mathbf{R}^n$, i.e. that there exist n distinct points, say $x_1, \dots, x_n$ in $(a, b]$ such that the vectors $\Phi(x_1), \dots, \Phi(x_n)$ are linearly dependent. Because $f_1, \dots, f_n$ form CS_0 on (a, b) , it follows that $\Phi((a, b))$ is an n -vectorial-independent subset of $\mathbf{R}^n$, and therefore $x_n = b$, if it is supposed that the zeros are arranged increasingly. Let $\mathbf{R}^{n-1}$ be the subspace of $\mathbf{R}^n$ which is spanned by $\Phi(x_1), \dots, \Phi(x_{n-1})$; according to our assumption, $\Phi(b)$ is contained in $\mathbf{R}^{n-1}$. Let be now f a linear combination of the form (2), where $c = (c_1, \dots, c_n)$ is a normal vector of $\mathbf{R}^{n-1}$. Then $x_1, \dots, x_{n-1}$ are simple zeros of f , and therefore the subspace $\mathbf{R}^{n-1}$ is pierced by the arc $\Phi(a, b)$ in the points $\Phi(x_1), \dots, \Phi(x_{n-1})$. If $\Phi(b)$ is linearly dependent on each set of vectors of form (3) for $i = 1, \dots, n-1$, then it would be follow that $\Phi(b) = 0$, in contradiction with the assumption of the theorem. Suppose that the system of vectors (3) and $\Phi(b)$ form a set of linearly independent vectors, and therefore they span the subspace $\mathbf{R}^{n-1}$. Let now be $x_0 \in (a, b)$ and let $x_0 \rightarrow b$. For x_0 sufficiently closed to b the vectors (3) and $\Phi(x_0)$ span a subspace $\mathbf{R}_1^{n-1}$ of $\mathbf{R}^n$ with the property that it meets the set $\Phi((a, b))$ at a point $\Phi(x')$ closed to $\Phi(x_i)$ and distinct from the points (3) and $\Phi(x_0)$. But then the subspace $\mathbf{R}_1^{n-1}$ will contain n distinct points of the set $\Phi((a, b))$, in contradiction with the n -vectorial-independency of this set. The theorem is proved.

THEOREM 3. If the elements $f_1, \dots, f_n$ form a CS_0 on (a, b) , then it is always possible a redefinition of these functions on a point α in $\mathbf{R}$, outside to (a, b) (and if we require continuity, then generally outside to $[a, b]$), in order to form a CS_0 on $(a, b) \cup \{\alpha\}$.

Proof. Denote by S^{n-1} the surface of the unit sphere in $\mathbf{R}^n$ with its centre in the origin, and let be Φ the mapping of (a, b) in $\mathbf{R}^n$ defined by (1). Let Ψ denotes the radial projection of $\mathbf{R}^n - \{0\}$ into S^{n-1} . We put for simplicity $\Psi\Phi(x) = \Psi(x)$. The set $\Psi((a, b))$ will be obviously n -vectorial-independent. Consider the sequence $\{x_v\} \rightarrow b$, $v \rightarrow \infty$, $x_v \in (a, b)$, and denote by β a cluster point of the sequence $\{\Psi(x_v)\}$. Suppose for simplicity that $\{\Psi(x_v)\} \rightarrow \beta$, $v \rightarrow \infty$. The set $\Phi((a, b)) \cup \{\beta\}$ is n -vectorial-independent in $\mathbf{R}^n$. To prove this, suppose the contrary: there exist the distinct points $x_1, \dots, x_{n-1}$ in (a, b) , such that the vectors $\Phi(x_1), \dots, \Phi(x_{n-1}), \beta$ are linearly de-

pendent. The vectors (3) and β cannot be linearly dependent for all $i = 1, \dots, n-1$, because it would follow that $\beta = 0$, which is a contradiction. Suppose that the vectors (3) and β are linearly independent. Then they span the subspace $\mathbf{R}^{n-1}$ of $\mathbf{R}^n$ which contains the point $\Phi(x_i)$. This subspace is pierced by $\Phi((a, b))$ at $\Phi(x_i)$. The vectors (3) and $\Psi(x_v)$ span a subspace $\mathbf{R}_v^{n-1}$. For sufficiently great v , $\mathbf{R}_v^{n-1}$ will be pierced by $\Phi((a, b))$ at a point $\Phi(x')$ closed to $\Phi(x_i)$ and distinct from (3) and $\Phi(x_v)$. This means that the points (3), $\Phi(x_v)$, $\Phi(x')$ are contained in $\mathbf{R}_v^{n-1}$, in contradiction with the n -vectorial-independency of the set $\Phi((a, b))$. Implicitly was proved that $\beta \notin \Phi((a, b))$. If we define now the functions $f_1, \dots, f_n$ on α by setting $\Phi(\alpha) = \beta$, the theorem follows.

A repeated application of the above theorem and the Lemma 3 of [2], gives us the

Corollary. *An n -dimensional Chebyshev space of order 0 spanned by continuous functions defined on an open interval, has Chebyshev subspaces of order 0 and of any dimension r , $r \leq n$.*

This property of Chebyshev spaces spanned by continuous functions which are defined on an open interval was observed by M. G. KREIN (see for example in the paper [4]).

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OPTIMAL PATHS IN NETWORKS AND GENERALIZATIONS (I)

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Introduction

The theory of graphs has recently asserted itself as an important branch in mathematics, comprising two major qualities: a high level of abstraction and a wide field of applications. As far as the applications in economy are concerned the section of the theory of graphs dealing with the optimal paths has been proved to be one of the most important. The planning of construction — assembling works on the building sites, the capital repairs, the setting of production schedules, the organization of transports, the delivery of information, the traffic in networks, the new organization of enterprises by means of the electronic processing of data, the introductions of new technology, the replacing of the old equipment, the planning and the delivery of a new product, are only some of the activities which by mathematical modelling lead to problems of paths in networks. There have been elaborated new methods by means of which one can determine the shortest or the longest path between two nodes in a network, or the path with the maximum probability of going from a node to another, or the path with the maximum or minimum capacity etc. Each of these problems has been studied separately, taking into account the peculiarities it presents. On the other hand there are also surveys that study some groups of problems of paths in networks, but a general study of these problems has not been done so far. Such a study makes the subject of this work.

The following discussions are based on the remark that there is a bijective mapping between the directed finite graphs and certain matrices defined on an algebraical structure which is organized as a partially ordered semigroup. Considering some particular cases of sets, laws of composition and order relations, we obtain algebraical structures which are specific