

P.295

U N I V E R Z I T E T U B E O G R A D U

PUBLIKACIJE

ELEKTROTEHNIČKOG FAKULTETA

SERIJA:

MATEMATIKA I FIZIKA

№ 461 — № 497

(1974)

BEOGRAD

P.6571 B

494. FUNCTIONAL EQUATIONS FOR WALLIS AND
GAMMA FUNCTIONS*

Ilija B. Lazarević and Alexandru Lupuş

The aim of this note is to find all convex solutions of the functional equation

$$(1) \quad f(x+1) = \frac{x+1}{x+\theta} f(x), \quad x \in [0, +\infty)$$

where $f: \mathbb{R}_+ \rightarrow \mathbb{R}$, $\mathbb{R}_+ = [0, +\infty)$ and θ is a prescribed number on $(0, 1)$. Setting $f(0) = 1/\Gamma(\theta)$ we shall see that the unique solution is the „Wallis function” $W(\cdot, \theta): \mathbb{R}_+ \rightarrow \mathbb{R}$ defined as

$$W(x, \theta) = \frac{\Gamma(x+1)}{\Gamma(x+\theta)}.$$

Further we establish some inequalities for the WALLIS function. At the end of this paper a new characterization of the Gamma function through functional equation is given.

Theorem 1. Let $A \in \mathbb{R}_+$, $\alpha \in (0, +\infty)$, $\theta \in (0, 1)$ be fixed elements. There is a unique solution $f: \mathbb{R}_+ \rightarrow \mathbb{R}$ of (1) defined by

$$f(x) = \alpha \frac{\Gamma(\theta) \Gamma(x+1)}{\Gamma(x+\theta)}$$

which is convex on $(A, +\infty)$ and such that $f(0) = \alpha$.

Proof. It is easy to see that for a natural number n

$$f(x+n) = \frac{(x+n)(x+n-1)\cdots(x+1)}{(x+n+\theta-1)(x+n+\theta-2)\cdots(x+\theta)} f(x), \quad x \in \mathbb{R}_+$$

or

$$(2) \quad f(x+n) = \frac{\Gamma(x+n+1) \Gamma(x+\theta)}{\Gamma(x+n+\theta) \Gamma(x+1)} f(x), \quad x \in \mathbb{R}_+.$$

Therefore

$$f(y) = f([y] + \{y\}) = \frac{\Gamma(y+1) \Gamma(\{y\} + \theta)}{\Gamma(y+\theta) \Gamma(\{y\} + 1)} f(\{y\}), \quad y \in [1, +\infty)$$

* Presented May 5, 1974 by D. D. ADAMOVIĆ.

which confirms that it is sufficient to suppose $x \in [0, 1)$. Let $n \geq 2 + [A]$. Since f is convex (non-concave) on $(A, +\infty)$ we may write

$$[n-1, n; f] \leq [n, n+x; f] \leq [n, n+1; f], \quad x \in (0, 1),$$

where the symbol $[a, b; f]$ denotes the divided difference.

Using (1) and the above inequalities one obtains

$$\frac{1-\theta}{n} f(n) \leq \frac{f(n+x)-f(n)}{x} \leq \frac{1-\theta}{n+\theta} f(n)$$

which can be written as

$$\frac{n+x(1-\theta)}{n} f(n) \leq f(n+x) \leq \frac{n+\theta+x(1-\theta)}{n+\theta} f(n).$$

But from these inequalities as well as from (2), for $x \in (0, 1)$ we have

$$\frac{n+x(1-\theta)}{n} f(n) \leq \frac{\Gamma(n+x+1) \Gamma(x+\theta)}{\Gamma(n+x+\theta) \Gamma(x+1)} f(x) \leq \frac{n+\theta+x(1-\theta)}{n+\theta} f(n).$$

From these inequalities we may write

$$(3) \quad \frac{\alpha(n+x(1-\theta)) \Gamma(\theta) \Gamma(x+1)}{n \Gamma(x+\theta)} F_n(x, \theta) \leq f(x) \\ \leq \frac{\alpha(n+\theta+x(1-\theta)) \Gamma(\theta) \Gamma(x+1)}{(n+\theta) \Gamma(x+\theta)} F_n(x, \theta)$$

where

$$F_n(x, \theta) = \frac{\Gamma(n+1)}{\Gamma(n+\theta)} \cdot \frac{\Gamma(n+x+\theta)}{\Gamma(n+x+1)}.$$

We remark that (3) is trivially verified at the point $x=0$, i.e., the inequalities (3) are valid on $[0, 1)$.

According to a well-known theorem by H. BOHR and I. MOLLERUP [3] (see also [1-2]) the restriction at $(0, +\infty)$ of the Gamma function, is the unique logarithmic-convex function on $(B, +\infty)$, $B \geq 0$, which satisfies

$$\Gamma(x+1) = x \Gamma(x), \quad \Gamma(1) = 1.$$

This means that for $0 < a < b < c < +\infty$ the inequality

$$[a, b, c; \ln \Gamma(\cdot)] > 0$$

implies

$$(\Gamma(b))^{c-a} < (\Gamma(a))^{c-b} (\Gamma(c))^{b-a}.$$

Setting $a=z$, $b=z+\xi$, $c=z+1$, $\xi \in (0, 1)$, we get

$$\frac{(\Gamma(z+1))^\xi}{\Gamma(z+\xi)} > (\Gamma(z))^{\xi-1},$$

or

$$(4) \quad \frac{\Gamma(z+1)}{\Gamma(z+\xi)} > z^{1-\xi}, \quad z \in (0, 1),$$

which holds also at $z=0$.

On the other hand the following may be proved (see [4], Lemma 1): If $g: \mathbf{R}_+ \times (0, 1) \rightarrow (0, +\infty)$ is such that for all $(z, \theta) \in \mathbf{R}_+ \times (0, 1)$

$$\frac{\Gamma(z+1)}{\Gamma(z+\theta)} > g(z, \theta),$$

then

$$\frac{\Gamma(z+1)}{\Gamma(z+\theta)} < \frac{z+\theta}{g(z+\theta, 1-\theta)}, \quad \theta \in (0, 1).$$

In this manner, by means of (4), we conclude with the inequalities

$$(5) \quad z^{1-\theta} < \frac{\Gamma(z+1)}{\Gamma(z+\theta)} < (z+\theta)^{1-\theta}, \quad (z, \theta) \in \mathbf{R}_+ \times (0, 1).$$

Put $z=n$, $z=n+x$ respectively in (5); we find

$$n^{1-\theta} < \frac{\Gamma(n+1)}{\Gamma(n+\theta)} < (n+\theta)^{1-\theta}$$

and

$$\frac{1}{(n+x+\theta)^{1-\theta}} < \frac{\Gamma(n+x+\theta)}{\Gamma(n+x+1)} < \frac{1}{(n+x)^{1-\theta}}.$$

From these inequalities as well as from (2), by mutual multiplications of the corresponding members we get

$$\left(\frac{n}{n+x+\theta}\right)^{1-\theta} < F_n(x, \theta) < \left(\frac{n+\theta}{n+x}\right)^{1-\theta}, \quad (x, \theta) \in \mathbf{R}_+ \times (0, 1).$$

In conclusion, (3) implies that for $x \in \mathbf{R}_+$, $\theta \in (0, 1)$,

$$\begin{aligned} \frac{\alpha(n+x(1-\theta))}{n} \left(\frac{n}{n+x+\theta}\right)^{1-\theta} \cdot \frac{\Gamma(\theta)\Gamma(x+1)}{\Gamma(x+\theta)} &\leq f(x) \\ &\leq \frac{\alpha(n+\theta+x(1-\theta))}{n+\theta} \left(\frac{n+\theta}{n+x}\right)^{1-\theta} \cdot \frac{\Gamma(\theta)\Gamma(x+1)}{\Gamma(x+\theta)}. \end{aligned}$$

For $n \rightarrow +\infty$ the general convex solution of (1)

$$f(x) = \alpha \frac{\Gamma(\theta)\Gamma(x+1)}{\Gamma(x+\theta)}$$

is found and the theorem is proved thereby.

We note that a similar functional equation was treated by J. ANASTASIADIS [1] by a different method and supposing that f is positive and logarithmic-convex.

Theorem 2. Let $W(\cdot, \theta): \mathbf{R}_+ \rightarrow \mathbf{R}_+$ be the Wallis function. There exists a decreasing, convex function $\varepsilon_\theta: \mathbf{R}_+ \rightarrow (a, b]$ where

$$a = \frac{\theta}{2}, \quad b = (\Gamma(\theta))^{\frac{1}{\theta-1}}, \quad \theta \in (0, 1)$$

such that

$$(6) \quad W(x, \theta) = (x + \varepsilon_\theta(x))^{1-\theta}, \quad x \in \mathbf{R}_+.$$

Proof. On account of (5) we observe that there is a function $\varepsilon_\theta: \mathbf{R}_+ \rightarrow \mathbf{R}$ with

$$W(x, \theta) = (x + \varepsilon_\theta(x))^{1-\theta}, \quad 0 < \varepsilon_\theta(x) < \theta, \quad x \in \mathbf{R}_+.$$

Since

$$(7) \quad \varepsilon_\theta(x) = W(x, \theta)^{\frac{1}{1-\theta}} - x,$$

it is clear that $\varepsilon_\theta: \mathbf{R}_+ \rightarrow (0, \theta)$ is convex on its domain. Indeed according to the first theorem $W(\cdot, \theta)$ is convex on $\mathbf{R}_+$. On the other hand, for $x > 0$ we have

$$\begin{aligned} \varepsilon_\theta'(x) &= \frac{x + \varepsilon_\theta(x)}{1-\theta} \left(\frac{\Gamma'(x+1)}{\Gamma(x+1)} - \frac{\Gamma'(x+\theta)}{\Gamma(x+\theta)} \right) - 1 \\ &< (x+\theta) \psi'(x+\theta) - 1 < \frac{1}{x+\theta-1}. \end{aligned}$$

where $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$. Further, ε_θ' is increasing on $\mathbf{R}_+$ and

$$\varepsilon_\theta'(x) \leq \lim_{x \rightarrow +\infty} \varepsilon_\theta'(x) \leq 0,$$

that is ε_θ is a decreasing function. This implies

$$\varepsilon_\theta(x) \leq \varepsilon_\theta(0), \quad x \in \mathbf{R}_+,$$

with equality only for $x=0$. From (7), $\varepsilon_\theta(0) = (\Gamma(\theta))^{\frac{1}{1-\theta}}$, and in this way

$$\varepsilon_\theta(x) \leq b, \quad x \in \mathbf{R}_+.$$

Finally, by means of STIRLING series we prove that

$$\lim_{x \rightarrow +\infty} \varepsilon_\theta(x) = \frac{\theta}{2},$$

i.e.,

$$a = \frac{\theta}{2} = \lim_{x \rightarrow +\infty} \varepsilon_\theta(x) < \varepsilon_\theta(x), \quad x \in \mathbf{R}_+,$$

which completes the proof.

Corollary. For $(x, \theta) \in \mathbf{R}_+ \times (0, 1)$ we have

$$\left(x + \frac{\theta}{2}\right)^{1-\theta} < \frac{\Gamma(x+1)}{\Gamma(x+\theta)} < \left\{x + (\Gamma(\theta))^{\frac{1}{1-\theta}}\right\}^{1-\theta}.$$

The case $\theta = 1/2$ leads to G. N. WATSON's result [6], namely

$$\left(x + \frac{1}{4}\right)^{1/2} < \frac{\Gamma(x+1)}{\Gamma\left(x + \frac{1}{2}\right)} \leq \left(x + \frac{1}{\pi}\right)^{1/2}, \quad x \in \mathbf{R}_+.$$

For other information regarding the inequalities involving Gamma function see [5].

Now we intend to find all solutions of the functional equation

$$f(x+1) = xf(x), \quad x \in (0, +\infty)$$

in some classes of functions.

Theorem 3. *There exists a unique function $f: (0, +\infty) \rightarrow \mathbb{R}$, positive on $(0, 1)$ and satisfying:*

- (i) $f(x+1) = xf(x)$, $x \in (0, +\infty)$;
- (ii) f is logarithmic-concave of the second order on $(A, +\infty)$, A being a prescribed non-negative number;
- (iii) $f(1) = 1$.

More precisely, this function coincides with the restriction at $(0, +\infty)$ of the Gamma function.

Proof. If n is a natural number, then any solution of (i) has the property

$$(8) \quad f(n+x) = x(x+1) \cdots (x+n-1)f(x).$$

Let $y \in (1, +\infty)$, $y = [y] + \{y\}$; it is clear that

$$f(y) = \{y\}(\{y\}+1) \cdots (y-1)f(\{y\}).$$

Therefore the positivity of f at $(0, 1)$ implies that $f: (0, +\infty) \rightarrow \mathbb{R}$ is positive on its domain.

As usual, a logarithmic-concave function f of the second order on $(A, +\infty)$, has the properties: f is positive on $(A, +\infty)$ and $[a, b, c, d; \ln f] < 0$ for any points $A < a < b < c < d < +\infty$, which may be written as

$$(9) \quad (f(b))^{(d-a)(c-a)(d-c)} \cdot (f(d))^{(c-b)(b-a)(c-a)} \\ < (f(a))^{(c-b)(d-c)(d-b)} \cdot (f(c))^{(d-a)(d-b)(b-a)}.$$

If we select

$$a = n-1, \quad b = n+x-1, \quad c = n, \quad d = n+x, \quad x \in (0, 1), \quad n \geq 2 + [A]$$

we obtain

$$(10) \quad (f(n+x-1))^{1+x} (f(n+x))^{1-x} < (f(n-1))^{1-x} (f(n))^{1+x}.$$

Taking into account (i) as well as the fact that $f(n) = (n-1)!$ and $f(n+x-1) = \frac{1}{n+x-1} f(n+x)$, from (10) we get

$$[f(n+x)]^2 < ((n-1)!)^2 \frac{(n-x-1)^{1+x}}{(n-1)^{1-x}} \\ = ((n-1)!)^2 n^{2x} \left(1 + \frac{1}{n-1}\right)^{1-x} \left(1 + \frac{1-x}{n+x-1}\right)^{-1-x}.$$

On the other hand [5, p 262, § 3.6.3] we have

$$\left(1 + \frac{1-x}{n+x-1}\right)^{1+x} > e^{\frac{2(1-x^2)}{2n+x-1}}, \quad \left(1 + \frac{1}{n-1}\right)^{1-x} < e^{\frac{1-x}{n-1}}.$$

Therefore

$$\begin{aligned} f(n+x) &< n^x (n-1)! \sqrt{\left(1 + \frac{1}{n-1}\right)^{1-x} \left(1 + \frac{1-x}{n+x-1}\right)^{-1-x}} \\ &< n^x (n-1)! \sqrt{e^{\frac{1-x}{n-1} - \frac{2(1-x)^2}{2n+x-1}}} \\ &< n^x (n-1)! e^{\frac{1-x}{2(n-1)}} \end{aligned}$$

i.e.,

$$(11) \quad f(n+x) < n^x (n-1)! e^{\frac{1}{2(n-1)}}, \quad x \in (0, 1), \quad n \geq 2 + [A].$$

Further, with

$$a = n+x-1, \quad b = n, \quad c = n+x, \quad d = n+1, \quad x \in (0, 1), \quad n \geq 2 + [A],$$

from (9) we have

$$(f(n))^{2-x} (f(n+1))^x < (f(n+x-1))^x (f(n+x))^{2-x},$$

i.e.,

$$(n-1)! (n-1)^x < f(n+x), \quad x \in (0, 1), \quad n \geq 2 + [A].$$

This last inequality holds also if n is substituted by $n+1$. Therefore we have

$$(12) \quad \frac{n! n^x}{n+x} < f(n+x), \quad x \in (0, 1), \quad n \geq 2 + [A].$$

On account of (8), (11) and (12) we conclude

$$(13) \quad \frac{n! n^x}{x(x+1) \cdots (x+n)} < f(x) < \frac{n! n^x}{x(x+1) \cdots (x+n)} \cdot \frac{n+x}{n} e^{\frac{1}{2(n-1)}}.$$

Since

$$\Gamma(x) = \lim_{n \rightarrow +\infty} \frac{n! n^x}{x(x+1) \cdots (x+n)},$$

if $n \rightarrow +\infty$, the inequalities (13) show that

$$f(x) = \Gamma(x) \quad \text{on } (0, 1).$$

It is clear that from the above remarks (see (8)) we have

$$f(x) = \Gamma(x) \quad \text{on } (0, +\infty).$$

Now let k be a natural number, and let us denote

$$F_k(x) = (-1)^{k+1} \ln f(x)$$

where the function $f: (0, +\infty) \rightarrow \mathbb{R}$ is positive. In a similar way, with the proof of the above theorem, we may establish

Theorem 4. If $f: (0, +\infty) \rightarrow \mathbb{R}$ is positive on $(0, 1)$ and satisfies the following conditions

$$(i) \quad f(x+1) = xf(x), \quad x \in (0, +\infty),$$

(ii) F_k is a convex function of the order k on $(A, +\infty)$, A being a fixed non-negative number,

$$(iii) \quad f(1) = 1,$$

then we have

$$f(x) = \Gamma(x) \quad \text{for } x \in (0, +\infty).$$

REFERENCES

1. J. ANASTASSIADIS: *Définition des fonctions eulériennes par des équations fonctionnelles*. Mémoires des Sciences Mathématiques, fascic. CLVI. 1964.
2. E. ARTIN: *The Gamma Function*. New York 1964.
3. H. BOHR, I. MOLLERUP: *Leerebog i matematisk Analyse*, t. III. Kopenhagen, 1922.
4. A. LUPAŞ: *Inequalities involving Gamma-function*. To appear.
5. D. S. MITRINOVIĆ (in cooperation with P. M. VASIĆ): *Analytic Inequalities*. Berlin — Heidelberg — New York, 1970.
6. G. N. WATSON: *A note on Gamma functions*. Proc. Edinburgh Math. Soc. (2) **11** (1958/59); Edinburgh Math. Notes **42** (1959), 7—9.

Katedra za matematiku
Elektrotehnički fakultet
11000 Beograd, Jugoslavija

Institutul de calcul
Cluj, R. S. România