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THE STUDY OF THE FIRST ORDER PERTURBATION FOR MISES' EQUATION FROM THE BOUNDARY LAYER THEORY BY FINITE-DIFFERENCES METHOD

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An explicit method with finite differences for the perturbation of first order of Mises' equation from the theory of boundary layer of a viscous incompressible fluid has been studied in this paper. The paper is concluded with an application to the calculation of the speed and skin friction on a circular cylinder with slight deformation. The calculations were scheduled in FORTRAN IV language on the electronic computer FELIX-C 256 and the results are given in tables and graphics.

SYMBOLS

$\bar{x}$, $\bar{\psi}$	Mises coordinates for boundary layer ($\bar{\psi}$ = the stream function)
$\bar{u}$, $\bar{u}_1$	velocities in x direction on inner and on external edge of the boundary layer, respectively
u_∞,	constant velocity of the fluid to infinite
ν,	kinematic viscosity
U_{10}^*, U_{11}^*,	nondimensional speeds in the neighbourhood of stagnation point
C_0, C_1,	constants (2)
(i, j)	denote the mesh point (X_i, ψ_j); i and j are integers
I, J,	integer and positive numbers greater than unity
X, ψ	nondimensional variables given by (1)
G, W	unknown main functions, considered in (3)
$g_{i,j}$; $w_{i,j}$	net functions, which are approximations in (i, j) points, of G, W functions, corresponding to equation (8)
a_i, b_i,	unknown coefficients in (23)
$V(X)$,	function given in (2)
r,	parameter of D_Δ net, equal with $\Delta X/(\Delta\psi)^2$
ΔX, $\Delta\psi$,	the grid-point spacing in the X and ψ directions
Re,	Reynolds number
$c_{\tau w}$,	nondimensional skin-friction coefficient
$\bar{x}_0$, x_0,	coordinate of initial station
L,	reference length

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INTRODUCTION

In [5] the method of small perturbations has been applied to Mises' equation from the theory of incompressible laminar boundary layer [2], [3]

$$\frac{\partial \bar{u}^2}{\partial \bar{x}} = \frac{d\bar{u}_1^2}{d\bar{x}} + \nu \bar{u} \frac{\partial^2 \bar{u}^2}{\partial \bar{\psi}^2}$$

$$\bar{u}(\bar{x}, 0) = 0, \ \bar{u}(\bar{x}, \bar{\psi}_\infty) = \bar{u}_1(\bar{x}), \ \bar{x}_0 < \bar{x} \leqslant \bar{x}_I$$

$$\bar{u}(\bar{x}_0, \bar{\psi}) = \bar{u}^0(\bar{\psi}), \qquad 0 \leqslant \bar{\psi} \leqslant \infty, \ \bar{\psi}_\infty$$

with expressions

$$\bar{x} = x + \varepsilon x_1(x) + 0(\varepsilon^2) \tag{0.1}$$

$$\bar{\Psi} = \Psi + \varepsilon \Psi_1(\Psi) + 0(\varepsilon^2), \ (\bar{\psi} = \bar{\Psi} \sqrt{\nu}) \tag{0.2}$$

$$\bar{u}_1(\bar{x}) = u_{10}(x) + \varepsilon u_{11}(x) + 0(\varepsilon^2) \tag{0.3}$$

$$\bar{u}(\bar{x}, \bar{\Psi}) = u_0(x,\Psi) + \varepsilon u_1(x,\Psi) + 0(\varepsilon^2) \tag{0.4}$$

in the closed domain

$$\bar{D}^* \equiv \{x,\Psi \,|\, 0 \leqslant x_0 \leqslant x \leqslant x_I, \ 0 \leqslant \Psi \leqslant \infty, \ \Psi_\infty\}$$

corresponding to the flow field in the physical plane $(\bar{x}, \bar{y})$.

For the zero order u_0 and, respectively, first order u_1 perturbation the following equations have been obtained [5]

$$\frac{\partial u_0^2}{\partial x} = \frac{du_{10}^2}{dx} + u_0 \frac{\partial^2 u_0^2}{\partial \Psi^2}, \ (x, \Psi) \in D^* \tag{0.5}$$

$$\frac{\partial}{\partial x}(u_0 u_1) = u_0 \frac{\partial^2}{\partial \Psi^2}(u_0 u_1) + \frac{u_1}{2} \frac{\partial^2 u_0^2}{\partial \Psi^2} + \frac{d}{dx}(u_{10} u_{11}) +$$

$$+ \frac{1}{2}\left(\frac{dx_1}{dx} - 2\frac{d\Psi_1}{d\Psi}\right) u_0 \frac{\partial^2 u_0^2}{\partial \Psi^2}, \ (x, \Psi) \in D^* \tag{0.6}$$

with initial and boundary values (initial station $x = x_0$)

$$u_i(x, 0) = 0, \ u_i(x, \infty) = u_{1i}(x), \ x_0 < x \leqslant x_I \tag{0.7}$$

$$u_i(x_0,\Psi) = u_i^0(\Psi), \ 0 \leqslant \Psi \leqslant \infty, \ \Psi_\infty \tag{0.8}$$

where $u_i^0(\Psi)$ is a given (calculated) function in the initial station (starting station) $x = x_0$, in the neighbourhood of the stagnation point, so that $u_i^0(0) = 0$ and $u_i^0(\infty) = u_{1i}(x_0)$.

This paper describes the numerical study of the boundary-value problem (0.6)—(0.8) which we then apply to the boundary layer which is formed on a circular cylinder with slight deformation, considering that the motion domain is divided in three subdomains (D^s, D^i, D^∞).

1. SOLUTION IN THE INNER DOMAIN (D^i)

1.1 *Equation with finite differences (explicit scheme with 4 points)*
Let's take again the boundary problem (0.6)—(0.8) and let us do the transformations for variables and functions

$$x = LX, \quad x_1 = LX_1, \Psi = \psi\sqrt{LU_\infty}, \quad \Psi_1 = \psi_1\sqrt{LU_\infty} \tag{1}$$

$$u_{10} = u_\infty U_{10}, u_{11} = u_\infty U_{11}, U^*_{10} = C_0 X, U^*_{11} = C_1 X, U^2_{10} \equiv V(x) \tag{2}$$

$$\frac{d\psi_1}{d\psi} = \frac{C_1}{2C_0} (C_0, \ C_1 = \text{constants})$$

$$u_0 = u_\infty U = u_\infty \sqrt{U_{10}{}^2 - G(X, \psi)}, \ u_0 u_1 = u^2_\infty [U_{10} U_{11} - W(X, \psi)] \tag{3}$$

The continous and unknown function $W(X, \psi)$ with respect to (X, ψ) in the closed domain $\bar{D}$ satisfies a linear and inhomogeneus equation, of parabolic type, subject to the following initial and boundary conditions

$$L(W), \equiv \frac{\partial W}{\partial X} - \sqrt{U^2_{10} - G}\,\frac{\partial^2 W}{\partial \psi^2} + \frac{1}{2}\,\frac{\partial G/\partial x}{U^2_{10} - G}\,W =$$

$$= \frac{1}{2}\,\frac{\partial G}{\partial X}\left(\frac{U_{10} U_{11}}{U^2_{10} - G} + \frac{dX_1}{dX} - \frac{C_1}{C_0}\right) \tag{4}$$

$$(X, \ \psi) \in D \equiv \{X, \psi \mid X_0 < X \leqslant X_I \,; 0 < \psi < \infty, \ \psi_\infty\}$$

$$W(X, 0) = U_{10} U_{11}, \ W(X, \psi_\infty) = 0, \ X_0 < X \leqslant X_I \tag{5}$$

$$W(X_0, \psi) = W^0(\psi) \text{ given function}, \ 0 \leqslant \psi \leqslant \infty, \ \psi_\infty \tag{6}$$

$$(G(X, 0) = U^2_{10}(\equiv V(x)), \ G(X, \psi_\infty) = 0)$$

Starting from the singularity

$$\sqrt{U^2_{10} - G} \to 0, \quad \frac{\partial^2 W}{\partial \psi^2} \to \infty \text{ for } \psi \to 0$$

like in Mises's equation, we shall divide the domain from (X, ψ) plane corresponding to the viscous fluid moving in three subdomains D^s, D^i

and D^∞. These subdomains will represent the body's neighbourhood, the inner boundary layer and, respectively, the infinite point domain.

We shall use the finite differences method for solving this problem. We replace the derivative with respect to X by a forward difference. Then we substitute the second order derivative with respect to ψ by a second order central difference. Using Taylor's series and introducing the notations $W(X_i, \psi_j) \equiv W_{i,j}$, $G(X_i, \psi_j) \equiv G_{i,j}$ the formulas result

$$\frac{\partial W}{\partial X}(X_i, \psi_j) = \frac{W_{i+1,j} - W_{i,j}}{\Delta X} - \frac{\Delta X}{2}\frac{\partial^2 W}{\partial X^2}(\tilde{X}_i, \psi_j) \approx \frac{w_{i+1,j} - w_{i,j}}{\Delta X} \tag{7_1}$$

$$\frac{\partial^2 W}{\partial \psi^2}(X_i, \psi_j) = \frac{W_{i,j+1} - 2W_{i,j} + W_{i,j-1}}{\Delta\psi^2} - \frac{\Delta\psi^2}{12}\frac{\partial^4 W}{\partial\psi^4}(X_i, \tilde{\psi}_j) \approx$$

$$\approx \frac{w_{i,j+1} - 2w_{i,j} + w_{i,j-1}}{\Delta\psi^2} \tag{7_2}$$

and similar formulas concerning the $G(X, \psi)$ function.

We represent the finite differences boundary problem associating to (4)—(6), on $\bar{D}_\Delta$ net, by the following explicit scheme with 4 points

$$w_{i+1,j} = c^1_{i,j}(w_{i,j-1} + w_{i,j+1}) + c^0_{i,j}w_{i,j} + d_{i,j} \tag{8}$$

$$(i = 0, 1, 2, \ldots, I-1;\ j = 1, 2, \ldots, J-1)$$

$$w_{i,0} = (U_{10}U_{11})_i,\ w_{i,J} = 0\ (w_{0,J} = 0),\ i = 1, 2, \ldots, I \tag{9}$$

$$w_{0,j} = W^0_j,\ j = 0, 1, 2, \ldots, J \tag{10}$$

where

$$\left.\begin{aligned}
c^1_{i,j} &= r\sqrt{V_i - g_{i,j}}\ (= c^{-1}_{i,j}) \\
c^0_{i,j} &= 1 - 2r\sqrt{V_i - g_{i,j}} - \frac{1}{2}\frac{g_{i+1,j} - g_{i,j}}{V_i - g_{i,j}} \\
d_{i,j} &= \frac{1}{2}\frac{g_{i+1,j} - g_{i,j}}{V_i - g_{i,j}}\left\{(U_{10}U_{11})_i + \left[\left(\frac{\mathrm{d}X_1}{\mathrm{d}X}\right)_i - \frac{C_1}{C_0}\right](V_i - g_{i,j})\right\} \\
&(i = 0, 1, 2, \ldots, I;\ j = 0, 1, 2, \ldots, J)
\end{aligned}\right\} \tag{11}$$

$$\Delta X = \frac{X_I - X_0}{I},\ \Delta\psi = \frac{\psi_\infty}{J},\ \psi_0 = 0,\ \psi_J = \psi_\infty,\ r = \frac{\Delta X}{(\Delta\psi)^2} \tag{12}$$

$$\bar{D}_\Delta \equiv \{X_i, \psi_j \mid X_i = X_0 + i\Delta X,\ \psi_j = j\Delta\psi;\ i = 0, 1, \ldots, I;\ j = 0, 1, \ldots, J\} \tag{13}$$

We add the zero order perturbation scheme to (8)—(13) scheme

$$g_{i+1,j} = g_{i,j} + c^1_{i,j}(g_{i,j+1} - 2g_{i,j} + g_{i,j-1}) \tag{14}$$

$$(i = 0, 1, 2, \ldots, I - 1;\ j = 1, 2, \ \ldots, \ J - 1)$$

$$g_{i,0} = V_i,\ g_{i,J} = 0,\ g_{0,j} = G_{0,j},\ \text{(given values)} \tag{15}$$

$$(i = 1, 2, \ \ldots, \ I;\ j = 0, 1, \ \ldots, \ J)$$

which we can obtain starting from the (0.5) equation and using the transformations of the form (3); $G_{0,j}$ are initial given values.

It has been considered that

$$g_{i,0} \geqslant g_{i,j},\ g_{i,j} < g_{i+1,j} \tag{16}$$

We must determine the initial solution of these explicit schemes and analyse their consistency and stability for their electronic computer programming.

1.2. *Initial solution.* According to [5] wich studies the initial solution determination $X = X^0 \equiv X_0$ section, we have, using (3), the formulas

$$\frac{u^0_1}{u^0_0} = \frac{u^0_{11}}{u^0_{10}} = \frac{U^0_{11}}{U^0_{10}} = \frac{U^0_{10}U^0_{11} - W^0}{U^{02}_{10} - G^0} \tag{17}$$

For initial values calculation $W_{0,j}$ of W function in $X = X_0$ section, the following formulas result

$$W_{0,j} = \frac{C_1}{C_0} G_{0,j},\ j = 0, 1, 2, \ldots, \ J(W_{0,J} \equiv 0) \tag{18}$$

$$(W^0_j \equiv W_{0,j},\ G^0_j \equiv G_{0,j})$$

where the initial values $G_{0,j}$ are calculated in zeroth order approximation (in [5] the $W_0 = G/U^2_{10}$ function has been used instead of G).

1.3 *Consistency.* Using the (7) type formulas we show easily that the truncation error $\tilde{\tau}(X_i, \psi_j) \equiv \tilde{\tau}_{i,j}$ of (8)—(13) scheme has the expression

$$\tilde{\tau}_{i,j} = \frac{\Delta X}{2}\left\{\frac{\partial^2 W}{\partial X^2} + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}\left[\frac{W}{V - G} - \left(\frac{U_{10}U_{11}}{V - G} + \frac{\mathrm{d}X_1}{\mathrm{d}X} - \frac{C_1}{C_0}\right)\right]\right\}_{(i,j)} -$$

$$- \frac{\Delta\psi^2}{12}\left(\sqrt{V - G}\,\frac{\partial^4 W}{\partial\psi^4}\right)_{(i,j)}. \tag{19}$$

The scheme with finite differences (8)—(13) approximates the boundary value problem (4)—(6) with an error of $\Delta X + \Delta\psi^2$ order, hence it is consistent on D^i_Δ grid (D^i_Δ = grid of points from D^i).

1.4 *Stability*. Consideration of some linear schemes of positive type, that is of some schemes in which the coefficients of $w_{i,j-1}$, $w_{i,j}$, $w_{i,j+1}$ must be nonnegative in D^i_Δ, is the starting point in the study of the stability of some parabolic schemes of (8)—(13) type, inhomogeneous and with variable coefficients. In these conditions the explicit scheme (8)—(13) is stable. Indeed, equation (8) is of positive type if the condition

$$r \leqslant \frac{2g_{i,0} - g_{i,j} - g_{i+1,j}}{4(g_{i,0} - g_{i,j})^{3/2}}, \ (r > 0) \tag{20}$$

$$(i = 0, 1, 2, \ldots, I - 1 \ ; \ j = 1, 2, \ldots, J)$$

is verified in each point $(i, j) \in D^i_\Delta$.

★

Obs. If we admit that the net function $g_{i,j}$ satisfies the (15)—(16) conditions on the D_Δ net and that it is monotonously decreasing with respect to ψ and increasing with respect to X, then condition (20) can be replaced by

$$r \leqslant \frac{1}{2U_{10}(X)} \text{ or } r \leqslant \min_{X \in [X_0, X_I]} \frac{1}{2U_{10}(X)} = \frac{1}{2U_{10}(X_I)} \tag{20'}$$

if we adopt a constant value for r and admit that $U(X)$ is an increasing function for $X \in [X_0, X_I]$.

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Moreover, let's suppose that the coefficients sum of $w_{i,j-1}$, $w_{i,j}$, $w_{i,j+1}$ does not exceed the unity. As it has been found directly, this is true if

$$\sum_s c^s_{i,j} = 1 - \frac{1}{2}\frac{g_{i+1,j} - g_{i,j}}{g_{i,0} - g_{i,j}} \leqslant 1, \ (s = -1, 0, 1) \tag{21}$$

which is verified in every point with conditions (16).

-Let's metion that using the approximative method of determination of the initial solution and formulas (18), we introduce errors. So, we begin the calculation with the approximate values $w^*_{0,j}$, not with the exact values $w_{0,j}$. Without introducing other errors, we calculate, in station $X = X_{i+1}$, the values $w^*_{i+1,j}$, depending on $w^*_{i,j-1}$, $w^*_{i,j}$, $w^*_{i,j+1}$ with an equation which has been obtained from (8), replacing w by w^*. The calculation error (the stability error) $z_{i,j} = w_{i,j} - w^*_{i,j}$ [$w_{i,j}$ — the exact solution of equation (8)] satisfies equation

$$z_{i+1,j} = c^1_{i,j}(z_{i,j-1} + z_{i,j+1}) + c^0_{i,j} z_{i,j}$$

$$z_{i,0} = 0, \ z_{i,J} = 0, \ z_{0,j} = e_j, \ i = 0, 1, \ldots, I$$

where e_j are the initial errors.

As a result of inequalities (20) and (21) one can obtain

$$|z_{i+1,j}| \leqslant \max_{(k=-1,0,1)} |z_{i,j+k}|, \ j = 1, 2, \ldots, J \tag{22}$$

which proves the stability of scheme (8) in comparison with the step in X direction. The error $z_{i,j}$, due to the initial error e_j, doesn't increase in comparison with the step ΔX, if condition (20) is satisfied.

2. SOLUTION IN THE NEIGHBOURHOOD OF THE BODY'S SURFACE (D^s)

2.1 *Determination of perturbation speed of first order.* We look, in the body neighbourhood, for the solution of the following form

$$U_0 = \sum_1^\infty a_n(X)\, \psi^{n/2}, \ U_1 = \sum_1^\infty b_n(X)\, \psi^{n/2} \tag{23}$$

$$(\psi = \text{little values})$$

The perturbation of zero in order U_0 ($\equiv U$ from [5]) is determined in [6] by the aid of the coefficients

$$a_2 = -\frac{4A}{3a_1^2}, \ a_3 = -\frac{14A^2}{9a_1^5}, \ a_4 = -\frac{80A^3}{27a_1^8}, \left(A = U_{10}\frac{\mathrm{d}U_{10}}{\mathrm{d}X}\right) \tag{24}$$

knowing that a_1 satisfies an algebraic equation of 3rd degree.

For the determination of the $b_i(X)$ coefficients we use compatibility conditions (on the body), deduced from (0.6), of the form

$$\left.\begin{aligned} Z(X, \psi) &= -B(X) \\ U_0\frac{\partial Z}{\partial \psi} &= 0 \end{aligned}\right\} \psi = 0 \tag{25}$$

$$U_0\frac{\partial}{\partial \psi}\left(U_0\frac{\partial Z}{\partial \psi}\right) = 0, \ \psi = 0$$

where

$$\left.\begin{aligned} Z(X, \psi) &= U_0\frac{\partial^2}{\partial \psi^2}(U_0U_1) + \frac{1}{2}U_1\frac{\partial^2 U_0^2}{\partial \psi^2} \\ B(X) &= \frac{\mathrm{d}}{\mathrm{d}X}(U_{10}U_{11}) - \frac{1}{2}\frac{\mathrm{d}U_{10}^2}{\mathrm{d}X}\left(\frac{\mathrm{d}X_1}{\mathrm{d}X} - 2\frac{\mathrm{d}\psi_1}{\mathrm{d}\psi}\right). \end{aligned}\right\} \tag{26}$$

We observe that

$$\frac{\partial}{\partial \psi}(U_1U_1) = \sum_{n=2}^\infty \sum_{k=1}^{n-1} \frac{n}{2}\, a_k b_{n-k}\, \psi^{\frac{n}{2}-1}$$

Using the above conditions, we shall obtain, after elementary calculations, the following formulas for the $b_i(X)$ coefficients

$$\left.\begin{aligned} b_1(X) &= \frac{1}{a_1} \lim_{\psi \to 0} \frac{\partial (U_0 U_1)}{\partial \psi} \\ b_2(X) &= \frac{4}{3a_1^3}(2b_1 A - a_1 B) \\ b_3(X) &= \frac{14A}{9a_1^6}(5b_1 A - 2a_1 B) \end{aligned}\right\} \tag{27}$$

We shall use a method of successive approximations to determine the b_1 coefficient. For the W function, we have the expression

$$\begin{aligned} W(X, \psi) = U_{10} U_{11} - [a_1 b_1 \psi + (a_1 b_2 + a_2 b_1)\psi^{3/2} + \\ + (a_1 b_3 + a_2 b_2 + a_3 b_1)\psi^2] \end{aligned} \tag{28}$$

Considering the points $\psi = 0$, $\psi = \Delta\psi$ and $\psi = 2\Delta\psi$ and using only the first three terms of Taylor formula, there results the approximation formula of zeroth order

$$(a_1 b_1)^{(0)} = \frac{3w_{i,0} - 4w_{i,1} + w_{i,2}}{2\Delta\psi} \tag{29}$$

Considering four terms from Taylor's series of W function and taking into consideration the coefficients expressions a_2 and b_2 we find the approximation of first order

$$a_1 b_1 = \frac{(a_1 b_1)^{(0)} + \dfrac{\theta^{-3/2}\sqrt{\Delta\psi}}{6} \dfrac{B}{a_1}}{1 + \dfrac{\theta^{-3/2}\sqrt{\Delta\psi}}{6} \dfrac{A}{a_1^3}} \left(\equiv \left(\frac{\partial W}{\partial \psi} \right)_{\psi=0} \right) \tag{30}$$

$$(0 < \theta < 1)$$

We have the following expression for the perturbation speed u_1

$$\frac{u_1}{u_\infty} = U_1 = b_1 \psi^{1/2} + b_2 \psi + b_3 \psi^{3/2} \tag{31}$$

2.2 *Calculation of skin friction* $\tau_w(X)$. The local skin friction, on the body surface, τ_w, has been calculated in the first approximation of the small perturbations method, with the formula

$$\tau_w(X) = \mu \left(\frac{\partial \bar{u}}{\partial \bar{y}} \right)_{\bar{y}=0} = \frac{\mu}{2} \left(\frac{\partial \bar{u}^2}{\partial \bar{\psi}} \right)_{\bar{\psi}=0} = \frac{\mu}{2\sqrt{\nu L u_\infty}} \left(1 - \varepsilon \frac{\psi_1}{\psi} \right) \left(\frac{\partial \bar{u}^2}{\partial \psi} \right)_{\psi=0}$$

Taking into account that $\bar{u} = u_0 + \varepsilon u_1$ it results the formula

$$\tau_w(X) = \frac{\rho u_\infty^2}{2\sqrt{\mathrm{Re}}}\left[\left(1 - \varepsilon\frac{\psi_1}{\psi}\right)\left(\frac{\partial U_0^2}{\partial \psi}\right)_{\psi=0} - 2\varepsilon\left(\frac{\partial W}{\partial \psi}\right)_{\psi=0}\right] \tag{32}$$

and, for the local skin friction coefficient $c_{\tau w}$, we have the formula

$$\sqrt{\mathrm{Re}}\; c_{\tau w}(X) = T_1 + \varepsilon T_2 \tag{33}$$

where

$$T_1 = a_1^2,\; T_2 = 2\, a_1 b_1 - \frac{C_1}{2C_0} a_1^2 \tag{34}$$

$$\left(c_{\tau w} = \frac{2\tau_w}{\rho u_\infty^2},\; \mathrm{Re} = \frac{\rho L u_\infty}{\mu}\right)$$

APPLICATION

INCOMPRESSIBLE BOUNDARY LAYER ON A CIRCULAR CYLINDER SLIGHTLY DEFORMED

3.1 *Expression of the $\bar{x}$ coordinate.* Let be a circular cylinder representing a slight (small) deformation. The perpendicular section on the surface is the deformed circle (C) having the following equation in the polar coordinates r^* and θ (fig. 1)

$$r^* = R(1 - \varepsilon \sin^2 \theta), \qquad (\varepsilon \geqslant 0) \tag{35}$$

where the small quantity ε is the perturbation parameter. The deformed circle (C) can be considered as the first approximation of an ellipse with semi-axes $\overline{O_1A} = R$ and $\overline{O_1B} = R(1 - \varepsilon)$. Let be $\bar{x} = \widehat{OP}$, where P is a point on (C) in the second quadrant, the coordinate of boundary layer. We observe that

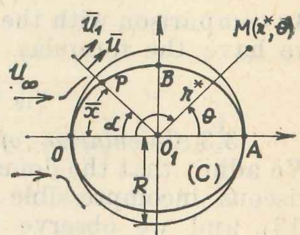

Fig. 1 3.1

$$dr^* \sim 0(\varepsilon),\; dr^{*2} \sim 0(\varepsilon^2),\; d\bar{x} \approx -r^* d\theta = -R(1 - \varepsilon \sin^2 \theta)\, d\theta$$

Integrating, it results

$$\bar{x} = -R\int_\pi^\theta (1 - \varepsilon \sin^2 \theta)\, d\theta = R\left[\alpha - \frac{\varepsilon}{2}\left(\alpha - \frac{\sin 2\alpha}{2}\right)\right]$$

$$(\alpha = \pi - \theta)$$

If compared with the general formula considered in (0.1) we infer that

$$\left.\begin{aligned} x &= R\alpha \\ x_1 &= -\frac{R}{2}\left(\alpha - \frac{1}{2}\sin 2\alpha\right)\end{aligned}\right\} \tag{36}$$

3.2 *Determination of external velocity u_1 of boundary layer.* The external speed is given by the distribution of ideal fluid speed on the cylinder surface. Let us suppose, therefore, that the cylinder is perpendicularly attacked on its generatrices by an ideal incompressible stream, which is in uniform translation to infinity with u_∞ speed in the direction of big axis OA. We can use the small perturbations method [1], [3] in the study of motion, looking for the stream function ψ a harmonic function like

$$\psi(r^*,\ \theta,\ \varepsilon) = \psi_0(r^*,\ \theta) + \varepsilon\, \psi_1(r^*,\ \theta) + \cdots$$

where the harmonic function ψ_0 coresponds to the motion around the circular cylinder ($\varepsilon = 0$), and function ψ_1, satisfying Laplace equation (in polar coordinates), has been determined as part of boundary problem by the form

$$\nabla^2 \psi_1(r^*,\ \theta) = 0$$

$$\psi_1(r^* \to \infty,\ \theta) = 0$$

$$\psi_1(r^* = R,\ \theta) = \frac{Ru_\infty}{2}(3 \sin \theta - \sin 3\theta)$$

After determination of ψ_1, using expressions like $\psi_1 = A(r^*) \sin \theta + B(r^*) \sin 3\theta$ we find the following formula for the distribution of the speed on the cylinder [3], [1]

$$\bar{u}_1 = u_\infty(2 \sin \theta + \varepsilon \sin 3\theta + \ldots)$$

By comparison with the expression $\bar{u}_1$, given by the general theory (0.3), we have the formulas

$$u_{10} = 2u_\infty \sin \alpha,\ u_{11} = u_\infty \sin 3\alpha \tag{37}$$

3.3 *Calculation of speed and skin friction in the boundary layer.* We admit that the domain of the plane $OX\,\psi$ of the boundary layer of viscous incompressible fluid is covered by the net $\bar{D}_\Delta$, considered in (13), and we observe that, taking $L = R$, there results

$$\left.\begin{aligned}
&x = R\alpha = RX,\ X_1 = -\frac{1}{2}\left(X - \frac{1}{2}\sin 2X\right)\\
&(X \equiv \alpha,\ X_1 = RX_1)\\
&U_{10} = 2 \sin X,\ U_{11} = \sin 3X,\ U^*_{10} = 2X,\ U^*_{11}, = 3X
\end{aligned}\right\}$$

x_1

$$(U_{10}U_{11})_i = 2 \sin X_i(3 \sin X_i - 4 \sin^3 X_i) = \frac{V_i}{2}(3 - V_i) \tag{38}$$

$$(g_{i.0} = V_i = (U^2_{10})_i = 4 \sin^2 X_i)$$

$$\left(\frac{\mathrm{d}X_1}{\mathrm{d}X}\right)_i = -\sin^2 X_i = -\frac{1}{4} g_{i,0},\ r = \frac{\Delta X}{\Delta\psi^2} = 0.33$$

$$\Delta\psi = 0.1\,;\ \Delta X = 0.0033\,;\ \frac{\psi_1}{\psi} = \frac{3}{4}\,;\ C_0 = 2\,;\ C_1 = 3$$

Let's denote the approximative values of function $W(X, \psi)$ in the nodes (X_i, ψ_j) by $w(X_i, \psi_j) \equiv w_{i,j}$. These approximative values have been calculated by finite differences, according with the scheme (8)—(12) and (14)—(15) taking into account formulas (38).

We add to the schemes (14)—(15) and (8)—(12) the following data :

$$V_i = 4 \sin^2 (0.1745 + 0.0033i), \; i = 0, 1, \ldots, 265 \qquad (39)$$

Schemes (14) — (15) and (8) — (12) with these data were scheduled in FORTRAN IV language (Appendix 1) and calculated on the electronic computer FELIX-C 256. The results of the calculation, relating to G and W, are given in tables 2 and 3 (for 8 cross sections of the boundary layer).

Table ~~1~~ 3.1

j	$G_{0,j} \equiv Aj$	$W_{0,j} \equiv B_j$
0	0.1208	0.1812
1	0.0555	0.0833
2	0.0230	0.0345
3	0.0091	0.0137
4	0.0033	0.0050
5	0.0009	0.0014
6	0.0002	0.0003
7	0.0000	0.0000
8	0.0000	0.0000

— The D^s domain of the body's neighbourhood. We use formula (30) in the calculation of the product $a_1 b_1$, where

$$\theta^{-3/2} = 6{,}6668, \; \left(\theta = \frac{1}{3.6}\right), \; \Delta\psi = 0.1$$

$$A(X_i) \equiv A_i = 2 \sin 2X_i$$

$$B(X_i) \equiv B_i = \frac{1}{2}(18 - 28 \sin^2 X_i) \sin 2X_i = (-7g_{i,0} + 18)\frac{A_i}{4}$$

$$\theta^{-3/2} \sqrt{\Delta\psi}/6 = 0.3515$$

approximation $(a_1 b_1)^{(0)}$ is given by formula (29), where the values $w_{i,j} (j = 0, 1, 2)$ came from the Table 3. For the calculation of the coefficients b_2 and b_3 we use formulas (27) where coefficient a_1, determined in the study of zeroth approximation [6], has the values given in Table 4. Function W has been calculated, then, for $\psi = 0.1$ and $\psi = 0.2$ with the help of formula (28), and its values are given in Table 5. These values can be compared with the corresponding ones in Table 3. We can consider the values W from Table 3, for $0 \leqslant \psi \leqslant 2\Delta\psi$ as initial approximative values in the method of determination with successive approximations of function W in the domain near the body.

We have formulas (33)—(34) for the calculation of the local skin friction on the deformed cylinder surface. Table 4 contains the resulted values. Functions T_1, T_2 and $c_{\tau w}\sqrt{\mathrm{Re}}$ are represented by points in the figures 2 and 3.

3-2

Table 2

$g_{i,j} = g(X_i, \psi_j)$

ψ_j \ X_i	0.1745	0.3494	0.5408	0.7157	0.7850	0.8411	0.9599	1.0490
0.0	0.1208	0.4688	1.0602	1.7221	1.9984	2.2223	2.6839	3.0062
0.1	0.0555	0.3132	0.8134	1.4089	1.6648	1.8750	2.3180	2.6333
0.2	0.0230	0.2103	0.6292	1.1607	1.3958	1.5913	2.0112	2.3155
0.3	0.0091	0.1396	0.4861	0.9570	1.1713	1.3519	1.7465	2.0374
0.4	0.0033	0.0910	0.3737	0.7876	0.9817	1.1474	1.5156	1.7916
0.5	0.0009	0.0581	0.2854	0.6462	0.8209	0.9719	1.3134	1.5736
0.6	0.0002	0.0362	0.2162	0.5281	0.6842	0.8211	1.1359	1.3798
0.7	0.0000	0.0219	0.1622	0.4296	0.5682	0.6915	0.9801	1.2075
0.8	0.0000	0.0129	0.1205	0.3477	0.4700	0.5803	0.8435	1.0543
0.9		0.0074	0.0886	0.2798	0.3870	0.4852	0.7238	0.9184
1.0		0.0041	0.0644	0.2239	0.3171	0.4040	0.6193	0.7980
1.1		0.0022	0.0462	0.1781	0.2586	0.3349	0.5281	0.6915
1.2		0.0011	0.0328	0.1407	0.2098	0.2765	0.4489	0.5975
1.3		0.0006	0.0230	0.1104	0.1692	0.2271	0.3802	0.5148
1.4		0.0003	0.0158	0.0860	0.1357	0.1857	0.3208	0.4422
1.5		0.0001	0.0108	0.0666	0.1082	0.1510	0.2697	0.3787
1.6		0.0000	0.0062	0.0485	0.0822	0.1180	0.2206	0.3232
1.7		0.0000	0.0017	0.0307	0.0566	0.0854	0.1720	0.2750
1.8			0.0008	0.0130	0.0311	0.0529	0.1235	0.2286
1.9			0.0008	0.0065	0.0056	0.0204	0.0750	0.1827
2.0			0.0004	0.0028	0.0028	0.0102	0.0266	0.1369
2.1			0.0002	0.0014	0.0014	0.0056	0.0133	0.0912
2.2			0.0000	0.0005	0.0004	0.0024	0.0088	0.0456
2.3			0.0000	0.0001	0.0002	0.0012	0.0064	0.0228
2.4				0.0000	0.0001	0.0005	0.0046	0.0057
2.5				0.0000	0.0000	0.0002	0.0029	0.0029
2.6					0.0000	0.0000	0.0014	0.0015
							0.0007	0.0003

Table 3

$w_{i,j} = w(X_i, \psi_j)$

ψ_j \ X_i	0.1745	0.3494	0.5408	0.7157	0.7850	0.8411	0.9599	1.0490
0.0	0.1812	0.5932	1.0283	1.1003	1.0008	0.8641	0.4242	−0.0094
0.1	0.0833	0.3980	0.7912	0.9028	0.8361	0.7309	0.3654	−0.0126
0.2	0.0345	0.2672	0.6102	0.7395	0.6952	0.6130	0.3052	−0.0275
0.3	0.0137	0.1768	0.4680	0.6019	0.5734	0.5088	0.2475	−0.0468
0.4	0.0050	0.1146	0.3558	0.4861	0.4686	0.4173	0.1941	
0.5	0.0014	0.0726	0.2677	0.3889	0.3789	0.3379	0.1458	
0.6	0.0003	0.0447	0.1990	0.3079	0.3028	0.2696	0.1032	
0.7	0.0000	0.0267	0.1459	0.2410	0.2388	0.2114	0.0663	
0.8	0.0000	0.0155	0.1055	0.1863	0.1855	0.1625	0.0349	
0.9	0.0000	0.0087	0.0750	0.1419	0.1416	0.1218	0.0088	
1.0		0.0047	0.0525	0.1064	0.1059	0.0885		
1.1		0.0024	0.0361	0.0783	0.0772	0.0615		
1.2		0.0012	0.0243	0.0564	0.0545	0.0401		
1.3		0.0006	0.0161	0.0396	7.0369	0.0235		
1.4		0.0002	0.0104	0.0270	0.0234	0.0108		
1.5		0.0001	0.0065	0.0176	0.0134	0.0015		
1.6		0.0001	0.0040	0.0108	0.0062			
1.7		0.0000	0.0023	0.0060	0.0011			
1.8		0.0000	0.0014	0.0028				
1.9		0.0000	0.0007	0.0007				
2.0			0.0003					
2.1			0.0002					
2.2			0.0000					
2.3			0.0000					
2.4			0.0000					
2.5			0.0000					

Table 4

X	10° / 0.1745	20°1′ / 0.3494	31° / 0.5408	41°1′ / 0.7157	45° / 0.7850	48°12′ / 0.8411	55° / 0.9599	60°6′ / 1.0490
$(a_1b_1)°$	1.2245	2.2740	2 15	2.1460	1.7660	1.4085	0.5810	−0.0265
a_1b_1	1.8316	2.9850	3.1872	2.4520	1.9525	1.4993	0.4872	−0.2288
a_1	1.1170	1.5500	1.8572	2.0350	2.0818	2.1101	2.1384	2.1386
a_2	−0.7309	−0.7139	−0.6361	−0.6377	−0.6153	−0.5952	−0.5485	−0.5046
a_3	−0.4185	−0.2877	−0.1906	−0.1748	−0.1591	−0.1469	−0.1231	−0.1042
b_1	1.6397	1.9258	1.7161	1.2049	0.9379	0.7105	0.2278	−0.1070
b_2	−0.9887	−0.8529	−0.5067	−0.1927	−0.0626	0.0337	0.2248	0.3334
b_3	−0.5179	−0.3300	−0.1275	−0.0021	0.0393	0.0678	0.1140	0.1325
$T_1 = a_1^2$	1.2477	2.4025	3.4492	4.1412	4.3339	4.4525	4.5728	4.5736
T_2	2.7274	4.1681	3.7875	1.7981	0.6546	−0.3408	−2.4552	−3.8878

Table 5

$W(x, \psi)$

ψ \ X	0.1745	0.3494	0.5408	0.7157	0.7850	0.8411	0.9599	1.0490
0.1	0.0762	0.3846	0.7762	0.8927	0.8281	0.7249	0.3633	−0.0120
0.2	0.0423	0.2557	0.5823	0.7173	0.6746	0.5944	0.2913	−0.0372

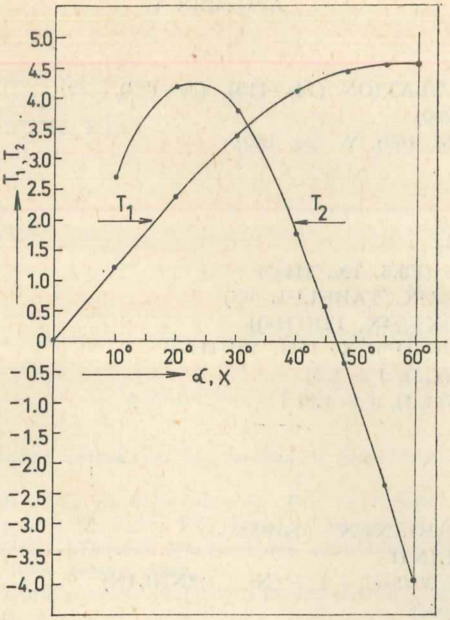

Fig. 2

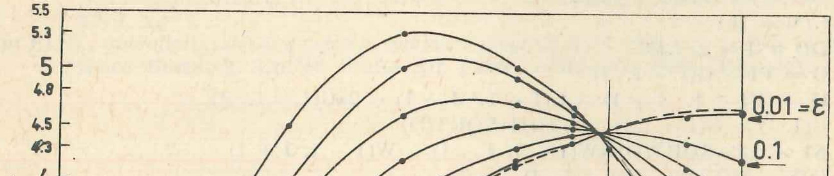

Fig. 3

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APPENDIX 1

```
C     FORTRAN IV
C     SCHEMES CALCULATION (14)-(15), (8)-(12), (39)
      DIMENSIØN X(200)
      DIMENSIØN G(26, 160), W (26, 160)
      M = 160
      NRLIN = 25
      NRLINII = 26
    1 FØRMAT (8F6. 4)
   11 FØRMAT (4X, 11 (F9.6, 1X, 1H*))
   13 FØRMAT(1H1, ///30X, 'TABELUL W')
   14 FØRMAT (1H1, 5X////4X, 121(1H*))
  325 FØRMAT (4X, 121(1H*)/3X, 1H*, 121(1H*))
      READ (105.1) (G(1,J), J = 1,8)
      READ (105.1) (W(1,J), J = 1,8)
      DO 2 K = 9,160
      G(1, K) = 0
    2 W(1, K) = 0
      DO 4 N = 1,18
      X(1) = 0.1745 + NRLIN*(N - 1)*0.0033
      DO 3 I = 2, NRLINII
      X(I) = 0.1745 + 0.0033*(I - 1 + (N - 1)*NRLIN)
      VI = 4*SIN(X(I))**2
      G(I, 1) = VI
      W(I, 1) = VI*(3 - VI)/2
      LIM = 161 - I
      DO 5 J = 2, LIM
      D = VI - G(I - 1, J)
      H = G(I - 1, J + 1) + G(I - 1, J - 1) - 2*G(I - 1, J)
      G(I, J) = G(I - 1,J) + 0.33*H*SQRT(D)
      S1 = 0.33*SQRT(D)*(W(I - 1, J - 1) + W(I - 1, J + 1)
      DIF = G(I, J) - G(I - 1, J)
      S2 = 1 - 0.66*SQRT(D) - DIF/(2*D)
      S2 = S2*W(I - 1, J)
      S3 = DIF*((6 + VI)*G(I - 1, J) - 3*VI**2)/(8*D)
      W(I, J) = S1 + S2 + S3
    5 CØNTINUE
    3 CØNTINUE
      NCRES = 3
      LIMITA = 31 + 5*(N/NCRES)
      DØ 8 L = 1, NRLIN
      MR = NRLIN*(N - 1) + L
      WRITE (108, 325)
 9876 FØRMAT (4X, 1H*, I5, F8.4)
      WRITE (108,9876)MR, X(L)
      WRITE (108,1) (G (L, J), J = 1, LIMITA)
    8 CØNTINUE
      WRITE (108,13)
      DØ 888 L = 1, NRLIN
      MR = (N - 1)*NRLIN + L

  326 FØRMAT (4X, 1H*, I5)
      WRITE (108, 326)MR
      WRITE (108,11)X(L), (W(L, J), J = 1, LIMITA)
```

Appendix 1 — continued

```
888 CØNTINUE
    DØ 20 K = 1, LIM
    G(1, K) = G(NRLINII, K)
 20 W(1, K) = W(NRLINII, K)
    DØ 21 K = LIM, 160
    G(1, K) = 0
 21 W(1, K) = 0
    WRITE (108, 14)
  4 CØNTINUE
    STØP
    END
```

REFERENCES

1. M. Van Dyke, *Perturbation methods in fluid mechanics*. Acad. Press, New-York and London, 1964
2. Teodor Oroveanu, *Mecanica fluidelor vîscoase*. Ed. Academiei R.S.R., București, 1967
3. Petre Brădeanu *Mecanica fluidelor*, Ed. Tehnică, București, 1973
4. F. B. Hildebrand, *Finite-difference equations and simulations*. Prentice-Hall, Inc. Englewood Cliffs, New Jersey, 1968.
5. Doina Brădeanu, *O metodă a micilor perturbații pentru ecuația stratului limită*, St. cerc. mec. apl., 1974 **33**, *2*.
6. Doina Brădeanu, *Rezolvarea numerică a ecuației lui Mises pentru stratul limită incompresibil. Aplicație la cilindrul circular*. Studia Univ. Babeș-Bolyai, Series Math.-Mechanica 1974, *2*.
7. Nabi Faiza, *Operation instr. and descrip. computer program finite diff. equ...*, Office of Aerospace Research, United States Air Force (ARL), 67—0093, May, 1967.