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383. A REMARK ON THE SCHWEITZER AND KANTOROVICH INEQUALITIES*

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In 1914 P. SCHWEITZER proved the following theorem: If $\{a_1, \dots, a_n\}$ is a set of real numbers with the property

$$0 < m \leq a_k \leq M \quad (k = 1, \dots, n),$$

then

$$(1) \quad \left(\frac{1}{n} \sum_{k=1}^n a_k \right) \left(\frac{1}{n} \sum_{k=1}^n \frac{1}{a_k} \right) \leq \frac{(M+m)^2}{4Mm}.$$

In [4] (see also [3]) G. PÓLYA and G. SZEGÖ have obtained the following generalization: If $\{A_1, \dots, A_n\}$, $\{B_1, \dots, B_n\}$ are real numbers which verify

$$0 < m_1 \leq A_k \leq M_1, \quad 0 < m_2 \leq B_k \leq M_2 \quad (k = 1, \dots, n),$$

then

$$(2) \quad \frac{\left(\sum_{k=1}^n A_k^2 \right) \left(\sum_{k=1}^n B_k^2 \right)}{\left(\sum_{k=1}^n A_k B_k \right)^2} \leq \left(\frac{\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}}{2} \right)^2.$$

Then they showed that the equality is attained in (2) if and only if the following conditions are simultaneously fulfilled:

$$I. \quad \alpha := \frac{\frac{M_1}{m_1}}{\frac{M_1}{m_1} + \frac{M_2}{m_2}} n, \quad \beta := \frac{\frac{M_2}{m_2}}{\frac{M_1}{m_1} + \frac{M_2}{m_2}} n$$

are natural numbers.

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II. α of the numbers $A_1, \dots, A_n$ are equal to m_1 , and β of these numbers equal to M_1 , and if the corresponding numbers B_k are equal to M_2 and m_2 respectively.

If $A_k := \sqrt{a_k}$, $B_k := \frac{1}{\sqrt{a_k}}$, $M_1 := \sqrt{M}$, $m_1 := \sqrt{m}$, $M_2 := \frac{1}{\sqrt{m}}$, $m_2 := \frac{1}{\sqrt{M}}$, then (2) is the same inequality as (1). On the other hand in this case

$$\alpha = \beta = \frac{n}{2}$$

such that for $n = 2p + 1$ the SCHWEITZER's result does not give the best bound, taking into account that in (1) the equality is not attained (except some trivial situations as: $m = a_k = M = 1$, $k = 1, \dots, n$).

The object of this note is to improve SCHWEITZER's inequality.

Theorem 1. If $\{a_1, \dots, a_n\}$ is a set of real numbers with

$$0 < m \leq a_k \leq M \quad (k = 1, \dots, n),$$

then

$$\left(\sum_{k=1}^n a_k \right) \left(\sum_{k=1}^n \frac{1}{a_k} \right) \leq \frac{\left(\left[\frac{n}{2} \right] M + \left[\frac{n+1}{2} \right] m \right) \left(\left[\frac{n+1}{2} \right] M + \left[\frac{n}{2} \right] m \right)}{Mm}$$

where the symbol $[\cdot]$ denotes the integral part.

The equality is attained if $\left[\frac{n}{2} \right]$ of the numbers $a_1, \dots, a_n$ are equal to m (or to M) and if $\left[\frac{n+1}{2} \right]$ of these numbers are equal to M (or to m).

Proof. It is stated in [1] (see also 3.2.26, page 205 in [3]) that if $\{A_1, \dots, A_n\}$, $\{B_1, \dots, B_n\}$ are real numbers with

$$m_1 \leq A_k \leq M_1, \quad m_2 \leq B_k \leq M_2 \quad (k = 1, \dots, n),$$

then

$$\left| \sum_{k=1}^n A_k B_k - \frac{1}{n} \sum_{k=1}^n A_k \sum_{k=1}^n B_k \right| \leq (M_1 - m_1)(M_2 - m_2) w(n)$$

where

$$w(n) = \begin{cases} \frac{n}{4} & \text{for } n \text{ even,} \\ \frac{n^2-1}{4n} & \text{for } n \text{ odd.} \end{cases}$$

If we take

$$A_k := a_k > 0, \quad B_k := \frac{1}{a_k}, \quad M_1 := M, \quad m_1 = m, \quad M_2 := \frac{1}{m}, \quad m_2 := \frac{1}{M},$$

then from the above inequality we obtain

$$\left(\sum_{k=1}^n a_k \right) \left(\sum_{k=1}^n \frac{1}{a_k} \right) \leq \frac{(M+m)^2}{4Mm} n^2 - g(n)$$

where

$$g(n) = \begin{cases} 0, & \text{for } n \text{ even} \\ \frac{(M-m)^2}{4Mm} & \text{for } n \text{ odd.} \end{cases}$$

An easy computation shows that

$$\frac{(M+m)^2}{4Mm} n^2 - g(n) = \frac{\left(\left[\frac{n}{2}\right]M + \left[\frac{n+1}{2}\right]m\right)\left(\left[\frac{n+1}{2}\right]M + \left[\frac{n}{2}\right]m\right)}{Mm}$$

and the theorem is proved.

In [2] P. HENRICI showed that the KANTOROVICH's inequality

$$\left(\sum_{k=1}^n a_k p_k\right) \left(\sum_{k=1}^n \frac{1}{a_k} p_k\right) \leq \frac{(M+m)^2}{4Mm}$$

$$0 < m \leq a_k \leq M, \quad p_k \geq 0, \quad \sum_{k=1}^n p_k = 1 \quad (k = 1, \dots, n),$$

is a simple consequence of SCHWEITZER's inequality (1).

By using the same method of proof which is described in [2], from the above theorem we obtain the following improvement of KANTOROVICH's result.

Theorem 2. Let $\{p_1, \dots, p_n\}$ and $\{a_1, \dots, a_n\}$ be real numbers which verify

$$p_k \geq 0, \quad \sum_{k=1}^n p_k = 1, \quad 0 < m \leq a_k \leq M \quad (k = 1, \dots, n).$$

Then

$$\left(\sum_{k=1}^n a_k p_k\right) \left(\sum_{k=1}^n \frac{1}{a_k} p_k\right) \leq \frac{\left(\left[\frac{n}{2}\right]M + \left[\frac{n+1}{2}\right]m\right)\left(\left[\frac{n+1}{2}\right]M + \left[\frac{n}{2}\right]m\right)}{Mmn^2}.$$

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