pendent. The vectors (3) and β cannot be linearly dependent for all $i = 1, \ldots, n-1$, because it would follow that $\beta = 0$, which is a contradiction. Suppose that the vectors (3) and β are linearly independent. Then they span the subspace $\mathbf{R}^{n-1}$ of $\mathbf{R}^n$ which contains the point $\Phi(x_i)$. This subspace is pierced by $\Phi((a, b))$ at $\Phi(x_i)$. The vectors (3) and $\Psi(x_\nu)$ span a subspace $\mathbf{R}_\nu^{n-1}$. For sufficiently great ν, $\mathbf{R}_\nu^{n-1}$ will be pierced by $\Phi((a, b))$ at a point $\Phi(x')$ closed to $\Phi(x_i)$ and distinct from (3) and $\Phi(x_\nu)$. This means that the points (3), $\Phi(x_\nu)$, $\Phi(x')$ are contained in $\mathbf{R}_\nu^{n-1}$, in contradiction with the n-vectorial-independentity of the set $\Phi((a, b))$. Implicitly was proved that $\beta \notin \Phi((a, b))$. If we define now the functions $f_1, \ldots, f_n$ on α by setting $\Phi(\alpha) = \beta$, the theorem follows.

A repeated application of the above theorem and the Lemma 3 of [2], gives us the

C o r o l l a r y. *An n-dimensional Chebyshev space of order* 0 *spanned by continuous functions defined on an open interval, has Chebyshev subspaces of order* 0 *and of any dimension r, $r \leq n$.*

This property of Chebyshev spaces spanned by continuous functions which are defined on an open interval was observed by M. G. KREIN (see for example in the paper [4]).

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OPTIMAL PATHS IN NETWORKS AND GENERALIZATIONS (I)

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Introduction

The theory of graphs has recently asserted itself as an important branch in mathematics, comprising two major qualities: a high level of abstraction and a wide field of applications. As far as the applications in economy are concerned the section of the theory of graphs dealing with the optimal paths has been proved to be one of the most important. The planning of construction — assembling works on the building sites, the capital repairs, the setting of production schedules, the organization of transports, the delivery of information, the traffic in networks, the new organization of enterprises by means of the electronic processing of data, the introductions of new technology, the replacing of the old equipment, the planning and the delivery of a new product, are only some of the activities which by mathematical modelling lead to problems of paths in networks. There have been elaborated new methods by means of which one can determine the shortest or the longest path between two nodes in a network, or the path with the maximum probability of going from a node to another, or the path with the maximum or minimum capacity etc. Each of these problems has been studied separately, taking into account the peculiarities it presents. On the other hand there are also surveys that study some groups of problems of paths in networks, but a general study of these problems has not been done so far. Such a study makes the subject of this work.

The following discussions are based on the remark that there is a bijective mapping between the directed finite graphs and certain matrices defined on an algebraical structure which is organized as a partially ordered semigroup. Considering some particular cases of sets, laws of composition and order relations, we obtain algebraical structures which are specific for various particular problems of paths in networks. In this way we come to results which are already known, but we find new results as well. The algorithms of various problems do not differ any longer depending on the nature of the problems, and thus we can make up standard programs for computers. At the end we should mention that the new perspective in which the problems of paths in networks are viewed gives the possibility of a further development of theoretical researches and practical applications.

This study is divided in seven chapters. In chapter 1 we make a brief survey of the main results which have been obtained by now in the algebraical study of the optimal paths in networks. In chapter 2 the C-semigroup is defined and some properties of its elements are studied. At the end of the chapter we give some examples of C-semigroups. Chapter 3 is concerned with matrices defined on a C-semigroup. Operations with matrices are defined and some of their properties are studied. In chapter 4 we study certain matricial equations connected with the optimal path problem. In chapter 5 we make a general study of the computation algorithms of the stable power of a matrix. Chapter 6 is concerned with the applications of the results obtained in the previous chapters, to graphs. Ultimately in chapter 7 some applications to economy are indicated.

CHAPTER 1

Former results

The use of algebraical methods in order to determine certain paths in networks has three major advantages as compared to the study of paths directly on the network. First, because of the formal character of the algebraical methods their programming for computers is extremely simple, second, they lead to more rigorous proofs and third, they maintain their character of generality without neglecting the principal properties of each individual problem of paths in networks. This is the explanation that lately the study of some algebraical structures for their usage in the determination of certain paths in networks has acquired an important place in the preoccupation of the mathematicians. Two tendencies make themselves clear in almost all the studies. The first is to study more and more general algebraical structures and the second is to establish the computation algorithms that need a restricted amount of operations. The aim of this chapter is to make a brief presentation of the most important results that have been obtained in this respect.

The first work in literature in which an axiomatically defined algebraical structure is used in the study of graphs belongs to A. G. LUNTZ [12] published in 1952. LUNTZ defines the Boolean algebra as being a set of elements with two binary operations $+$ and $\times$, which has the following properties:

$$(a + b) + c = a + (b + c), \qquad (a \times b) \times c = a \times (b \times c) \tag{1.1}$$

$$a + b = b + a \qquad a \times b = b \times a \tag{1.2}$$

$$(a + b) \times c = (a \times c) + (b \times c) \tag{1.3}$$

$$\exists 0 \text{ and } 1 \text{ with the properties } a + 0 = a,\ a \times 1 = a \tag{1.4}$$

$$a + a = a \tag{1.5}$$

(1.6) For any a there is an $\bar{a}$ so that $a + \bar{a} = 1$ and $a \times \bar{a} = 0$.

The author defines as follows the addition and multiplication of the $n \times n$ dimensional matrices whose elements belong to the Boolean algebra:

$$A + B = C \quad \text{where} \quad c_{ij} = a_{ij} + b_{ij}$$

and

$$A \times B = D \text{ where } d_{ij} = (a_{i1} \times b_{1j}) + \ldots + (a_{in} \times b_{nj}).$$

A matrix A is called normed if the elements of the main diagonal are equal to 1. It is shown that if A is a normed matrix, then $A^{n-1} = A^n$. The author uses the above results in the analysis and synthesis of switching circuits. He shows that if A is the matrix of direct conductibility between the nodes of a circuit, then A^{n-1} is the matrix of total conductibilities of that matrix.

GR. C. MOISIL [13] defined in 1960 an algebraical structure called a semilaticially ordered semigroup as being a set S with the following properties:

I. In S it is defined a law of internal composition „$\circ$" with neutral element e so that whichever should a, b and c of S be

$$a \circ b = b \circ a \tag{1.7}$$

$$a \circ (b \circ c) = (a \circ b) \circ c \tag{1.8}$$

$$a \circ e = e \circ a = a. \tag{1.9}$$

II. In S it is defined a relation of partial order so that each pair of elements belonging to S should have a lower boundary in S. If one denotes $\subset$

the semiserial relation and $a \wedge b$ the lower boundary of the elements a and b, then

(1.10) $$a \subset a$$

(1.11) $$a \subset b \text{ and } b \subset a \quad \text{implies } a = b$$

(1.12) $$a \subset b \text{ and } b \subset c \quad \text{implies } a \subset c$$

(1.13) $$a \wedge b \subset a, \; a \wedge b \subset b$$

(1.14) $$z \subset a \text{ and } z \subset b \quad \text{implies } z \subset a \wedge b.$$

III. In S the laws of distributivity and absobtion are valid,

(1.15) $$a \circ (b \wedge c) = (a \circ b) \wedge (a \circ c)$$

(1.16) $$a \wedge (a \circ b) = a.$$

Consider on S the square matrix $A = (a_{ij})$ and the matricial product $P = A \times B$ where $p_{ij} = \bigwedge\limits_h (a_{ih} \circ b_{hj})$. It is proved that if A is a $n \times n$ matrix on S having the property $a_{ij} = e$, $i = 1, 2, \ldots, n$ then $A^{n-1} = A^n$.

The author shows that certain problems of transport economy can be solved on the basis of the above results. Such problems are: the problem of determining the transport route on which the costs are minimum, the problem of determining the transport route with the greatest transport capacity and the problem of determining the route on which the probability of the transport outcome is maximum.

In 1961 M. YOELI [29] defined the algebraical structure which he called a Q-semiring. A Q-semiring is a set of elements Q which has two binary operations „+" and „∘" and the properties:

(1.17) $$a + b = b + a$$

(1.18) $$(a + b) + c = a + (b + c)$$

(1.19) $$(a \circ b) \circ c = a \circ (b \circ c)$$

(1.20) $$a \circ (b + c) = (a \circ b) + (a \circ c)$$

(1.21) $$(b + c) \circ a = (b \circ a) + (c \circ a)$$

and which contains the elements 0 and 1 with the following properties:

(1.22) $$a + 0 = a, \quad a \circ 0 = 0 \circ a = 0$$

(1.23) $$a \circ 1 = 1 \circ a = a$$

(1.24) $$a + 1 = 1.$$

The author defines matrices and operations with matrices on this algebraical structure. It is denoted by A the link matrix of a graph and by T the transmission matrix. The author proves that $A^m = T$ for $m \geq n - 1$, where A is a matrix of the order n. Other properties of matrices are also studied but they are of no interest for the subject of the present paper. It is worth mentioning that unlike the other authors quoted above, M. YOELI does not postulate the commutativity of multiplication.

In 1965, CRUON and HERVÉ's work [4] was published. It studies a particular algebraical structure, referring to the longest path in a network. Consider the set $E = \mathbf{R} \cup \{-\infty\}$ where $\mathbf{R}$ is the set of real numbers. On the set E define two operations $\dotplus$ and $*$, called symbolic addition and symbolic multiplication

$$a \dotplus b = \max (a, b).$$

$$a * b = a + b.$$

This algebraical structure surpasses the frame of those presented above, it does not verify either the property (1.24) of the Q-semiring or the property (1.16) of the semilaticially ordered semigroup. The authors study some equations of the form:

(1.25) $$T = T * A$$

where T is a vector which must be determined and A is a square matrix of the order n, which has all the elements of the main diagonal equal to 0. The equation (1.25) is an algebraical representation of the problem of potentials formulated by B. ROY [24] by means of graphs. ROY proves that the problem of potentials is compatible if the graph it represents does not contain circuits of strictly positive length. Taking into account these hypotheses CRUON and HERVÉ show that $A^n = A^{n+1}$ where A is the arcs length matrix in the corresponding graph. The solution of the equation is given by

$$T = U * A^n$$

where U is a vector with n arbitrary components from E.

In 1966 in my paper [18] I showed that $A^{n-1} = A^n$ is a necessary and sufficient condition for the compatibility of the equation (1.25). I also showed that for solving this equation the computation of the matrix A^{n-1} is not necessary, instead, due to the associativity of matricial multiplication, the algorithm

$$T_1 = U * A$$
$$T_2 = T_1 * A$$
$$\cdots\cdots\cdots\cdots$$
$$T_{n-1} = T_{n-2} \circ A$$
$$T = T_{n-1}$$

can be used. This algorithm needs a considerably smaller amount of computations.

In 1967 in [20] I generalized the algebraical structure suggested by CRUON and HERVÉ, by considering an ordered set having two binary operations, a set in which neither the property (1.24) nor the other equivalent ones were postulated. This structure comprises all the exemples of paths in networks given by GR. C. MOISIL and M. YOELI, but without including the above mentioned algebraical structures. If we limit ourselves to the study of path in networks, the hypothesis of total order is necessary because its absence would make impossible the comparison of several paths in order to choose one of them. I made another generalization together with F. RADO [21] in which I gave up the hypothesis of total order.

From the above discussion we notice that the matrix A^{n-1} and implicitly the computation of its elements present a special interest. Obviously this matrix can be calculated by repeatedly multiplying the matrix A by itself. But there are algorithms which permit the determination of A^{n-1} with a smaller amount of computations.

The first paper in which it is given an algorithm for the determination of the transitive closing of an application Γ defined on a finite set of points, belongs to B. ROY [23] and it was published in 1959. The results obtained by B. ROY can be algebraically formulated as follows: Consider an algebraical structure whose elements are the numbers 0 and 1 and whose operations are $a + b = \max(a, b)$ and $a \times b = \min(a, b)$. For the square matrices $M = (a_{hk})$ of the order n, whose elements on the main diagonal are all equal to 1, the following operators are defined:

$$T_i M = (b_{hk}) \text{ where } b_{hk} = a_{hk} + (a_{hi} \times a_{ik})$$

and it is proved a theorem which is equivalent to the property:

$$T_n T_{n-1} \ldots T_1 M = M^{n-1}.$$

In 1962, S. WARSHALL [28] obtained a result which was close to that of B. ROY's. It was shown in the following theorem:

THEOREM. *It is given a square matrix M of the order d which has the elements m_{ij}, 0 or 1. M' is defined as follows: $m'_{ij} = 1$ if and only if $m_{ij} = 1$ or if the indices $k_1, k_2, \ldots, k_n$ exist, so that $m_{ik_1} = m_{k_1k_2} = \ldots = m_{k_nj} = 1$.* In the opposite case $m'_{ij} = 0$. M^* is defined by means of the following construction

0. $M^* = M$

1. $i = 1$

2. For any j for which $m^*_{ij} = 1$ and for any k, m^*_{jk} is replaced by $\max(m^*_{jk}, m^*_{ik})$

3. i is increased with a unit.

4. If $i = d$, we pass to the second step, if it is not, we stop.

Then $M^* = M'$.

The algorithm suggested by B. ROY was generalized in 1968 by I. TOMESCU [27], in the case of an algebraical structure equivalent to that suggested by GR. C. MOISIL, for matrices whose elements on the main diagonal are all equal to e. TOMESCU introduced the operators $U_\rho(A)$, $\rho = 1, 2, \ldots, n$ defined as follows:

$$U_\rho(A) = B \text{ where } b_{ij} = a_{ij} \wedge (a_{i\rho} \circ a_{\rho j})$$

and he proved that

$$U_{\rho_1} U_{\rho_2} \ldots U_{\rho_n} A = A^{n-1}.$$

The study of S. WARSHALL has developed several generalizations. We mention R. W. FLOYD's [8] work published in 1962 and P. ROBERT & FERLAND's work [22] published in 1968. ROBERT and FERLAND make a generalization of WARSHALL's results in the case of YOELI's Q-semiring. They show that if A is a square matrix of the order n defined on a Q-semiring, then

$$A^{[n]} = A + A^2 + \ldots + A^n$$

where the matrix $A^{[n]}$ is recursively construed as follows:

$$A^{[0]} = A$$

$$A^{[k]} = (a_{ij}^{[k]}) \text{ where } a_{ij}^{[k]} = a_{ij}^{[k-1]} + (a_{ik}^{[k-1]} \times a_{kj}^{[k-1]}),\ k = 1, 2, \ldots, n.$$

ROBERT and FERLAND's results include TOMESCU's result, because on the one hand the algebraical structure is more general and on the other hand no assumption is made on the elements of the main diagonal of the matrix A. Obviously the algorithm is the same.

In chapter 5 we make a more general study of some algorithms of the types mentioned above.

CHAPTER 2

II. The routing semigroup

In this chapter we will define an algebraical structure containing in particular the algebraical structures which have been presented in the previous chapter. We will call it the routing semigroup or, shortly, the C-semigroup. We will reveal those properties of the C-semigroup which will be used in the following chapters.

D e f i n i t i o n 2.1. *A routing semigroup or C-semigroup is called a non-void set $\mathfrak{C}$, in which two internal composition-laws are defined „$\oplus$" and „$\circ$"*

denominated as addition and correspondingly multiplication, with the following properties:

(2.1) $$(a \oplus b) \oplus c = a \oplus (b \oplus c)$$

(2.2) $$a \oplus b = b \oplus a$$

(2.3) $$a \oplus a = a$$

(2.4) $$(a \circ b) \circ c = a \circ (b \circ c)$$

(2.5) $$a \circ (b \oplus c) = (a \circ b) \oplus (a \circ c)$$

(2.6) $$(b \oplus c) \circ a = (b \circ a) \oplus (c \circ a)$$

(2.7) $$\exists\, e \in \mathcal{C},\ a \circ e = e \circ a = a$$

whichever would a, b, c be, belonging to the set $\mathcal{C}$.

The element e is called unity element and it is unique. The property of idempotence (2.3) of the operation $\oplus$ can be replaced by

(2.3′) $$e \oplus e = e$$

the two operations being equivalent, due to the distributivity of multiplication relative to addition. Thus, from (2.3′) multiplying from the left by a we get

$$a \circ (e \oplus e) = a \circ e$$

hence

$$a \oplus a = a, \ \forall\, a \in \mathcal{C}$$

THEOREM 2.1. A *C-semigroup is a semilatice.*

Proof. We say that a is greater than or equal to b and we denote $a > b$ or $b < a$ if and only if $a \oplus b = a$. The relation „$>$" satisfies the three axioms of the order relation. Thus, by (2.3) we get

(2.8) $$a > a, \quad \forall\, a \in \mathcal{C}.$$

From the property (2.2) it results that

(2.9) $$a > b \text{ and } b > a, \quad \text{implies } a = b, \ \forall\, a,\, b \in \mathcal{C}.$$

Ultimately, the relation „$>$" is transitive, that is

(2.10) $a > b$ and $b > c$, implies $a > c$ because $a \oplus b = a$ and $b \oplus c = b$ implies $a \oplus c = (a \oplus b) \oplus c = a \oplus (b \oplus c) = a \oplus b = a.$

The properties (2.8), (2.9) and (2.10) attest the fact that the relation „$>$" is an order relation and hence the C-semigroup $\mathcal{C}$ is an ordered set.

Let now be $d \in \mathcal{C}$ an element for which the relations $d > a$ and $d > b$ that is $d \oplus a = d$ and $d \oplus b = d$ are satisfied. Hence $(d \oplus a) \oplus b = d$ or $d \oplus (a \oplus b) = d$ that is $d > a \oplus b$.

In conclusion:

(2.11) $$d > a \text{ and } d > b \text{ implies } d > a \oplus b$$

and because

(2.12) $$a \oplus b > a \text{ and } a \oplus b > b$$

results that $a \oplus b$ is the smallest majorant of the elements a and b. This fact proves the theorem.

The theorem 2.1 shows that the C-semigroup could have been defined as an algebraical structure which is a semilatice and in the same time a non-commutative semigroup with unit element. We have preferred, however, the above definition, as the Matricial operations which appear in the following chapters will thus have a formal analogy with the usual matricial operations.

THEOREM 2.2. *Two inequalities of the same kind can be added term by term.*

Proof. Suppose that

$$a > b \text{ and } c > d$$

that is

(2.13) $$a \oplus b = a, \ c \oplus d = c.$$

Adding term by term the above equalities we get:

(2.14) $$(a \oplus b) \oplus (c \oplus d) = a \oplus c$$

Taking into account that the addition is associative and commutative, the relation (2.14) can be rewritten:

$$(a \oplus c) \oplus (b \oplus d) = a \oplus c.$$

This means that $a \oplus c > b \oplus d$ whichever would be the elements a, b, c and d of the C-semigroup $\mathcal{C}$.

Corollary. *$a > b$ implies $a \oplus c > b \oplus c$, $\forall\, a,\, b,\, c \in \mathcal{C}$.*

THEOREM 2.3. *Two inequalities of the same kind can be multiplied term by term.*

Proof. Let be $a > b$ and $c > d$. From the first inequality results that

$$a \oplus b = a.$$

Multiplying by c both terms of the above equality, from the right, we obtain:

(2.15) $$(a \oplus b) \circ c = a \circ c.$$

According to the property (2.6) the relation (2.15) becomes

$$(a \circ c) \oplus (b \circ c) = a \circ c$$

that is

$$a \circ c > b \circ c. \tag{2.16}$$

On the other hand $c > d$ implies $c \oplus d = c$.

Multiplying by b both terms of the equality above, from the left, we obtain:

$$b \circ (c \oplus d) = b \circ c$$

which, according to (2.5) can be written

$$(b \circ c) \oplus (b \circ d) = b \circ c$$

that is

$$b \circ c > b \circ d. \tag{2.17}$$

From (2.16) and (2.17) results that $a \circ c > b \circ c > b \circ d$ This fact proves the theorem.

Remark. Throughout the proof of the theorem we have shown that $a > b$ implies $a \circ c > b \circ c$, which is an analogous property to the corollary of the previous theorem.

Definition 2.2. *An element a belonging to the C-semigroup $\mathcal{C}$ is called supraunitary if $a > e$. A C-semigroup whose elements are all supraunitary ones is called a supraunitary C-semigroup.* The element a is called subunitary element if $a < e$. The C-semigroup whose elements are all subunitary elements is called a subunitary C-semigroup.

Remark. The algebraical structures studied by GR. C. MOISIL [13] and M. YOELI [29] are subunitary C-semigroups. They were defined either by the supplementary condition

$$a \oplus e = e, \quad \forall a \in \mathcal{C} \tag{2.18}$$

attached to the properties (2.1) — (2.7), or by a law of absobtion of the form

$$b \oplus (b \circ a) = b \qquad \forall\ a,\ b \in \mathcal{C} \tag{2.19}$$

The relation (2.18) is equivalent to $a < e$. A similar conclusion is reached if in (2.19) we take $a = e$. In fact, both the above conditions are equivalent in a C-semigroup, because from (2.18), by multiplying from the left, we have

$$b \circ (a \oplus e) = b$$

and hence

$$(b \circ a) \oplus (b \circ e) = b$$

that is

$$(b \circ a) \oplus b = b$$

which, due to the commutativity of addition, represents the condition (2.19) proper. The notion of C- semigroup defined in this study represents an important generalization of the previously studied algebraical structures, enlarging considerably the class of the problems under discussion. For example, we mention that the algebraical structures studied by the authors quoted above, are not concerned with the study of the longest path in a graph and implicitly cannot be applied to the scheduling problems. But the C-semigroup comprises these problems too.

In what follows, we denote $a^{\circ} = e$, $a^{k} = a^{k-1} \circ a$ and

$$\underset{k=1}{\overset{n}{\boxed{+}}}\ a_k = a_1 \oplus a_2 \oplus \ldots \oplus a_n.$$

Property. 2.1. *If $a \in \mathcal{C}$ and $a > e$ then*

$$\underset{q=0}{\overset{p}{\boxed{+}}}\ a^q = a^p$$

because q being supraunitary according to the remark of the theorem 2.3, we have

$$e < a < a^2 < \ldots < a^p.$$

Similarly it is justified

Property 2.2 *If $a \in \mathcal{C}$ and $e > a$ then*

$$\underset{q=}{\overset{p}{\boxed{+}}}\ a^q = e.$$

THEOREM. 2.4. *If $a \in \mathcal{C}$, $x \in \mathcal{C}$ and $a > e$, the following relations are equivalent*

$$x > x \circ a \tag{2.20}$$

$$x = x \circ a \tag{2.21}$$

$$x = x \circ a^p \tag{2.22}$$

$$x > x \circ a^p \tag{2.23}$$

whichever should be the natural number p.

Proof.
a). The relation (2.20) implies (2.21). Evidently, the inequality $a > e$ multiplied from the left by x becomes $x \circ a > x$. This inequality together with (2.20), implies (2.21).
b). The relation (2.21) implies (2.22). Multiplying from the right the relation (2.21) by a, a^2, a^3, $\ldots$, a^{p-1} we obtain the succession of equalities

$$x = x \circ a$$

$$x \circ a = x \circ a^2$$

$$x \circ a^2 = x \circ a^3$$

$$\cdots\cdots\cdots$$

$$x \circ a^{p-1} = x \circ a^p$$

which imply the relation (2.22).
c). The relation (2.22) implies (2.23). This implication is obvious.
d). The relation (2.23) implies (2.20). From $a > e$ results that $a^p > a$ hence $x \circ a^p > x \circ a$. From the last inequality and from (2.23) results (2.21).

Definition 2.3. *An element a belonging to the C-semigroup $\mathcal{C}$ is called a regular one if $a \circ b < a$ implies $b < e$, $\forall\, b \in \mathcal{C}$. A subset of the C-semigroup is called a regular one if all its elements are regular.*
Remark. Any C-semigroup contains regular elements because e is obviously a regular element. In a subunitary C-semigroup any element is regular because apriori $b < e$.

Definition 2.4. *An element a belonging to the C-semigroup $\mathcal{C}$ is called normal, if $a = a \circ a$ implies $a = e$. A C-semigroup is called normal if all its elements are normal.*

Definition 2.5. *An element a belonging to the C-semigroup $\mathcal{C}$ is called p-stable if $a^p = a^{p+1}$. The element a^p is called the stable power of the element a.*

THEOREM 2.5 *If $a \in \mathcal{C}$ is a p-stable element, then the equation*

$$x = x \circ a \tag{2.24}$$

is satisfied by

$$x_1 = x_0 \circ a^p \tag{2.25}$$

whichever should be $x_0 \in \mathcal{C}$.

Proof. As a is p-stable, $a^p = a^{p+1}$. Multiplying from the left this equation by x_0, we obtain

$$x_0 \circ a^p = x_0 \circ a^{p+1}. \tag{2.26}$$

The relation (2.26) can also be written:

$$x_0 \circ a^p = (x_0 \circ a^p) \circ a$$

and taking into account (2.25)

$$x_1 = x_1 \circ a.$$

This fact proves the theorem.

Definition. 2.6 *An element a belonging to the C-semigroup $\mathcal{C}$ is called weak p-stable if*

$$\mathop{\boxed{+}}\limits_{q=1}^{p+1} a^q = \mathop{\boxed{+}}\limits_{q=1}^{p} a^q. \tag{2.27}$$

THEOREM 2.6 *Any weak p-stable supraunitary element is p-stable.*
Proof. Let be $a \in \mathcal{C}$ a weak p-stable supraunitary element. Due to the fact that $a > e$ we have

$$\mathop{\boxed{+}}\limits_{q=1}^{p+1} a^q = a^{p+1}$$

and

$$\mathop{\boxed{+}}\limits_{q=1}^{p} a^q = a^p$$

and since a is weak p-stable

$$a^{p+1} = a^{p}.$$

This fact proves the theorem.

THEOREM 2.7. *If $a \in \mathcal{C}$ is weak p-stable, then*

$$\mathop{\boxed{+}}_{q=1}^{m} a^{q} = \mathop{\boxed{+}}_{q=1}^{p} a^{q}.$$

whichever should be $m \geqq p + 1$.

Proof. The fact that a is weak- p-stable is shown by the equality

$$\mathop{\boxed{+}}_{q=1}^{p+1} a^{q} = \mathop{\boxed{+}}_{q=1}^{p} a^{q}$$

which is equivalent to

$$a^{p+1} < \mathop{\boxed{+}}_{q=1}^{p} a^{q}. \tag{2.28}$$

Multiplying both terms of the inequality (2.28) by a we obtain:

$$a^{p+2} < \mathop{\boxed{+}}_{q=2}^{p+1} a^{q}. \tag{2.29}$$

Now adding the element a in both the terms of the inequality (2.29) we obtain

$$a^{p+2} \oplus a < \mathop{\boxed{+}}_{q=1}^{p+1} a^{q}. \tag{2.30}$$

This inequality is equivalent to

$$a^{p+2} \oplus a \oplus \mathop{\boxed{+}}_{q=1}^{p+1} a^{q} = \mathop{\boxed{+}}_{q=1}^{p+1} a^{q} \tag{2.31}$$

or, writing the first term shorter and taking into account that a is weak p-stable

$$\mathop{\boxed{+}}_{q=1}^{p+2} a^{q} = \mathop{\boxed{+}}_{q=1}^{p} a^{q}. \tag{2.32}$$

By repeating the above process the theorem will be proved.

D e f i n i t i o n 2.7. *The element θ belonging to the C-semigroup $\mathcal{C}$ with the properties*

$$a \oplus \theta = a, \; a \circ \theta = \theta \circ a = \theta, \; \forall a \in \mathcal{C} \tag{2.33}$$

is called the neuter element of the C-semigroup.

Remark. The neuter element, if any, is unique. If a C-semigroup does not have a neuter element, there exists an extension of the C-semigroup which contains the neuter element. The element θ is neither regular, nor normal. The first equality from (2.27) is equivalent to $\theta < a, \forall a \in \mathcal{C}$ hence θ is the smallest element of the C-semigroup.

Further, we shall give some examples of C-semigroups.

E x a m p l e 2.1. $\mathcal{C} = \{0, 1\}$, $\oplus = \vee$, $\circ = \wedge$, $e = 1$, $\theta = 0$ the usual Boolean algebra. It is a commutative, subunitary C-semigroup.

E x a m p l e 2.2. $\mathcal{C} = (-\infty, +\infty]$, $\oplus = \min$, $\circ = +$, $e = 0$, $\theta = +\infty$ "$>$" = "$\leq$". This C-semigroup is regular, normal and commutative. It does not contain p-stable elements except the elements e and θ.

E x a m p l e 2.3. $\mathcal{C}\ [0, +\infty]$, $\oplus = \min$, $\circ = +$, $e = 0$, $\theta = +\infty$ This C-semigroup is subunitary, regular, normal and commutative.

E x a m p l e 2.4. CRUON R, HERVÉ PH [4]). $\mathcal{C} = [-\infty, +\infty)$ $\oplus = \max$, $\circ = +$, $e = 0$, $\theta = -\infty$, "$<$" = "$\geqq$".

This C-semigroup is regular, normal and commutative.

E x a m p l e 2.5. $\mathcal{C} = M(k)$, $\oplus = \min$, $\circ = +$, $e = 0$, $\theta = +\infty$ where $M(k)$ is the set of the multiples of the number k reunited with $\{0, +\infty\}$. In particular we may have $k = 1$. SHIMBEL A. [26]). This C-semigroup is a subunitary, regular, normal and commutative one.

E x a m p l e 2.6. $\mathcal{C} = [0, 1]$, $\oplus = \max$, $\circ = \times$, $e = 1$, $\theta = 0$ where $\times$ represents the usual multiplication. This C-semigroup is subunitary, regular, normal and commutative.

E x a m p l e 2.7. $\mathcal{C} = [-\infty, +\infty]$, $\oplus = \max$, $\circ = \min$, $e = +\infty$ $\theta = -\infty$. This C-semigroup is subunitary and commutative. It is not normal since $\min(a, a) = a$ for any $a \in \mathcal{C}$.

E x a m p l e 2.8. $\mathcal{C} = A$, $+ = \cup$, $\circ = \cap$, $e = \bigcup_{a \in A} a$, $\theta = \Phi$ where A is a family of sets, $\cup$ is the reunion operation and $\cap$ is the operation of intersection of two sets. This C-semigroup is subunitary and commutative but not normal.

Example 2.9. Let M be a set in which it is defined an associative internal composition-law $*$ which has a neutral element denoted by 1. Then $\mathcal{C}=\mathfrak{L}(M)$, $\oplus = \cup$, $A \circ B = \{a * b,\ a \in A,\ b \in B\}$, $\forall\ A,\ B \in \mathfrak{L}(M)$, $e = 1$, $\theta = \Phi$, where $\mathfrak{L}(M)$ is the set of the parts of M and Φ is the void part, being a C-semigroup.

Example 2.10. $\mathcal{C}$ is the set of the functions defined and continuous on a closed interval $[0,1]$, taking in each point $x \in [0, 1]$ a real nonnegative value, $(f \oplus g)\ (x) = \sup\ [f(x),\ g(x)]$, $(f \circ g)\ (x) = f(x)\ g(x)$, $e = e(x) = 1$, $\theta = \theta(x) = 0$ This C-semigroup is regular, normal and commutative.

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