MATHEMATICA VOL. 11 (34), 1, 1969, pp. 135–136

KOROVKIN'S THEOREM FOR NONLINEAR 3-PARAMETER FAMILIES

by

A. B. NÉMETH

Cluj

The aim of the present note is to prove the theorem of P. P. KOROVKIN [1] for the general case of 3-parameter families (we use here this notion in the sense of [2]). Our theorem may be formulated as follows:

THEOREM 1. *Let be F a (linear or nonlinear) continuous 3-parameter family defined on the closed interval $[a, b]$. If the sequence of linear positive operators $\{A_n\}$ – which transform the space $C[a, b]$ of continuous functions on $[a, b]$ normed with the maximum norm, into the space $B[a, b]$ of bounded functions on $[a, b]$ normed with the supremum norm – has the property that $\lim A_n\varphi = \varphi$ for every φ in F then $\lim A_nf = f$ for every f in $C[a, b]$.*

This theorem for the particular case of the linear 3-parameter families (i.e. Chebyshev systems) is the above mentioned theorem of Korovkin. Reffering to Korovkin's proof we observe that there in fact the followings have been proved:

Let $\mathcal{F}$ be a family of continuous functions on $[a, b]$ with the properties

(i) for any $x_0 \in (a, b)$ there exists an element $\psi \in \mathcal{F}$ so that $\psi(x_0) = 0$, $\psi(x) > 0$ for $x \in [a, b] - \{x_0\}$;

(ii) there exists an element $\psi_0 \in \mathcal{F}$ so that $\psi_0(a) = \psi_0(b) = 0$, $\psi_0(x) > 0$, $x \in (a, b)$;

(iii) for any y_1 and y_2 there exists an element $\psi_1 \in \mathcal{F}$ so that $\psi_1(a) = y_1$ and $\psi_1(b) = y_2$.

THEOREM 2. *If the sequence $\{A_n\}$ of linear positive operators which transform $C[a, b]$ in $B[a, b]$ has the property that $\lim A_n\psi = \psi$ for each $\psi \in \mathcal{F}$ where $\mathcal{F}$ is the above defined family of functions, then $\lim A_nf = f$ for each $f \in C[a, b]$.*

Prove Theorem 1 showing that the 3-parameter family F may be transformed by addition of a function to each of its elements in a family $\mathcal{F}$ which verifies the conditions (i), (ii) and (iii) and then Theorem 2 may be applied.

It is easy to show that if F is an n-parameter family and g is a fixed continuous function, then the family of functions $\mathcal{F} = \{\varphi - g | \varphi \in F\}$ is also an n-parameter family.

Let now F be the 3-parameter family in Theorem 1 and let be $g \in F$. Form the new 3-parameter family $\mathcal{F} = \{\varphi - g | \varphi \in F\}$. Then the zero function will be an element in $\mathcal{F}$. From the properties of the 3-parameter family $\mathcal{F}$ it follows immediately that the conditions (ii) and (iii) are fulfilled.

Prove now that the condition (i) is also satisfied for our 3-parameter family $\mathcal{F}$. Let be $x_0 \in (a, b)$ and let be $a < x_1 < x_0 < x_2 < b$. There exists two elements ψ_1 and ψ_2 of $\mathcal{F}$ so that $\psi_1(a) = \psi_2(a) = 1$, $\psi_1(x_1) = \psi_1(x_0) = 0$, $\psi_2(x_0) = \psi_2(x_2) = 0$. Observe that on (x_0, b) is valid the inequality $\psi_2(x) < \psi_1(x)$. Let now $x_0 < x_3 < x_2$ and let $\psi_3 \in \mathcal{F}$ an element with the properties $\psi_3(a) = 1$, $\psi_3(x_0) = \psi_3(x_3) = 0$. Then on (x_0, b) we have $\psi_2(x) < \psi_3(x) < \psi_1(x)$. If $\{x_i\}_{i=3}^{\infty}$ is a monotonous sequence of points $x_i \in (x_0, x_2)$ converging to the point x_0, then the sequence of functions $\{\psi_i\}$ where ψ_i is determined by the conditions $\psi_i(a) = 1$, $\psi_i(x_0) = \psi_i(x_i) = 0$ converges uniformly to the function $\psi(x)$ of $\mathcal{F}$, which satisfies the condition (i).

Really, the sequence $\{\psi_i(b)\}$ is monotone increasing and bounded: $\psi_i(b) < \psi_1(b)$, $i = 3, 4, \ldots$ and therefore it is convergent; the sequence $\{\psi_i(x_1)\}$ is monotone decreasing and positive and then also convergent. The sequence $\{\psi_i(a)\}$ is convergent being constant and then from a theorem of L. TORNHEIM (see [2] Theorem 5.) it follows that the the sequence $\{\psi_i(x)\}$ converges uniformly to a function $\psi(x)$ of the family $\mathcal{F}$. It is obvious that $\psi(x_0) = 0$. Because $\psi_i(x) > 0$ on $[a, x_0)$ we have $\psi(x) \geqq 0$ on $[a, x_0)$ and because $\psi_i(x) > 0$ on $(x_j, b]$, $i = j, j+1, \ldots$ it follows $\psi(x) \geqq 0$ on $(x_j, b]$. But j is arbitrary and therefore $\psi(x) \geqq 0$ on $(x_0, b]$. So x_0 is a duble zero of $\psi(x)$ and therefore it must be the single zero of this function, which proves (i).

Finally, let be $\eta \in \mathcal{F}$. Then $\eta = \varphi - g$ for some φ in F and $\lim A_n \eta = \lim A_n \varphi - \lim A_n g = \varphi - g = \eta$ and Theorem 2 may be applied.

REFERENCES

[1] Коровкин П. П., *Линейные операторы и теория приближений*. Физматгиз, Москва, 1958.

[2] Tornheim L., *On n-parameter families of functions and associated convex functions*. Trans. Amer. Math. Soc. **69**, 457–467 (1950).

Received 27. XI. 1967.

MATHEMATICA VOL. 11 (34), 1, 1969, pp. 137–141

LA RÉSOLUTION DES SYSTÈMES D'ÉQUATIONS OPÉRATIONNELLES À L'AIDE DES MÉTHODES ITÉRATIVES

par

ION PĂVĂLOIU

à Cluj

1. Soient X et Y deux espaces de type Banach et $Z = X \times Y$ le produit cartésien de ces espaces.

Dans l'espace Z on considérera l'équation suivante:

$$(1) \qquad \begin{aligned} x &= \varphi(x, y) \\ y &= \psi(x, y). \end{aligned}$$

Dans la présente note on interprétera l'équation antérieure comme un système de deux équations à deux inconnues où φ et ψ sont des opérateurs définis sur Z et à valeurs respectivement X et Y. De cette manière on mettra en évidence un critérium de convergence du procédé de Gauss-Seidel appliqué à la résolution de ce système. Ensuite on montrera que ce criterium est plus général que ceux qui sont connus.

Enfin on appliquera les résultats obtenus à l'élaboration d'une nouvelle méthode de résolution des systèmes des équations lineaires.

Une partie des résultats de cette note ont été obtenus par nous dans un cas particulier ($X = Y = R$) dans le travail [1].

2. On suppose que les opérateurs φ et ψ satisfont les conditions suivantes:

a) Les opérateurs φ et ψ transforment le domaine $D \subset Z$ en lui même.

b) Il existe des constantes α, β, a et b telles que

$$||\psi(x_2, y_2) - \psi(x_1, y_1)|| \leqq a||x_2 - x_1|| + b||y_2 - y_1||$$

$$||\varphi(x_2, y_2) - \varphi(x_1, y_1)|| \leqq \alpha||x_2 - x_1|| + \beta||y_2 - y_1||$$

pour tout $(x_i, y_i) \in D$, $i = 1, 2$.