UNIVERZITET U BEOGRADU

PUBLIKACIJE

ELEKTROTEHNIČKOG FAKULTETA

SERIJA:

MATEMATIKA I FIZIKA

№ 381 — № 409
(1972)

BEOGRAD

PUBLIKACIJE ELEKTROTEHNIČKOG FAKULTETA UNIVERZITETA U BEOGRADU
PUBLICATIONS DE LA FACULTÉ D'ÉLECTROTECHNIQUE DE L'UNIVERSITÉ À BELGRADE

SERIJA: MATEMATIKA I FIZIKA — SÉRIE: MATHÉMATIQUES ET PHYSIQUE

№ 381 — № 409 (1972)

384. AN INTEGRAL INEQUALITY FOR CONVEX FUNCTIONS*

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In this paper we denote by F one of the following functionals which are well-defined by the relations

$$F(f):=\frac{1}{b-a}\int_a^b f(x)\,dx,\quad F(f):=\frac{\int_a^b p(x)f(x)\,dx}{\int_a^b p(x)\,dx}\quad (a<b),$$

$$F(f):=\sum_{k=0}^{n} p_k f(w_k)\quad (w_k\in[a,b];\ k=0,1,\ldots,n),\tag{1}$$

where $p:[a,b]\to\mathbf{R}$ is a positive, integrable function on $[a,b]$. Likewise we suppose that $p_k\geqq 0$ $(k=0,1,\ldots,n)$, $\sum_{k=0}^{n} p_k=1$. Clearly, if F is in such a manner defined then $F(1)=1$. Sometimes instead of F we write F_x in order to put in evidence the corresponding variable. For instance

$$F_x(f)=\frac{1}{b-a}\int_a^b f(x)\,dx\quad\text{and}\quad F_x\big(f(z)\big)=f(z).$$

Lemma. *If $f,g:[a,b]\to\mathbf{R}$ are convex functions on the interval $[a,b]$, then*

$$\begin{aligned}F(fg)\,[F(e^2)-F(e)^2]-F(f)\,F(g)\,F(e^2)\\ \geqq F(ef)\,F(eg)-[F(f)\,F(eg)+F(g)\,F(ef)]\cdot F(e)\end{aligned}\tag{2}$$

where $e(x)=x$, $x\in[a,b]$. If f or g is a linear function then the equality holds in (2).

Proof. Let $[x,y,z;f]$ be the divided difference of a certain function f. Under our conditions, for all distinct points x,y,z from $[a,b]$

$$[x,y,z;f]\cdot[x,y,z;g]\geqq 0$$

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* Presented December 31, 1971 by D. S. Mitrinović.

1) This paper was partially supported by the Alexander von Humboldt Foundation.

$\int_a^b q(x)\,dx=1$. The ordinary means of order r are the special cases of (2) obtained by setting all $q_k=\frac{1}{n}$. We note that if a is a constant sequence (f is a constant function) or if $n=1$, then $M(r)$ is a constant, and the middle term in (1) is not defined. For this reason we shall always assume that a is not a constant sequence (that f is not constant), and that $n\geqq 2$. In this case it is well-known (see [2, 16—18], [3, 26—27] and [6, 74—76]) that $M(r)$ is a continuous, strictly increasing function of r, and that $r\log M(r)$ is a strictly convex function of r. The left-hand inequality in (1) follows at once from the first of these results, and we shall now show that the strict convexity of $M(r^{-1})$ for $r>0$ follows from the second. In what follows, we deal with (2), the proof for (3) being essentially identical.

In order to prove that $f(r)\equiv M(r^{-1})$ is strictly convex for $r>0$, it suffices [3, 77] to show that $f''(r)>0$ for $r>0$, or since

$$f'(r)=-r^{-2}M'(r^{-1}),\qquad f''(r)=r^{-4}M''(r^{-1})+2r^{-3}M'(r^{-1}),$$

it suffices to prove that

$$g(y)\equiv y^2M''(y)+2yM'(y)>0\quad\text{for}\quad y>0. \tag{4}$$

Now we have

$$M'(y)=y^{-1}\left\{M^{1-y}(y)\sum_{k=1}^{n}q_k a_k^y\log a_k-M(y)\log M(y)\right\}, \tag{5}$$

and so

$$M''(y)=-y^{-2}\left\{M^{1-y}\sum_{1}^{n}q_k a_k^y\log a_k-M\log M\right\} \tag{6}$$

$$+y^{-1}\left\{(1-y)M^{-y}M'\sum_{1}^{n}q_k a_k^y\log a_k+M^{1-y}\sum_{1}^{n}q_k a_k^y\,(\log a_k)^2-M'-M'\log M\right\}.$$

From (4)—(6) it follows that

$$g(y)=M\left\{(\log M)^2-2M^{-y}\log M\sum_{1}^{n}q_k a_k^y\log a_k+M^{-2y}\left(\sum_{1}^{n}q_k a_k^y\log a_k\right)^2\right\} \tag{7}$$

$$+y\left\{M^{1-y}\log M\sum_{1}^{n}q_k a_k^y\log a_k-M^{1-2y}\left(\sum_{1}^{n}q_k a_k^y\log a_k\right)^2\right.$$

$$\left.+M^{1-y}\sum_{1}^{n}q_k a_k^y(\log a_k)^2\right\}.$$

The first term of (7) is

$$M\left(\log M-M^{-y}\sum_{1}^{n}q_k a_k^y\log a_k\right)^2,$$

and so it suffices to prove that the second term of (7) is positive. To prove this we use the strict convexity of $h(y) \equiv y \log M(y)$ (see the suggested proof of this in [2, p. 18]), or rather the fact that

$$h''(y) = 2\frac{M'}{M} + y\frac{MM''-M'^2}{M^2} > 0,$$

$$2MM' + yMM'' - yM'^2 > 0 \quad \text{for all } y. \tag{8}$$

Substituting from (5) and (6) into (8) we obtain finally

$$M^{2-y}\log M \sum_1^n q_k a_k^y \log a_k - M^{2-2y}\left(\sum_1^n q_k a_k^y \log a_k\right) + M^{2-y}\sum_1^n q_k a_k^y (\log a_k)^2 > 0$$

for all y. It follows from this that the second term of (7) is positive for all $y>0$, completing the proof of (1).

We note in passing that, since $(yM(y))'' = yM'' + 2M' = y^{-1}(y^2M'' + 2yM')$, *it follows that $yM(y)$ is also strictly convex for $y>0$.* (This also follows from 3, TH. 119]).

The proof of the right hand inequality of (1) is now almost immediate since it is equivalent to

$$M(s) < \frac{r(t-s)}{s(t-r)}M(r) + \frac{t(s-r)}{s(t-r)}M(t), \qquad 0<r<s<t. \tag{9}$$

Setting $\lambda = t(s-r)/\{s(t-r)\}$, we have $0<\lambda<1$ for the s in question. Hence, setting $r=1/y$, $t=1/x$, $s=1/\{\lambda x + (1-\lambda)y\}$, we see that (9) is equivalent to

$$M\left(\frac{1}{\lambda x + (1-\lambda)y}\right) < \lambda M\left(\frac{1}{x}\right) + (1-\lambda)M\left(\frac{1}{y}\right), \quad (0<x<y, \quad 0<\lambda<1), \tag{10}$$

and this follows from the strict convexity of $M(r^{-1})$ for $r>0$.

We also note that, using the strict convexity of $yM(y)$ for $y>0$, one can prove that M satisfies the inequality

$$(t-r)M(rt/s) < (s-r)M(r) + (t-s)M(t) \qquad (0<r<s<t),$$

or, setting $r=\alpha s$, the inequality

$$M(\alpha t) < \frac{(1-\alpha)s}{t-\alpha s}M(\alpha s) + \frac{t-s}{t-\alpha s}M(t) \qquad (0<s<t, \quad 0<\alpha<1). \tag{11}$$

L. C. Hsu also asserted that the inequalities (1) were *best possible.* Using the continuity of M at r this is obvious for the left hand inequality of (1) if we let s approach r. We shall prove that both inequalities of (1) are best possible for arbitrary (fixed) r, s, t such that $0<r<s<t$. To prove this we first take $a_1 = a_2 = \cdots = a_{n-1} = \varepsilon > 0$, $a_n = a > 0$, and note that for each $n \geq 2$,

$$M(r) = \left\{\varepsilon^r \sum_1^{n-1} q_k + q_n a^r\right\}^{1/r} \to q_n^{1/r} a \quad \text{as } \varepsilon \to 0+.$$

It follows that

$$\frac{M(t)-M(r)}{M(t)-M(s)} \to \frac{q_n^{1/t} - q_n^{1/r}}{q_n^{1/t} - q_n^{1/s}} = \frac{1-y^{R-T}}{1-y^{S-T}} = A(y) \quad \text{as } \varepsilon \to 0+, \tag{12}$$