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NONLINEAR DIFFERENTIAL n-PARAMETER FAMILIES

BY

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We prove that the continuous function $F(x, a)$ defined on $[\alpha, \beta] \times \mathbf{R}^n$ generates an n-parameter family in the sense of [7] iff for any distinct $x_1, \ldots, x_n$ in $[\alpha, \beta]$ the mapping with the coordinate functions $f^i(a) = F(x_i, a)$ is a homeomorphism of R^n onto R^n. The special case when this mapping is a diffeomorphism is considered and several results concerning n-parameter families with this property are obtained.

0. INTRODUCTION

Denote by $C[\alpha, \beta]$ the linear space of all continuous real-valued functions defined on the closed interval $[\alpha, \beta]$ of the real axis. In what follows an n-dimensional linear subspace L_n of $C[\alpha, \beta]$ will be said to be a Chebyshev space if only the zero element of L_n has more then $n-1$ distinct zeros in $[\alpha, \beta]$. A set of n elements of $C[\alpha, \beta]$ will be said to be a Chebyshev system if it spans an n-dimensional Chebyshev subspace of $C[\alpha, \beta]$.

The present paper is devoted to the study of n-parameter families. This notion is the non-linear generalization of the Chebyshev spaces and is defined as follows:

DEFINITION 1. A family F of functions in $C[\alpha, \beta]$ is said to be an *n-parameter family* if for any distinct points $x_1, \ldots, x_n$ in $[\alpha, \beta]$ and any real-values $y_1, \ldots, y_n$ there exists a single function f in F with the property

$$f(x_i) = y_i, \; i = 1, \ldots, n.$$

Investigations on the n-parameter families were made by T. Motzkin [5], L. Tornheim [7], M. I. Morozov [4], E. Moldovan [2, 3] and others.

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As observed by G. Meinardus in his book [1], a great inconvenience in the study of the n-parameter families is the small number of concrete exemples (loc. cit., p. 142).

The aim of this paper is to propose a topological method in the study of the n-parameter families which seems to be very efficient in revealing a lot of new interesting properties.

We consider the n-parameter families to be n-dimensional topological manifolds in the space $C[\alpha, \beta]$ which is topologized with the uniform norm. If such a manifold is differentiable and its tangent spaces are Chebyshev spaces of dimension n, then the n-parameter family is said to be differentiable. We concentrate ourselfs upon two problems :

(i) To characterise the differentiable n-parameter families which have in every point the same Chebyshev space as tangent space.

(ii) Given two Chebyshev spaces of dimension n, does then exist a differentiable n-parameter family for which these spaces are tangent?

The second problem is only partially solved. In particular this partial solution permits the construction of differentiable n-parameter families which do not have the property (i) and yield examples for non-linear n-parameter families.

1. THE NOTION OF DIFFERENTIAL n-PARAMETER FAMILIES

We need for our purpose the following equivalent definition of the n-parameter families :

DEFINITION 1′. Let $F(x, a)$ be a real-valued continuous function defined on $[\alpha, \beta] \times \mathbf{R}^n$ where $x \in [\alpha, \beta]$, $a \in \mathbf{R}^n$, $\mathbf{R}^n$ being the n-dimensional real Euclidean space. The function $F(x, a)$ *generates an n-parameter family* if for any distinct points $x_1, \ldots, x_n$ in $[\alpha, \beta]$ and any vector $b = (b^1, \ldots, b^n)$ in $\mathbf{R}^n$ there exists a single vector a in $\mathbf{R}^n$ such that

$$F(x_i, a) = b^i, \; i = 1, \ldots, n.$$

Now prove the following

THEOREM 1. *The function $F(x, a)$ which is continuous on $[\alpha, \beta] \times \mathbf{R}^n$ generates an n-parameter family if and only if for any distinct points $x_1, \ldots, x_n$ in $[\alpha, \beta]$ the map $b = \Phi(a)$ of $\mathbf{R}^n$ in $\mathbf{R}^n$ with coordinate functions*

$$b^i = F(x_i, a), \quad i = 1, \ldots, n \tag{1}$$

is a homeomorphism onto.

The sufficience of condition of Theorem 1 is obvious.

The necessity. From the definition of the continuous function $F(x, a)$ which generates an n-parameter family it follows the one-to-ones and the continuity of the mapping Φ. To demonstrate that Φ^{-1} is also continuous, it is sufficient to show that Φ is an open map. Let G be an open subset of $\mathbf{R}^n$ and $b \in \Phi(G)$, i.e. $b = \Phi(a)$, $a \in G$. Denote by V a bounded neighbourhood of a with the property that $V \subset G$. From the compacteness of $\overline{V}$ it follows that $\Phi \mid \overline{V}$ is a homeomorphism. Applying now the Brouwer theorem of invariance of the domain it follows that $\Phi(V)$ is open. But $\Phi(V) \subset \Phi(G)$ and $b \in \Phi(V)$ and therefore b belongs to $\Phi(G)$ together with the neighbourhood $\Phi(V)$. Because b was arbitrary in $\Phi(G)$ it follows that this set is open, which completes the proof. [1]

DEFINITION 2. If for any distinct points $x_1, \ldots, x_n$ in $[\alpha, \beta]$ the mapping Φ of $\mathbf{R}^n$ in $\mathbf{R}^n$ with coordinate functions (1) is also a diffeomorphism (i.e. it is a homeomorphism and the partial derivatives of $F(x_i, a)$ with respect to a^j are continuous functions and the Jacobian is everywhere different from zero), then the n-parameter family generated by $F(x, a)$ will be called *differential n-parameter family* (DnF).

Consider $F(x, a)$ as a manifold in $C[\alpha, \beta]$. If $F(x, a)$ generates a DnF, then its tangent space in any point $a \in \mathbf{R}^n$ forms a Chebyshev space of dimension n, which will be denoted by $L_n(a)$.

2. UNISPATIAL DIFFERENTIAL n-PARAMETER FAMILIES

DEFINITION 3. If the DnF generated by $F(x, a)$ has the property that its tangent space $L_n(a)$ in any point $a \in \mathbf{R}^n$ is the same Chebyshev space L_n, then it will be called *unispatial DnF* with the tangent space L_n.

THEOREM 2. *A function $F(x, a)$ generates a unispatial DnF with the tangent space L_n if and only if the function $F(x, a)$ may be represented in the form*

$$F(x, a) = \sum_{i=1}^{n} f^i(a)\, \varphi_i(x) + \psi(x) \tag{2}$$

where φ_i, $i = 1, \ldots, n$ is a basis of the space L_n, f^i, $i = 1, \ldots, n$ are the coordinate functions of a diffeomorphism of $\mathbf{R}^n$ onto $\mathbf{R}^n$ and $\psi(x)$ is an element in $C[\alpha, \beta]$.

The necessity. Suppose that the function $F(x, a)$ generates a unispatial DnF and show that it may be represented in the form (2).

The functions

$$\frac{\partial F(x, a)}{\partial a^i}, \quad i = 1, \ldots, n$$

form a basis of the space L_n. Fix now the distinct points $x_1, \ldots, x_n$ in $[\alpha, \beta]$ and suppose that $\varphi_1, \ldots, \varphi_n$ is a basis of L_n with the property

$$\varphi_i(x_j) = \delta_i^j, \quad i, j = 1, \ldots, n.$$

Then we have

$$\frac{\partial F(x, a)}{\partial a^i} = \sum_{j=1}^{n} a_{ij}(a)\, \varphi_j(x), \quad i = 1, \ldots, n. \tag{3}$$

[1] This simple proof was suggested by M. Bognár.

where $A(a) = \| a_{ij}(a) \|$ is a non-singular $n \times n$ matrix. Putting $x = x_k$ in the formula (3) we obtain

$$a_{ik}(a) = \frac{\partial F(x_k, a)}{\partial a^i}, \quad i, k = 1, \ldots, n$$

and then the formula (3) becomes

$$\frac{\partial F(x, a)}{\partial a^i} = \sum_{j=1}^{n} \frac{\partial F(x_j, a)}{\partial a^i} \varphi_j(x), \quad i = 1, \ldots, n.$$

By integration with respect to a^i we obtain

$$F(x, a) = \sum_{j=1}^{n} F(x_j, a) \varphi_j(x) + \varkappa_i(x, a^1, \ldots, a^{i-1}, a^{i+1}, \ldots, a^n),$$

$$i = 1, \ldots, n.$$

From the above equalities it follows that all the functions $\varkappa_i$, $i = 1, \ldots n$, are equal with a function $\psi(x)$ depending only on x, which according to the condition imposed on $F(x, a)$ must be continuous on $[\alpha, \beta]$. Finally, the map with the coordinate functions $F(x_j, a) = f^j(a)$, $j = 1, \ldots, n$, form a diffeomorphism of $\mathbf{R}^n$ onto $\mathbf{R}^n$ because $F(x, a)$ is supposed to generate a differential n-parameter family, which proves the necessity part of the theorem.

The sufficience. By representation (2) it follows that for fixed $x_1, \ldots, x_n$ in $[\alpha, \beta]$ the map with the coordinate functions $b^i = F(x_i, a)$ $i = 1, \ldots, n$ is the composition of the map with coordinate functions $b^i = f^i(a)$, $i = 1, \ldots, n$ and the map with coordinate functions $b^i = \sum_{j=1}^{n} a^j \varphi_j(x_i)$, $i = 1, \ldots n$, which are both diffeomorphisms onto, and an additional constant vector $(\psi(x_1), \ldots, \psi(x_n))$, and therefore it is also a diffeomorphism onto. Moreover, by (2) we have

$$\frac{\partial F(x, a)}{\partial a^i} = \sum_{j=1}^{n} \frac{\partial f^j(a)}{\partial a^i} \varphi_j(x), \; i = 1, \ldots, n,$$

i.e. every tangent space $L_n(a)$ is generated by $\varphi_1, \ldots, \varphi_n$, and therefore is identical with L_n, which completes the proof of sufficience.

3. PREPARATORY RESULT: AN INVERSE FUNCTION THEOREM OF GLOBAL CHARACTER

In the proof of our inverse function theorem we will use the following simple

LEMMA. *Let $f(a)$ be a function with continuous and bounded partial derivatives on the open, convex set D in $\mathbf{R}^n$. Then $f(a)$ may be extended by continuity to all the set $\overline{D}$.*

INVERSE FUNCTION THEOREM. *Consider the map* $\Phi : \mathbf{R}^n \to \mathbf{R}^n$ *with coordinate functions*

$$b^i = \Phi^i(a), \; i = 1, \ldots, n \tag{4}$$

which satisfy the conditions:

1) *The functions* Φ^i *have continuous and bounded partial derivatives on the whole space* $\mathbf{R}^n$.

2) *There exists a positive* δ *so that* $J(a) > \delta$ *on all the* $\mathbf{R}^n$, *where* $J(a)$ *is the Jacobian of the mapping* Φ *in the point* a.

Then Φ *is a diffeomorphism of* $\mathbf{R}^n$ *onto itself.*

Proof. Let $a_0 \in R^n$ and $b_0 = \Phi(a_0)$. Then from the classical inverse function theorem there exists a neighbourhood of a_0 which is mapped diffeomorphically onto a neighbourhood of b_0. Suppose that this second neighbourhood is the open cube $C(b_0)$ with its centre in b_0. The inverse function Ψ of Φ which is defined on $C(b_0)$ has coordinate functions Ψ^i with partial derivatives expressed in the form (see for example [8] pp. 462).

$$\left.\frac{\partial \Psi^i}{\partial b^j}\right|_{b} = -\frac{1}{J(a)} \frac{D(\eta^1, \ldots, \eta^n)}{D(a^1, \ldots, a^{i-1}, b^j, a^{i+1}, \ldots, a^n)}, \; i, j = 1, \ldots, n,$$

where $\eta^i = \Phi^i(a) - b^i$, $i = 1, \ldots, n$, and the vector a is that which corresponds to b by diffeomorphism Φ. From the conditions of the theorem it follows that the functions Ψ^i have continuous and bounded partial derivatives in $C(b_0)$. But then from the lemma it follows that they may be extended by continuity to the closed cube $\overline{C}(b_0)$. If we put Ψ^i for a^i in the equalities (4), they become identities on $C(b_0)$ and if we extend the functions Ψ^i by continuity to $\overline{C}(b_0)$, then from the continuity of the functions Φ^i it follows that (4) becomes an identity on all the set $\overline{C}(b_0)$, i.e. the map Ψ so extended is an inverse of Φ on $\overline{C}(b_0)$. This means that a boundary point b of $\overline{C}(b_0)$ satisfys (4) while $a = \Psi(b)$. Now, applying the classical inverse function theorem to the neighbourhoods of the points a which are counter-images under Φ of the boundary points of $\overline{C}(b_0)$, we observe that each a with this property has a neighbourhood which may be diffeomorphically maped in a neighbourhood of $b = \Phi(a)$. These neighbourhoods cover the boundary of $C(b_0)$ and because of compactness of this boundary, from the covering may be extracted a finite covering, and we conclude that there exists a cube $C_1(b_0) \supset \overline{C}(b_0)$ so that the functions Ψ^i may be extended to $C_1(b_0)$ and the map Ψ with these coordinate functions is a differentiable inverse of Φ. Applying the above method we may extend Ψ to $C_2(b_0) \supset \overline{C}_1(b_0)$, and so on. The family of such cubes has a maximal element which cannot be bounded because any bounded cube may be extended as above, and therefore this maximal element must coincide with the space $\mathbf{R}^n$.

4. COMPOSABLE CHEBYSHEV SYSTEMS

DEFINITION 4. Let $\varphi_1(x), \ldots, \varphi_n(x)$ and $\psi_1(x), \ldots, \psi_n(x)$ be Chebyshev systems on $[\alpha, \beta]$. They will be said to be *composable* if

(i) every sequence of n functions $\eta_1, \ldots, \eta_n$, where η_i is φ_i or ψ_i (generally depending on i) form a Chebyshev system;

(ii) the determinants

$$\det \| \eta_i (x_j) \|_{i, j = 1, \ldots, n}$$

are of the same sign independent of the manner of chosing φ_i or ψ_i for η_i $i = 1, \ldots, n$. Here $x_1, \ldots, x_n$ are distinct points in $[\alpha, \beta]$.

Example 1. Let be $\varphi_1(x), \ldots, \varphi_{n-1}(x)$ a Chebyshev system on $[\alpha, \beta]$. and let be φ_n and ψ_n two concave (or convex) functions in the sense of [2] with respect to this Chebyshev system. It is easy to see that the Chebyshev systems $\varphi_1(x), \ldots, \varphi_{n-1}(x), \varphi_n(x)$ and $\varphi_1(x), \ldots, \varphi_{n-1}(x), \psi_n(x)$ are composable.

Example 2. Let $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ be two systems of real numbers which satisfy

$$\alpha_1 \leqq \beta_1 < \alpha_2 \leqq \beta_2 < \ldots < \alpha_n \leqq \beta_n.$$

Then the Chebyshev systems $e^{\alpha_1 x}, \ldots, e^{\alpha_n x}$ and $e^{\beta_1 x}, \ldots, e^{\beta_n x}$ are composable. To show this let $x_1, \ldots, x_n$ be distinct points in $[\alpha, \beta]$. The determinants

$$\det \| e^{\gamma_i x_j} \|_{i, j = 1, \ldots, n}$$

are of the same sign independently of the choice of γ_i, which is α_i or β_i, $i = 1, \ldots, n$. We prove the following particular case of this assertion:

$$\operatorname{sgn} \det \| e^{\alpha_i x_j} \| = \operatorname{sgn} \det \| e^{\beta_i x_j} \|.$$

The general case may be proved similarly. Suppose the contrary:

$$\operatorname{sgn} \det \| e^{\alpha_i x_j} \| = - \operatorname{sgn} \det \| e^{\beta_i x_j} \|.$$

Now letting $\beta_i \to \alpha_i$, $i = 1, \ldots, n$ monotonically, it follows that the second determinant must vanish for some distinct $\beta_1, \ldots, \beta_n$ which is in contradiction with the fact that $e^{\beta_1 x}, \ldots, e^{\beta_n x}$ form a Chebyshev system for any distinct $\beta_1, \ldots, \beta_n$.

Other examples for composable Chebyshev systems may be constructed according to M. A. Rutman's results [6].

5. THE CONSTRUCTION OF A DIFFERENTIAL n-PARAMETER FAMILY WITH GIVEN TANGENT SPACES

THEOREM 3. *If $\varphi_1(x), \ldots, \varphi_n(x)$ and $\psi_1(x), \ldots, \psi_n(x)$ are composable Chebyshev systems on $[\alpha, \beta]$, then there exists a function $F(x, a)$ which generates a DnF and has the property that*

$$\frac{\partial F(x, 0)}{\partial a^i} = \psi_i(x) \text{ and } \frac{\partial F(x, 1)}{\partial a^i} = \varphi_i(x), \; i = 1, \ldots, n. \tag{5}$$

(Here 0 and 1 stand for the zero vector, respective for the vector with all the components 1.)

Proof. Construct the functions $f(x, a^i)$, $i = 1, \ldots, n$ in the following manner:

$$f(x, a^i) = \begin{cases} \psi_i(x) & a^i \leqq 0 \\ a^i \varphi_i(x) + (1 - a^i)\psi_i(x) & 0 < a^i < 1 \\ \varphi_i(x) & a^i \geqq 1 \end{cases} \qquad i = 1, \ldots, n.$$

The functions $f(x, a^i)$ have the following properties

(i) they are continuous functions of two variables;

(ii) for each $a = (a^1, \ldots, a^n)$ they form a Chebyshev system.

The property (i) is obvious. For (ii) we consider the distinct points $x_1, \ldots, x_n$ in $[\alpha, \beta]$. Let $a = (a^1, \ldots, a^n)$ be a vector in $\mathbf{R}^n$. With a reordering of the components (with respective reordering of the indices of the Chebyshev systems) we may realize $a^i \leqq 0$ $i = 1, \ldots, m$, $0 < a^i < 1$, $i = m + 1, \ldots, p$, $a^i \geqq 1$, $i = p + 1, \ldots, n$. We have then

$$\det \| f(x_j, a^i) \|_{i, j = 1, \ldots, n} =$$

$$\begin{vmatrix} \psi_1(x_1) \ldots \psi_m(x_1) & \nu(x_1, a^{m+1}) \ldots \nu(x_1, a^p) & \varphi_{p+1}(x_1) \ldots \varphi_n(x_1) \\ \psi_1(x_2) \ldots \psi_m(x_2) & \nu(x_2, a^{m+1}) \ldots \nu(x_2, a^p) & \varphi_{p+1}(x_2) \ldots \varphi_n(x_2) \\ \cdots & \cdots & \cdots \\ \psi_1(x_n) \ldots \varphi_m(x_n) & \nu(x_n, a^{m+1}) \ldots \nu(x_n, a^p) & \varphi_{p+1}(x_n) \ldots \varphi_n(x_n) \end{vmatrix}$$

where
$\nu(x_j, a^k) = a^k \psi_k(x_j) + (1 - a^k)\varphi_k(x_j)$, $j = 1, \ldots, n$, $k = m + 1, \ldots, p$.

If we decompose this determinant into the sum of determinants then from the composability of the Chebyshev systems $\varphi_1(x), \ldots, \varphi_n(x)$ and $\Psi_1(x), \ldots, \psi_n(x)$ (the composability remains after a simultaneous reordering of the indices of the Chebyshev systems) and the positivity of the numbers a^i and $1 - a^i$, $i = m + 1, \ldots, p$, it follows immediately that the above determinant is different from zero.

Let now

$$F(x, a) = \sum_{i=1}^{n} \int f(x, a^i) \mathrm{d}a^i.$$

The function $F(x, a)$ generates a DnF. Really, if $x_1, \ldots, x_n$ are distinct points in $[\alpha, \beta]$ then the map $\Phi : \mathbf{R}^n \to \mathbf{R}^n$ with coordinate functions

$$b^i = \Phi^i(a) = F(x_i, a), \; i = 1, \ldots, n$$

is a diffeomorphism of $\mathbf{R}^n$ onto itself. This follows from the inverse function theorem. The condition 1.) of this theorem is obviously satisfied and for condition 2) we have

$$\inf_{a \in R^n} |J(a)| = \inf_{a \in R^n} |\det \| f(x_j, a^i) \|| = \min_{\substack{0 \leqq a^i \leqq 1 \\ i = 1, \ldots, n}} |\det \| f(x_j, a^i) \|| > \delta > 0$$

because the determinant is a continuous function on the closed cube $0 \leqq a^i \leqq 1$, $i = 1, \ldots, n$ and attains on it its minimum, which cannot be zero. From the continuity of the partial derivatives, $J(a)$ must be everywhere positive or negative and therefore all conditions of the inverse function theorem are fulfilled.

Finally, we observe that the conditions (5) are satisfied and a simple application of the Theorem 1 completes the proof of our theorem.

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REFERENCES

1. G. Meinardus, *Approximation von Funktionen und ihre numerische Behandlung.* Springer V. Berlin, 1964.
2. E. Moldovan, *Sur une généralisation des fonctions convexes,* Mathematica, 1959, **1 (24)**, 49–80.
3. — *Introduction à l'étude comparative des ensembles de fonctions interpolatoires,* Mathematica 1964, **6 (29)**, 145–155.
4. М. И. Морозов, *О некоторых вопросах равномерного приблиэкения непрерывных функций посредством функции интерполяционных классов,* Изв. Акад. наук СССР, 1952, **16**, 75—100.
5. T. Motzkin, *Approximation by curves of a unisolvent family,* Bull. Amer. Math. Soc., 1949, **55**, 789–793.
6. М. А. Рутман, *Интегральное представление функций, образующих ряд Маркова,* Докл. Акад. наук СССР, 1965, **164**, 989—992.
7. L. Tornheim, *On n-parameter families of functions and associated convex functions,* Trans. Amer. Math. Soc., 1950, **69**, 457–467.
8. Г. М. Фихтенгольц, *Курс дифференциального и интегрального исчисления* I, Физматгиз, Москва, 1962.