

**DISCONJUGACY OF AN EQUATION BY THE DISCONJUGACY
OF ITS EQUATIONS IN VARIATION**

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Reprinted from the

GLASNIK MATEMATIČKI 8 (28) (1973), 229—245

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0. Definitions and notations

Denote by $C^\nu[0, 1]$ the linear normed space of all real valued functions which have continuous derivatives of order ν on the interval $[0, 1]$ (in the points 0 and 1 the one-sided derivatives are considered). In the case $\nu = 0$ we use the notation $C[0, 1]$. The norm in the space $C^\nu[0, 1]$ is introduced by the equality

$$\|g\|_\nu = \max_{t \in [0, 1]} \sum_{i=0}^{\nu} |g^{(i)}(t)|. \quad (1)$$

The space $C^\nu[0, 1]$ endowed with the norm (1) becomes a separable Banach space. In what follows we will also consider the direct product space

$$C_n^\nu = C^\nu[0, 1] \times \dots \times C^\nu[0, 1] \quad (2)$$

(n copies), in which the norm of the element $h = (h_1, \dots, h_n)$ is defined by

$$\|h\|_\nu = \sum_{j=1}^n \|h_j\|_\nu. \quad (3)$$

The direct product space (2) endowed with the norm (3) is also a separable Banach space.

DEFINITION 1. *The n -dimensional subspace L_n of the space $C^n[0, 1]$ is said to be an unrestricted Chebyshev space (abbreviated UCSp), if every nonzero element of L_n has at most $n-1$ zeros in $[0, 1]$, counting multiplicities. A basis in a UCSp is said to be an unrestricted Chebyshev system (abbreviated, UCS).*

Consider the n^{th} order differential equation

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)}). \quad (4)$$

In the whole paper we will suppose that the following two conditions are satisfied:

(H1) f is defined, continuous and has continuous partial derivatives with respect to its last n variables on the direct product $[0, 1] \times \mathbb{R}^n$, where $\mathbb{R}^n$ is the real n -dimensional Euclidean space.

(H2) All the solutions of (4) exist throughout on $[0, 1]$.

According to these conditions on (4), for any vector $a = (a^1, \dots, a^n)$ in $\mathbb{R}^n$, the Cauchy-type conditions

$$y^{(j-1)}(0) = a^j, \quad j = 1, \dots, n \quad (5)$$

determine a unique solution y of the differential equation (4). This solution depends continuously on the parameters a^i and has continuous partial derivatives with respect to the parameters a^i , $i = 1, \dots, n$. The solution of (4) satisfying the initial value conditions (5) will be denoted by $Y(t, a^1, \dots, a^n) = Y(t, a)$. Then Y is a function defined and continuous on the direct product $[0, 1] \times \mathbb{R}^n$ and has partial derivatives with respect to a^i , $i = 1, \dots, n$, which are continuous on this direct product. We will denote in what follows by $Y^{(j)}$ the partial derivative of order j of Y , $j = 1, \dots, n$ with respect to t . The partial derivatives $\frac{\partial Y^{(j)}(t, a)}{\partial a^i}$, $i, j = 1, \dots, n$ also exist and are continuous on $[0, 1] \times \mathbb{R}^n$ (Theorem 18 in [13]). Substituting Y in (4) we obtain an identity, which by derivation with respect to a^i furnish the relation

$$\frac{\partial Y^{(n)}}{\partial a^i} = \sum_{j=0}^{n-1} \frac{\partial f}{\partial y^{(j)}} \frac{\partial Y^{(j)}}{\partial a^i}. \quad (6)$$

For a fixed parameter a , the linear differential equation

$$x^{(n)} = \sum_{j=0}^{n-1} \frac{\partial f(t, Y(t, a), Y'(t, a), \dots, Y^{(n-1)}(t, a))}{\partial y^{(j)}} x^{(j)} \quad (7)$$

will be said to form the equation in variation of (4) in the point a . Because we have

$$\frac{\partial Y^{(j-1)}(0, a)}{\partial a^i} = \delta_i^j, \quad i, j = 1, \dots, n, \quad (8)$$

where δ_i^j is the symbol of Kronecker, and because of (6), it follows that the set of functions

$$\frac{\partial Y(t, a)}{\partial a^i}, \quad i = 1, \dots, n$$

for a fixed, form a fundamental system of solutions of the equation in variation (7) in the point a .

We will say that the set of numbers m , t_i , k_i , $i = 1, \dots, m$ satisfies the condition (I), if

$$m, k_i \leq N, \quad t_i \in [0, 1], \quad i = 1, \dots, m, \\ 1 \leq m \leq n, \quad \sum_{i=1}^m k_i = n, \quad t_i \neq t_j, \quad i \neq j,$$

where N denotes the natural numbers.

DEFINITION 2. (i) The differential equation (4) is said to be disconjugate if for any set of numbers m, t_i, k_i satisfying the condition (I), and for any given real numbers $y_i^{(j)}, j = 0, 1, \dots, k_i - 1, i = 1, \dots, m$, there exists a single solution y of (4) satisfying the many point boundary value conditions

$$y^{(j)}(t_i) = y_i^{(j)}, \quad j = 0, 1, \dots, k_i - 1, i = 1, \dots, m. \quad (9)$$

(ii) The differential equation (4) is said to be strongly disconjugate if it is disconjugate and if its equations in variation are all disconjugate.

(iii) The differential equation (4) is said to be weakly disconjugate, if all its equations in variation are disconjugate.

We observe that for the case of linear differential equations these three notions: the notion of disconjugacy; the notions of strong and of the weak disconjugacy, coincide. In this particular case the condition that the differential equation be disconjugate is equivalent with the condition that a fundamental system of solutions of the respective linear differential equation forms on UCS, or equivalently, that the space of solutions of this differential equation forms an UCSp (see Definition 1).

Consider now the linear differential equation

$$x^{(n)} = \sum_{j=0}^{n-1} p_{n-j}(t) x^{(j)}, \quad (10)$$

where $p_i, i = 1, \dots, n$ are elements of $C[0, 1]$. In the set of all linear differential equations of the form (10) we introduce a topology by identification of (10) with the element $(p_1, \dots, p_n)$ in the product space C_n^0 . If we denote after this identification by $\mathfrak{M}$ the set of all disconjugate differential equations, then $\mathfrak{M}$ is open and contractible (see Proposition 9 Chapt. 3 in [2] and respectively [9]).

Applying this notation, the notion of weak disconjugacy may be formulated as follows: The differential equation (4) is weakly disconjugate if the set $\mathfrak{M}(f)$ of all its equations in variation is contained in $\mathfrak{M}$.

The present paper aims to give an independent exposition of the problems concerning strong disconjugacy of nonlinear differential equations. Some results (as that of A. Lasota and Z. Opial [5] and [6], and those in our paper [11]) are reconsidered for this special case of disconjugacy. The principal result of our paper is contained in Theorem 6, which establishes a sufficient condition in order to a weakly disconjugate differential equation be strongly disconjugate. In the proof a special topological technique is used which has its origin in our note [8].

1. Related linear differential equations

Parallel with the equations in variation of (4) we will consider some related differential equations, which are also linear, and which will be used in some strong disconjugacy criterions. We will consider the inhomogeneous linear differential equation of the form

$$x^{(n)} = \sum_{j=0}^{n-1} \frac{\partial f(t, y, y', \dots, y^{(n-1)})}{\partial y^{(j)}} x^{(j)} + R(t, y, y', \dots, y^{(n-1)}), \quad (11)$$

where $y \in C^{n-1}[0, 1]$, and the function R is given by

$$R(t, y, y', \dots, y^{(n-1)}) = f(t, y, y', \dots, y^{(n-1)}) - \sum_{j=0}^{n-1} \frac{\partial f(t, y, y', \dots, y^{(n-1)})}{\partial y^{(j)}} y^{(j)}. \quad (12)$$

From the condition (H1) about f it follows that R is continuous.

Let y_1 and y_2 be elements of $C^{n-1}[0, 1]$ and consider the linear differential equation

$$x^{(n)} = \sum_{j=0}^{n-1} \left(\int_0^1 \frac{\partial f(t, \lambda y_1 + (1-\lambda)y_2, \dots, \lambda y_1^{(n-1)} + (1-\lambda)y_2^{(n-1)})}{\partial y^{(j)}} d\lambda \right) x^{(j)}. \quad (13)$$

In this paragraph we will present a result of A. Lasota and Z. Opial ([5] and [6]) which in a particular form was obtained independently by G. A. Bessmertnyh and Yu. A. Levin ([1]), and which establishes relations between the disconjugacy properties of the differential equations (11) and (13) and the disconjugacy properties of (4). Because of the conditions (H1) and (H2) this result may be formulated in the following form:

THEOREM 1. Suppose that the differential equation (4) has the properties (H1) and (H2).

(A) If

(i) all the differential equations (11) with y in $C^{n-1}[0, 1]$ are disconjugate;

(ii) for any given m, t_i, k_i satisfying (I) the solutions of all the differential equations of form (11) which satisfy the many point boundary value conditions (9) with any given real numbers y_i^j , form a bounded set in the space $C^n[0, 1]$; then any many point boundary value problem (I), (9) for (4) has at least a solution.

(B) If in addition

(iii) each linear differential equation of form (13) with y_1 and y_2 solutions of (4) is disconjugate, then the differential equation (4) is strongly disconjugate.

Proof. (A) Let us consider the many point boundary value problem (9). Denote by M the subset of $C^{n-1}[0, 1]$ of the elements which satisfy the condition (9). The set M is obviously convex and complete. We define a mapping T of M into itself in the following way: Suppose that y is an element of M . Consider the differential equation (11). According the condition (i) of the theorem this differential equation is disconjugate. Let $x = Ty$ be the solution of (11) which satisfies the boundary value conditions (9). By the condition (ii) of the theorem, the set $T(M)$ is bounded in $C^n[0, 1]$ and therefore it is (relatively) compact in the space $C^{n-1}[0, 1]$. We will now prove the continuity of the mapping T . Let $\{y_\nu\}_{\nu=1}^\infty$ be a sequence in M converging to the element y_0 and let be $x_\nu = Ty_\nu$, $\nu = 0, 1, \dots$. The set of vectors $\{(x_\nu(0), x'_\nu(0), \dots, x_\nu^{(n-1)}(0))\}_{\nu=1}^\infty$ is bounded in $\mathbf{R}^n$ and therefore it is compact. Suppose that this set has a limit point, say $(\xi, \xi', \dots, \xi^{(n-1)})$ which is different from $(x_0(0), x'_0(0), \dots, x_0^{(n-1)}(0))$, and suppose that $\{x_{\nu_i}\}_{i=1}^\infty$ is a subsequence of $\{x_\nu\}_{\nu=1}^\infty$ with the property that

$$x_{\nu_i}^{(j)}(0) \rightarrow \xi^{(j)}, \quad \text{as } i \rightarrow \infty, j = 0, 1, \dots, n-1.$$

Let be $\tilde{x}$ the solution of (11) with $y = y_0$, which satisfies the initial value conditions

$$\tilde{x}^{(j)}(0) = \xi^{(j)}, \quad j = 0, 1, \dots, n-1$$

According to the continuity of the functions $\frac{\partial f}{\partial y^{(j)}}, j = 0, 1, \dots, n-1$ and R , it follows that

$$\frac{\partial f(t, y_{\nu_i}, y'_{\nu_i}, \dots, y_{\nu_i}^{(n-1)})}{\partial y^{(j)}} \rightarrow \frac{\partial f(t, y_0, y'_0, \dots, y_0^{(n-1)})}{\partial y^{(j)}}$$

and

$$R(t, y_{\nu_i}, y'_{\nu_i}, \dots, y_{\nu_i}^{(n-1)}) \rightarrow R(t, y_0, y'_0, \dots, y_0^{(n-1)})$$

uniformly in $[0, 1]$ as $i \rightarrow \infty$. Therefore $x_{\nu_i} \rightarrow \tilde{x}$ in the metric of $C^n[0, 1]$, as $i \rightarrow \infty$. But this means that $\tilde{x}$ satisfies (9) and because $x_0 \neq \tilde{x}$, it follows that the differential equation (11) with $y = y_0$ has two distinct solutions, x_0 and $\tilde{x}$, both satisfying (9), which contradicts the condition (i) of the theorem. From this contradiction it follows that the sequence $\{(x_\nu(0), x'_\nu(0), \dots, x_\nu^{(n-1)}(0))\}_{\nu=1}^\infty$ tends to the point $(x_0(0), x'_0(0), \dots, x_0^{(n-1)}(0))$, and therefore x_ν tends to x_0 as $\nu \rightarrow \infty$ in the norm of the space $C^n[0, 1]$. This proves the continuity of T .

Because M is convex, T continuous and $T(M)$ compact, from the Schauder's fixed point theorem it follows that T has a fixed point, i. e., there exists a point y in M such that $Ty = y$. According to the definition of the operator T , y is a solution of the many point boundary value problem (4), (9).

(B) Suppose now that y_1 and y_2 are two solutions of the many point boundary value problem (4), (9). Then we have

$$\begin{aligned} y_1^{(n)} - y_2^{(n)} &= f(t, y_1, y_1', \dots, y_1^{(n-1)}) - f(t, y_2, y_2', \dots, y_2^{(n-1)}) = \\ &= \int_0^1 \left(\frac{d}{d\lambda} f(t, \lambda y_1 + (1-\lambda)y_2, \dots, \lambda y_1^{(n-1)} + (1-\lambda)y_2^{(n-1)}) \right) d\lambda = \\ &= \sum_{j=0}^{n-1} \left(\int_0^1 \frac{\partial f(t, \lambda y_1 + (1-\lambda)y_2, \dots, \lambda y_1^{(n-1)} + (1-\lambda)y_2^{(n-1)})}{\partial y^{(j)}} d\lambda \right) \\ &\quad (y_1^{(j)} - y_2^{(j)}). \end{aligned}$$

Because $y = y_1 - y_2$ is a solution of (13) satisfying homogeneous many point boundary value conditions, from the condition (iii) of the theorem it follows that $y_1 = y_2$, i. e., the differential equation (4) is disconjugate. To prove that it is also strongly disconjugate we observe that from the condition (i) it follows that all the equations in variation of (4) are disconjugate.

THEOREM 1'. [6] Suppose that the function f satisfies on $[0, 1] \times \mathbb{R}^n$ the inequality

$$|f(t, y, y', \dots, y^{(n-1)})| \leq K + \sum_{j=0}^{n-1} P_{n-j}(t) |y^{(j)}|,$$

where K is a real number, $K \geq 0$, and $P_i(t) \geq 0$, $t \in [0, 1]$, $i = 1, \dots, n$ are continuous functions. If there exists a positive ε such that each linear differential equation of the form (10) with the coefficients $p_i(t)$ satisfying the inequalities

$$|p_i(t)| \leq P_i(t) + \varepsilon, \quad t \in [0, 1], \quad i = 1, \dots, n,$$

is disconjugate, then any many point boundary value problem for (4) has at least a solution. If in addition holds

$$\left| \frac{\partial f(t, y, y', \dots, y^{(n-1)})}{\partial y^{(j)}} \right| \leq P_{n-j}(t), \quad j = 0, 1, \dots, n-1, \quad (14)$$

then (4) is strongly disconjugate.

In the original version of the theorem of A. Lasota and Z. Opial, f is supposed to be only continuous and the condition (14) is given in the form of a Lipschitz condition. The proof constitutes the verification of the conditions in Theorem 1.

Concrete examples for the functions $P_i(t)$ with the property in Theorem 1' are furnished by various de la Vallée Poussin type theorems (see in the same paper [6]).

2. Unrestricted differential n -parameter families

Suppose that the differential equation (4) is strongly disconjugate and denote by $Y(t, a)$ the integral variety of this differential equation which is parametrized by the initial value conditions (5). Then the integral variety $Y(t, a)$ is an unrestricted differential n -parameter family in the sense of [10], which is defined as follows:

DEFINITION 3. Let $Y(t, a)$ be a function having continuous partial derivatives of the form $\frac{\partial^j}{\partial t^i} \frac{\partial}{\partial a^i} Y(t, a)$, $j = 0, 1, \dots, n$, $i = 1, \dots, n$ on the direct product space $[0, 1] \times \mathbb{R}^n$. $Y(t, a)$ is said to form an unrestricted differential n -parameter family (abbreviated UDnF), if for any system of numbers m, t_i, k_i satisfying the condition (I), there exists a single point a in $\mathbb{R}^n$ such that

$$Y^{(j)}(t_i, a) = y_i^j, \quad i = 1, \dots, m, \quad j = 0, 1, \dots, k_i - 1,$$

where y_i^j are any given real numbers and $Y^{(j)}$ denotes the j^{th} partial derivative of Y with respect to t , and if in addition, the partial derivatives $\frac{\partial Y}{\partial a^i}$, $i = 1, \dots, n$ form an UCS (see Definition 1), for any a in $\mathbb{R}^n$.

The notion of the UDnF's is closely related to the strong disconjugacy. This relation is given by the following theorem (see also [11]).

THEOREM 2. The function $Y(t, a)$ forms an UDnF if and only if it is an integral variety of a strongly disconjugate differential equation of the form (4).

In the proof of this theorem we will use the following result, which will be important also in our technique developed later:

THEOREM 3. The function $Y(t, a)$ with continuous partial derivatives of form $\frac{\partial}{\partial a^i} Y^{(j)}(t, a)$, $i = 1, \dots, n$, $j = 0, 1, \dots, n$ on $[0, 1] \times \mathbb{R}^n$, forms an UDnF if and only if for any system of numbers m, t_i, k_i satisfying the condition (I) the mapping Φ with the coordinate functions

$$\begin{aligned} \Phi^{k_1 + \dots + k_{i-1} + 1 + j} &= Y^{(j)}(t_i, \dots), \\ i &= 1, \dots, m, \quad j = 0, 1, \dots, k_i - 1 \end{aligned} \quad (15)$$

is a diffeomorphism of $\mathbb{R}^n$ onto itself.

Proof. From the definition of the UDnF-s it follows that the mapping Φ defined above is onto and one-to-one. The Jacobian of the mapping Φ in the point $a \in \mathbf{R}^n$ is of the form

$$J(a) = \det \left\| \frac{\partial Y^{(j)}(t_i, a)}{\partial a^l} \right\|$$

$$i = 1, \dots, m; j = 0, 1, \dots, k_i - 1; l = 1, \dots, n$$

and it is different from zero because the system of functions $\frac{\partial Y}{\partial a^i}$, $i = 1, \dots, n$ forms an UCS. Therefore Φ is an immersion which is one-to-one and onto, and therefore it is a diffeomorphism onto.

Conversely, if Φ is a diffeomorphism of $\mathbf{R}^n$ onto itself for any system of numbers m, t_i, k_i satisfying (I), then all the conditions of the Definition 3 are satisfied and it follows that $Y(t, a)$ forms a UDnF. This completes the proof.

Proof of Theorem 2. If $Y(t, a)$ is an integral variety of a strongly disconjugate differential equation of form (4), then it has partial derivatives of the form $\frac{\partial}{\partial a^i} Y^{(j)}(t, a)$, $j = 0, 1, \dots, n$, $i = 1, \dots, n$ which are continuous on the direct product $[0, 1] \times \mathbf{R}^n$ (Theorem 18, [13]). The functions $\frac{\partial Y}{\partial a^i}$, $i = 1, \dots, n$ form a fundamental system of solutions of the equation in variation of (4) in the point a , and therefore they form a UCS. This means that the integral variety $Y(t, a)$ forms a UDnF, which proves the sufficiency part of our theorem.

To prove the necessity we begin with the following

LEMMA. Suppose that

- (i) $\varphi : [0, 1] \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ is a continuous mapping;
- (ii) all the partial derivatives of the coordinate functions of φ with respect to its last n variables are continuous on $[0, 1] \times \mathbf{R}^n$;
- (iii) for any fixed t_0 in $[0, 1]$ the mapping

$$a \rightarrow \varphi(t_0, a)$$

is a diffeomorphism of $\mathbf{R}^n$ onto itself.

Then the relation

$$\varphi(t, a) = b$$

determines an implicit function

$$a = \psi(t, b),$$

which maps the set $[0, 1] \times \mathbf{R}^n$ onto $\mathbf{R}^n$ and has the properties (i), (ii) and (iii).

Proof. Denote by χ the mapping defined by

$$\chi : (t, a) \rightarrow (t, \varphi(t, a)).$$

Then χ will be a homeomorphism of $[0, 1] \times \mathbb{R}^n$ onto itself. Really, the mapping χ is one-to-one according to the condition (iii) of the lemma. It is also continuous by (i). To prove the continuity of χ^{-1} , we will prove in fact that χ is an open mapping. Let U be an open set in $[0, 1] \times \mathbb{R}^n$ and let be (t_0, a_0) a point in U . Then (t_0, a_0) has a neighbourhood U_0 of the form $U_0 = V \times W$, which is contained in U and has the property that V is open in $[0, 1]$ and W is open in $\mathbb{R}^n$. We have

$$\chi(U_0) = V \times \varphi(V, W)$$

The set $\varphi(V, W)$ is open. Really, we have

$$\varphi(V, W) = \bigcup_{t \in V} \varphi(t, W)$$

and the openness follows from the condition (iii). This proves that χ is an open mapping and completes the proof of the continuity of χ^{-1} .

Denote by

$$\psi(t, b) = \text{pr}_2 \circ \chi^{-1}(t, b),$$

where pr_2 denotes the projection onto $\mathbb{R}^n$ in the direct product $[0, 1] \times \mathbb{R}^n$. Then we have

$$\varphi(t, \psi(t, b)) = \varphi(t, \text{pr}_2 \circ \chi^{-1}(t, b)).$$

But $\chi^{-1}(t, b) = (t, a)$, where $b = \varphi(t, a)$ and therefore

$$\varphi(t, \psi(t, b)) = \varphi(t, a) = b, \quad (16)$$

that is, $\psi(t, b)$ is the required implicit function. The condition (i) for $\psi(t, b)$ is obviously fulfilled. To prove (ii) and (iii) denote by $\varphi = (\varphi^1, \dots, \varphi^n)$ and $\psi = (\psi^1, \dots, \psi^n)$ and let be

$$\eta^i(t, a, b) = \varphi^i(t, a) - b^i, \quad i = 1, \dots, n.$$

Applying these notations, from the implicit function theorem we have for a fixed t

$$\left. \frac{\partial \psi^i}{\partial b^j} \right|_b = - \frac{1}{J(a)} \frac{D(\eta^1, \dots, \eta^n)}{D(a^1, \dots, a^{i-1}, b^j, a^{i+1}, \dots, a^n)}, \quad (17)$$

$$i, j = 1, \dots, n,$$

where $J(a)$ is the Jacobian of the mapping

$$a \rightarrow \varphi(t, a)$$

and the second expression on the right hand side of (17) denotes the Jacobian of the mapping $\eta = (\eta^1, \dots, \eta^n)$ with respect to $a^1, \dots, a^{i-1}, b^j, a^i, \dots, a^n$. But ψ is continuous with respect to t and b and therefore if we put $a = \psi(t, b)$ in the above formula, we obtain the partial derivatives of ψ^i as functions of t and b . Because all the functions on the right hand side of (17) are continuous and $J(\psi(t, b)) \neq 0$, the continuity on $[0, 1] \times \mathbb{R}^n$ of the partial derivatives

of the coordinate functions of ψ with the respect to the last n variables, follows. This proves the condition (ii) for ψ .

Let now t_0 be fixed in $[0, 1]$. From the condition (iii) and from (16) it follows that the mapping

$$b \rightarrow \psi(t_0, b)$$

is a diffeomorphic mapping of $\mathbf{R}^n$ onto itself. This proves the condition (iii) for ψ , and completes the proof of the lemma.

We prove now the necessity of the condition of Theorem 2. Let be t fixed for a moment and introduce the notations

$$Y^{(j)}(t, a) = y^{(j)}, \quad j = 0, 1, \dots, n-1 \quad (18)$$

and

$$Y^{(n)}(t, a) = y^{(n)}. \quad (19)$$

If $(y, y', \dots, y^{(n-1)})$ is considered to be a point in $\mathbf{R}^n$, then the equalities (18) determine a mapping of $\mathbf{R}^n$ into itself, which by Theorem 3 is a diffeomorphism onto. According to the lemma, there exists a continuous function $g: [0, 1] \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ which for a fixed t will be the inverse of the mapping defined by (18). That is, we may write

$$a = g(t, y, y', \dots, y^{(n-1)}),$$

where g is continuous, has partial derivatives with respect to its last n variables which are also continuous on $[0, 1] \times \mathbf{R}^n$; and for any fixed t it is a diffeomorphism of $\mathbf{R}^n$ onto itself. By substitution in (19) we get

$$\begin{aligned} y^{(n)} &= Y^{(n)}(t, g(t, y, y', \dots, y^{(n-1)})) = \\ &= f(t, y, y', \dots, y^{(n-1)}). \end{aligned} \quad (20)$$

But for a fixed a , the values $y, y', \dots, y^{(n)}$ may be interpreted as the consecutive derivatives of a function of t , and therefore (20) is a differential equation. According to the differentiability of $Y^{(n)}$ and g with respect to a^i , it follows that f in (20) has partial derivatives with respect to $y^{(j)}$, $j = 0, 1, \dots, n-1$, which are continuous on $[0, 1] \times \mathbf{R}^n$. From the construction it follows also that $Y(t, a)$ serves as integral variety for the differential equation (20). But this means that (20) is a strongly disconjugate differential equation, and the proof of Theorem 2 is completed.

3. Criteria for strong disconjugacy

THEOREM 4. Suppose that (4) satisfies (H1) and (H2) and that it is weakly disconjugate. Then (4) is also strongly disconjugate if and only if for any m, t_i, k_i satisfying (I), for any sequence of parameters $\{a_\nu\}$ such that $\|a_\nu\| \rightarrow \infty$, if $\nu \rightarrow \infty$, it holds $\|\Phi(a_\nu)\| \rightarrow \infty$ for $\nu \rightarrow \infty$, where

$$\Phi = (\Phi^1, \dots, \Phi^n)$$

and

$$\begin{aligned} \Phi^{k_1+\dots+k_{i-1}+1+j} &= Y^{(j)}(t_i, \dots), \\ i &= 1, \dots, m, \quad j = 0, 1, \dots, k_i - 1, \end{aligned} \quad (21)$$

$Y(t, a)$ being the integral variety of (4) defined by the initial value conditions (5).

Proof. Because of the weak disconjugacy of (4), it follows that the mapping Φ with the coordinate functions (21) is an immersion of $\mathbf{R}^n$ into itself. By a result of R. S. Palais (Corollary 4.1 in [12]), it follows that Φ will be a diffeomorphism onto, if and only if the condition in the theorem is fulfilled. Applying now Theorem 3 the assertion follows.

THEOREM 5. Any differential equation (4) with the property (H1) and with f satisfying the condition

$$\left| \frac{\partial f(t, y, y', \dots, y^{(n-1)})}{\partial y^{(j)}} \right| \leq K, \quad j = 0, 1, \dots, n-1$$

which is weakly disconjugate and for which any many point boundary value problem has at most a solution, is also strongly disconjugate.

Proof. The assertion follows from the theorems II.1.1 and II.1.2 of Ph. Hartman [4] and Theorem 1' of A. Lasota and Z. Opial, given above.

Let us denote

$$T_n^{n-1} = C^{n-1}[0, 1] \otimes \dots \otimes C^{n-1}[0, 1]$$

the tensor product of n copies of the space $C^{n-1}[0, 1]$ endowed with projective topology (i. e., with the strongest topology on the tensorial product T_n^{n-1} in which the canonical n -linear mapping $(x_1, \dots, x_n) \rightarrow x_1 \otimes \dots \otimes x_n$ is continuous). Denote the canonical projection of C_n^{n-1} onto T_n^{n-1} by π .

Suppose that the system of numbers $m, t_i, k_i, i = 1, \dots, m$ has the property (I). We define the n -linear mapping $J(m, t_i, k_i)$ of C_n^{n-1} in $\mathbf{R}$ in the following way: If $(x_1, \dots, x_n)$ is an element in the space C_n^{n-1} , we put

$$\begin{aligned} J(m, t_i, k_i) x &= \\ &= \det \| x_i(t_1) \dots x_i^{(k_i-1)}(t_1) \dots x_i(t_m) \dots x_i^{(k_m-1)}(t_m) \| \end{aligned}$$

where the determinant in the right hand side is represented by the i^{th} row. Denote by $J^*(m, t_i, k_i)$ the linear functional defined on the space T_n^{n-1} , which occurs in the factorization

$$J(m, t_i, k_i) = J^*(m, t_i, k_i) \circ \pi.$$

Let be Ω the set in T_n^{n-1} with the property that for any x in Ω , $J^*(m, t_i, k_i) x \neq 0$ for any m, t_i, k_i having the property (I).

If we denote by i the inclusion mapping of C_n^n in C_n^{n-1} then it is obvious that Ω contains the images under $\pi \circ i$ of all the n -tuples $(x_1, \dots, x_n)$ in C_n^n for which the functions $x_1, \dots, x_n$ form an UCS (Definition 1), and conversely, if some $x = (x_1, \dots, x_n)$ in C_n^n is mapped by $\pi \circ i$ in Ω , then $x_1, \dots, x_n$ form an UCS.

We define now the mapping $\varphi: C_n^0 \rightarrow C_n^n$ as follows: If $p = (p_1, \dots, p_n)$ is an element of C_n^0 , then let be $x = (x_1, \dots, x_n) = \varphi(p)$ the element of C_n^n for which $x_1, \dots, x_n$ is the fundamental system of solutions of the differential equation (10) (having the functions $p_1, p_2, \dots, p_n$ as coefficients), which satisfy the initial value conditions

$$x_i^{(j-1)}(0) = \delta_i^j, \quad i, j = 1, \dots, n. \quad (22)$$

The mapping φ so defined is an imbedding of C_n^0 into C_n^n .

Applying the introduced notations we have the following theorem:

THEOREM 6. *Suppose that the differential equation (4) satisfying (H1) and (H2) has the property that the weak closure of the set $\pi \circ i \circ \varphi(\mathfrak{M}(f))$ in T_n^{n-1} , where $\mathfrak{M}(f)$ denotes the set in C_n^0 of all the equations in variation of (4), is weakly compact and is contained in Ω . Then (4) is strongly disconjugate.*

We observe that (4) is weakly disconjugate. Really, from the conditions of the theorem it follows that each equation in variation of (4) has a fundamental system of solutions which is mapped by $\pi \circ i$ in Ω , i. e., according to the above observation it is an UCS, and therefore the respective equation in variation is disconjugate. Thus Theorem 6 gives a sufficient condition in order to a weakly disconjugate differential equation be strongly disconjugate.

Proof. Suppose that $Y(t, a)$ is the integral variety of (4) parametrized by the initial value conditions of type (5). Then for the functions $\frac{\partial Y}{\partial a^i}$, $i = 1, \dots, n$ are satisfied the initial value conditions (8), and from the definition of the mapping φ of C_n^0 into C_n^n given above, we have for a given value of the parameter a the equality

$$\begin{aligned} \varphi \left(\frac{\partial f(t, Y, Y', \dots, Y^{(n-1)})}{\partial y^{(n-1)}}, \dots, \frac{\partial f(t, Y, Y', \dots, Y^{(n-1)})}{\partial y} \right) &= \\ &= \left(\frac{\partial Y}{\partial a^1}, \dots, \frac{\partial Y}{\partial a^n} \right). \end{aligned} \quad (23)$$

Denote by Δ the weak closure of $\pi \circ i \circ \varphi(\mathfrak{M}(f))$ in T_n^{n-1} . Because $\Delta \subset \Omega$, from the definition of Ω it follows that for any m, t_i, k_i satisfying the condition (I), and for each element x in Δ we have

$$J^*(m, t_i, k_i) x \neq 0,$$

and because Δ is weakly compact and $J^*(m, t_i, k_i)$ is a continuous linear functional on T_n^{n-1} , there exist positive numbers ε and A such that

$$A > |J^*(m, t_i, k_i)(\Delta)| > \varepsilon.$$

This means that we have also

$$A > |J^*(m, t_i, k_i) \circ \pi \circ i \circ \varphi(\mathfrak{M}(f))| > \varepsilon. \quad (24)$$

Consider a given element of $\pi \circ i \circ \varphi(\mathfrak{M}(f))$. It may be written according (23) in the form

$$\pi \circ i \left(\frac{\partial Y}{\partial a^1}, \dots, \frac{\partial Y}{\partial a^n} \right), \quad (25)$$

for a fixed value of the parameter a . Applying now to (25) the linear functional $J^*(m, t_i, k_i)$ and employing the factorization $J(m, t_i, k_i) = J^*(m, t_i, k_i) \circ \pi$, we have the equality

$$\begin{aligned} & J^*(m, t_i, k_i) \circ \pi \circ i \left(\frac{\partial Y}{\partial a^1}, \dots, \frac{\partial Y}{\partial a^n} \right) = \\ & = J(m, t_i, k_i) \circ i \left(\frac{\partial Y}{\partial a^1}, \dots, \frac{\partial Y}{\partial a^n} \right) = \\ & = \det \left\| \frac{\partial Y(t_1, a)}{\partial a^i} \dots \frac{\partial Y(t_{i-1}, a)}{\partial a^i} \dots \frac{\partial Y(t_m, a)}{\partial a^i} \dots \frac{\partial Y(t_m^{(k_m-1)}, a)}{\partial a^i} \right\|. \end{aligned}$$

But this is the Jacobian of the mapping Φ defined by (15) in Theorem 3. From (24) it follows that

$$A > |J(m, t_i, k_i) \circ i \circ \varphi(\mathfrak{M}(f))| > \varepsilon, \quad (26)$$

i. e., the Jacobian of the mapping Φ is everywhere different from zero and satisfies the inequality similar to (26). An immersion of $\mathbb{R}^n$ into itself having the property that its Jacobian is in each point of $\mathbb{R}^n$ between two given numbers of the same sign, is in fact diffeomorphism onto (see [8] and [14]). It follows then that for any m, t_i, k_i satisfying (I), the mapping Φ is a diffeomorphism of $\mathbb{R}^n$ onto itself. According Theorem 3, $Y(t, a)$ is then an UDnF which combined with Theorem 2 gives the assertion of our theorem.

Theorem 6 is a disconjugacy criterion formulated in the most general conditions which permit the employing of the theorems 2 and 3 and the above cited condition about an immersion of $\mathbb{R}^n$ into itself, in order to be a diffeomorphism onto, and uses standard notions of the functional analysis. It may be expected that using cha-

racterizations of the diffeomorphisms of $\mathbf{R}^n$ onto itself as those given by R. S. Palais [12], Corollary 4.3 or I. Vidav [14], other disconjugacy criterions may be derived. We present in what follows a corollary of this theorem, in which the conditions are given directly on the set $\mathcal{M}(f)$ of all the equations in variation of (4).

COROLLARY 1. ([11], Theorem 5) *Suppose that the differential equation (4) with the properties (H1) and (H2) has the property that the set $\mathcal{M}(f)$, where $\mathcal{M}(f)$ denotes the set of all the equations in variation of (4), is compact and is contained in $\mathcal{M}$, the set of all linear disconjugated differential equations of form (10). Then (4) is strongly disconjugated.*

We remind (see the paragraph 0) that the set $\mathcal{M}$ of all disconjugate differential equations (10) is considered to be topologized by the identification of the differential equation (10) with the n -tuple $(p_1, \dots, p_n)$ of its coefficients in C_n^0 .

Proof. Because the mappings φ , i and π are all continuous, the set $\pi \circ i \circ \varphi(\mathcal{M}(f))$ will be compact. It will be contained in Ω because $\mathcal{M}(f) \subset \mathcal{M}$ and $\pi \circ i \circ \varphi(\mathcal{M}) \subset \Omega$, and such the Theorem 6 may be applied.

4. Remarks

Theorem 6 has merely a theoretical meaning. Because of its character: the establishing of a connection between the disconjugacy of a nonlinear differential equation and the disconjugacy of its equations in variation, its effectivity is related to the existence of handable disconjugacy criterions for the linear differential equations, and from the conditions of this theorem it follows that a special importance would have the criterions in some families of linear differential equations with discontinuous coefficients. As far as we know there are few disconjugacy criterions even in the class of linear differential equations with continuous coefficients. We observe that the application of the criterions of de la Vallée Poussin type gives essentially no more than the theorem of A. Lasota and Z. Opial (Theorem 1' above). Another disconjugacy criterion for linear differential equations with continuous coefficients is that due to Ph. Hartman [3] and to Yu. A. Levin [7], which will be used below and has the following formulation:

(C) Consider the linear differential equation (10) and denote by $\lambda_1(t_0), \lambda_2(t_0), \dots, \lambda_n(t_0)$ the roots of the characteristic algebraic equation of the differential equation with the constant coefficients $p_i(t_0)$, $i = 1, \dots, n$. If for any t in $[0, 1]$ these roots are real numbers and there exist numbers $\mu_1, \dots, \mu_{n-1}$ such that

$$\lambda_1(t) \leq \mu_1 \leq \lambda_2(t) \leq \mu_2 \leq \dots \leq \mu_{n-1} \leq \lambda_n(t), \quad (27)$$

for any t in $[0, 1]$, then the differential equation (10) is disconjugate.

Corollary 1 contains a strong restriction about the differential equation (4). But even in this case it may be used to derive the unicity of the solution of a many point boundary value problem for some differential equations of particular form. Such we have following result:

COROLLARY 2. *Let the differential equation (4) be given and suppose that*

(i) *f is bounded function;*

(ii) *the partial derivatives $\frac{\partial f}{\partial y^{(j)}}$, $j = 0, 1, \dots, n-1$ are bounded functions;*

ded functions;

(iii) *$\frac{\partial f}{\partial y^{(j)}}$, $j = 0, 1, \dots, n-1$ are uniformly continuous functions;*

(iv) *each equation in variation of (4) has a property of the type (C) given above.*

Then the differential equation (4) is strongly disconjugate.

Proof. The existence of a solution for any many point boundary value problem in the conditions (i), (H2) and the continuity of f is well known (it follows for example from the original form of the theorem of A. Lasota and Z. Opial [6], and the theorem of de la Vallée Poussin). To prove the strong disconjugacy we will prove in fact that in the conditions of our corollary the set $\mathfrak{M}(f)$ of the equations in variation is relatively compact and holds $\overline{\mathfrak{M}(f)} \subset \mathfrak{M}$.

For proving the relatively compactness of the set $\mathfrak{M}(f)$ it is sufficient to show that the set of functions

$$M_j = \left\{ \frac{\partial f(t, y, y', \dots, y^{(n-1)})}{\partial y^{(j)}} : y \text{ solution of (4)} \right\} \quad (28)$$

is equally continuous ($j = 0, 1, \dots, n-1$). From the boundedness of f it follows that the set of the n^{th} order derivatives of the solutions of (4) is bounded and therefore the set of the continuous functions

$$N_j = \{y^{(j)} : y \text{ solution of (4)}\} \quad (29)$$

is equally continuous for each $j = 0, 1, \dots, n-1$.

From the uniform continuity of $\frac{\partial f}{\partial y^{(j)}}$ on its domain, it follows that for any $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ such that

$$\left| \frac{\partial f(t', y, y', \dots, y^{(n-1)})}{\partial y^{(j)}} - \frac{\partial f(t'', x, x', \dots, x^{(n-1)})}{\partial y^{(j)}} \right| < \varepsilon$$

as far as $|t' - t''| < \delta_\varepsilon$ and $|y^{(j)} - x^{(j)}| < \delta_\varepsilon$, $j = 0, 1, \dots, n-1$. From the equal continuity of the sets (29) it follows that there exists an $\eta_\varepsilon > 0$ such that

$$|y^{(j)}(t') - y^{(j)}(t'')| < \delta_\varepsilon, \quad j = 0, 1, \dots, n-1,$$

for any solution of (4), if $|t' - t''| < \eta_\varepsilon$. It follows that

$$\left| \frac{\partial f(t', y(t'), y'(t'), \dots, y^{(n-1)}(t'))}{\partial y^{(j)}} - \frac{\partial f(t'', y(t''), y'(t''), \dots, y^{(n-1)}(t''))}{\partial y^{(j)}} \right| < \varepsilon$$

as far as $|t' - t''| < \min(\delta_\varepsilon, \eta_\varepsilon)$, and it follows the equal continuity of the sets (28). This, together with (ii) assures the relative compactness of $\mathfrak{M}(f)$.

Let now l_ν , $\nu = 1, 2, \dots$ form a Cauchy sequence of elements in $\mathfrak{M}(f)$. From the relative compactness of $\mathfrak{M}(f)$ it follows that $l_\nu \rightarrow l$, as $\nu \rightarrow \infty$, where l is a differential equation with continuous coefficients. According to (iv) for any ν there exist the numbers

$$\mu_1^\nu \leq \mu_2^\nu \leq \dots \leq \mu_{n-1}^\nu,$$

such that l_ν has the property (C) with respect them. It is easy to see that the sequences $\{\mu_i^\nu\}$, have limit points μ_i such that

$$\mu_1 \leq \mu_2 \leq \dots \leq \mu_{n-1}.$$

and that l has the property (C) with respect to these numbers. This means that $l \in \mathfrak{M}$. The inclusion $\mathfrak{M}(f) \subset \mathfrak{M}$ is then proved and our assertion follows from Corollary 1.

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(Received April 23, 1973)

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DISKONJUGACIJA JEDNADŽBE POMOĆU DISKONJUGACIJE VARIRANIH JEDNADŽBI

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Sadržaj

Kaže se da je diferencijalna jednačba n -tog reda

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)})$$

diskonjugirana na $[0, 1]$ ako svaki problem interpolacije hermitskog tipa ima jedinstveno rješenje u skupu rješenja te jednačbe. U članku se pretpostavlja da f ima parcijalne derivacije po posljednjih n varijabli, ako su one neprekidne u $[0, 1] \times \mathbb{R}^n$, te da su sva rješenja diferencijalne jednačbe definirana na čitavom segmentu $[0, 1]$. Kaže se da je diferencijalna jednačba s navedenim svojstvima slabo diskonjugirana, ako su sve njoj odgovarajuće jednačbe u varijacijama diskonjugirane. Koristeći jedan teorem I. Vidava [14], kojeg je nezavisno dokazao i autor [8], daje se dovoljan uvjet da slabo diskonjugirana diferencijalna jednačba bude diskonjugirana.