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CONTINUOUS AND LOCALLY LIPSCHITZ CONVEX FUNCTIONS

by

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1. Introduction

A well known classical theorem asserts that every convex function defined on an open interval of the real axis satisfies the Lipschitz condition on every compact subinterval of its domain of definition ([4], Ch. III § 18). A. I. PEROV [7] and recently, A. W. ROBERTS and D. E. VARBERG [8] have extended this theorem to the case of convex functions defined on convex and open subsets of a finite dimensional Euclidean space and of an arbitrary normed linear space, respectively. Similar results appear in [5] p. 29, [11] and [12].

In this paper we consider convex functions defined on convex open subsets of a topological vector space and investigate continuity properties of these functions. Another question we consider is when such a function is locally Lipschitz. We say that a topological vector space X has the convex-continuity property if every convex function defined on a convex open subset of X is continuous. In §2 it is shown that this property is characteristic for the finite dimensional normed spaces (Proposition 2.3) but not for the finite dimensional locally convex Hausdorff spaces (Proposition 2.4). The extension theorem 3.1 in §3 is used in §4. Theorem 4.1 asserts that the convex functions defined on a convex open subset of a locally convex space are locally Lipschitz. A similar result, with a different definition of the local Lipschitz condition, was obtained A. J. BRANDÃO LOPES PINTO [2]. The main result of this paper is contained in Theorem 4.2, which extends the classical result mentioned at the begining: a continuous convex function defined on a convex open subset of a locally convex Hausdorff space satisfies the Lipschitz condition on every compact subset of its domain of definition. In §5 it is shown that the continuous convex functions, defined on convex and open subsets of some metrizable topological vector spaces, are locally Lipschitz.

All the spaces will be considered over the field of real numbers and all the functions will be supposed real-valued.

2. Continuity of convex functions

Let C be a non-void convex subset of a vector space X. A function $f: C \to R$ is called *convex* if $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$ for all $x, y \in C$ and $0 < \alpha < 1$. It is obvious that a function $f: C \to R$ is convex if and only if its *epigraph*, epi $f = \{(x, a) \in C \times R : f(x) \leq a\}$, is a convex subset of the Cartesian product $X \times R$.

If C is an open subset of a topological space then a function $f: C \to R$ is called *locally bounded from above (from below)* at $x_0 \in C$, if there exist a neighborhood $V \subset C$ of x_0 and a real number a such that $f(x) \leqslant a$ (respectively, $f(x) \geq a$) for all $x \in V$. The function f is called *locally bounded* on C if it is locally bounded from above and from below at every point $x \in C$.

The following theorem appears in [1], II. § 2.10 and [6] §3.2.3. in a slightly modified form:

2.1. THEOREM. *If C is a non-void convex open subset of a topological vector space X and $f: C \to R$ is a convex function, then the following conditions are equivalent:*

(i) *f is locally bounded from above at some point in C;*
(ii) *f is locally bounded on C;*
(iii) *f is continuous at some point in C;*
(iv) *f is continuous on C;*
(v) int (epi f) $\neq \emptyset$ *in the topological product $X \times R$. Furthermore, if one of these conditions holds, then* int (epi f) $= \{(x, a) \in C \times R : f(x) < a\}$.

The conditions in 2.1 are always fulfilled when X is a finite dimensional Hausdorff topological vector space ([5], p. 28):

2.2. Proposition *If f is a convex function defined on a non-void open convex subset C of a finite dimensional Hausdorff topological vector space, then f is continuous on C.*

The following example shows that the separation assumption in 2.2 is essential. Consider the locally convex non-separate space R^2 with the topology generated by the semi-norm $p(x, y) = |x|$, $(x, y) \in R^2$. The function $f(x, y) = y$ is linear but it is not locally bounded from above at $(0, 0)$

The property in 2.2 is characteristic for the finite dimensional normed spaces but not for the finite dimensional locally convex Hausdorff spaces, as it is shown by the following two propositions.

2.3. Proposition *If every linear functional defined on a normed linear space X is continuous, then X is finite dimensional.*

Proof. Suppose X is an infinite dimensional normed space. Let $E = \{e_n : n \in N\}$ be a linearly independent countable subset of X such that $\|e_n\| = 1$, for $n \in N$, and let $B \supset E$ be an algebraic basis of X. The linear functional f on X, defined by $f(e_n) = n$, $n \in N$, and $f(x) = 0$ for $x \in B \setminus E$, is not locally bounded from above at 0, since $f(r \cdot e_n) = rn \to \infty$, and $r \cdot e_n \in \{x \in X : \|x\| \leq r\}$, $n \in N$, for all $r > 0$.

2.4 Proposition *There exists an infinite dimensional locally convex complete Hausdorff space X such that every convex function defined on an arbitrary non-void convex open subset of X is continuous.*

Proof. Let X be an infinite dimensional vector space. Equipe the Cartesian product $Y = X \times R$ with the finest locally convex topology. Then the absolutely convex and absorbing subsets of Y form a basis of neighborhoods of the origin for this topology. The topology induced on $X = X \times \{0\} \subset Y$ will be also the finest locally convex topology on X. (Concerning, vector spaces with the finest locally convex topologies, see [10], p. 56 and Exercise 7 on page 69). It is easy to see that, every convex and absorbing subset U of Y is a neighborhood of the origin with respect to this topology. Indeed, let $\{z_i : i \in I\}$ be an algebraic basis of Y such that $[-z_i, z_i] = \{\alpha z_i + (1-\alpha)(-z_i) : \alpha \in [0, 1\} \subset U$. Then the set $V = \text{conv}(\{z_i : i \in I\} \cup \{-z_i : i \in I\})$ is absolutely convex and absorbing. It follows that V is a neighborhood of the origin and, by the convexity of U, we get $V \subset U$, so that U is also a neighborhood of the origin.

Let now C be a non-void convex open subset of X and let $f: C \to R$ be a convex function. Without loosing the generality (taking if necessary, $C - x_0$ instead of C, for a fixed $x_0 \in C$, and $f_0(x) = f(x + x_0) - f(x_0) - 1$, $x \in C - x_0$, instead of f), we can suppose that $0 \in C$ and $f(0) < 0$. The epigraph of f is a convex subset of $Y = X \times R$. We shall show that epi f is also an absorbing set in Y. Let $(x, a) \in Y$. The set $T = \{t \in R : tx \in C\}$ is an open interval of the real axis, and the function $g: T \to R$ defined by $g(t) = f(tx)$ is convex. By 2.2, g is continuous and by 2.1, $(0, 0)$ is an interior point of epi g, since $g(0) < 0$. It follows that epi g is an absorbing subset of R^2. Therefore, there exists $\lambda > 0$, such that $\lambda(1, a) \in$ epi g, and successively we have:

$$(\lambda, \lambda a) \in \text{epi } g \Leftrightarrow g(\lambda) \leq \lambda a \Leftrightarrow f(\lambda x) \leq \lambda a \Leftrightarrow$$

$$\Leftrightarrow (\lambda x, \lambda a) \in \text{epi } f \Leftrightarrow \lambda(x, a) \in \text{epi } f.$$

The last relation shows that epi f is an absorbing set in Y. The set epi f, being convex and absorbing, will be a neighborhood of the origin in Y. Therefore, int (epif) $\neq \emptyset$ and, by 2.1, the function f is continnous on C.

3. Extensions of continuous convex functions

Let X be a locally convex space and $\bar{\mathfrak{P}}$ a directed family of semi-norms generating the topology of X. If Y is a vector subspace of X, then the family $\mathfrak{P}$ of all restrictions of the semi-norms in $\bar{\mathfrak{P}}$ to Y, generates the induced topology for Y. Denote by p the restriction of a semi-norm $\bar{p} \in \bar{\mathfrak{P}}$ to Y. If Y is dense in X, then every $p \in \mathfrak{P}$ is uniformly continuous and hence has a unique extension $\bar{p} \in \bar{\mathfrak{P}}$. Furthermore, we have:

$$\overline{\{y \in Y : p(y - y_0) \leq r\}} = \{x \in X : \bar{p}(x - y_0) \leqslant r\}, \tag{1}$$

where $y_0 \in Y$, $r > 0$ and the closure is taken in X. If C is an open subset of Y (in the relative topology) then $C \subset \operatorname{int} \bar{C}$ (the interior and the closure being taken in X). Indeed, if $y_0 \in C$, there exist $p \in \mathfrak{P}$ and $r > 0$ such that $\{y \in Y : p(y - y_0) \leq r\} \subset C$, so that, by (1), $\{x \in X : \bar{p}(x - y_0) \leq r\} = \overline{\{y \in Y : p(y - y_0) \leq r\}} \subset \bar{C}$, which shows that y_0 is an interior point of $\bar{C}$.

We are now prepared to state the main result of this section:

3.1. THEOREM *Let X be a locally convex Hausdorff space, Y be a dense vector subspace of X, and C be a non-void convex open subset of Y. Then every continuous convex function $f : C \to R$ has a unique continuous convex extension $F : \operatorname{int} \bar{C} \to R$.*

Proof. Put

$$E = \overline{\operatorname{epi} f}, \quad E_x = \{a \in R : (x, a) \in E\},$$

and define $F : \operatorname{int} \bar{C} \to R$ by

$$F(x) = \inf E_x = \inf \{a \in R : (x, a) \in E\}. \tag{2}$$

The function F is well defined, convex, continuous and $F|_C = f$, as follows from the following lemma:

3.2. Lemma *a)* $(x, a) \in E$ *implies* $(x, b) \in E$ *for all* $b > a$;

b) $x \in \operatorname{int} \bar{C}$ *implies* $E_x \neq \emptyset$ *and* $\inf E_x > -\infty$;

c) F *is convex and continuous*;

d) $F|_C = f$.

Proof of 3.2 *a)* If $(x, a) \in E = \overline{\operatorname{epi} f}$, then there exists a net $\{(x_i, a_i)\}$ in epi f converging to (x, a). But $(x_i, a_i + \varepsilon)$ is also in epi f for $\varepsilon > 0$, and the net $\{(x_i, a_i + \varepsilon)\}$ converges to $(x, a + \varepsilon)$, which proves *a)*.

b) Firstly, we prove that $E_x \neq \emptyset$ for $x \in \operatorname{int} \bar{C}$. Let x' be an arbitrary, but fixed, point in C. Since $x \in \operatorname{int} \bar{C}$, there is $\alpha \in]0, 1[$ such that $x'' = \alpha^{-1}(x - (1 - \alpha)x') \in \operatorname{int} \bar{C}$. By the local boundedness of f at x'

(Theorem 2.1), it follows that there exist $p \in \mathfrak{P}$, $r > 0$ and $a \in R$ such that $V = \{y \in Y : p(y - x') \leq r\} \subset C$, and

$$f(y) \leq a, \text{ for all } y \in V. \tag{3}$$

As $U = \{y \in X : \bar{p}(y - x'') < (2\alpha)^{-1}(1 - \alpha)r\}$ is a neighborhood of $x'' \in \bar{C}$, the set $U \cap C$ is non-void. Let $y_0 \in U \cap C$, i.e. $p(y_0 - x'') < (2\alpha)^{-1}(1 - \alpha)r$. From $x \in \operatorname{int} \bar{C} \subset \bar{C}$, it follows that there exists a net $\{x_i : i \in I\}$ in C, which converges to x. Choose $i_0 \in I$ such that $\bar{p}(x_i - x) < 2^{-1}(1 - \alpha)r$ for $i \geq i_0$. Then $z_i = (1 - \alpha)^{-1}(x_i - \alpha y_0) \in V$, for $i \geq i_0$, since $p(z_i - x') = \bar{p}((1 - \alpha)^{-1}(x_i - y_0) + (1 - \alpha)^{-1}x'' - (1 - \alpha)^{-1}x'' - x') = \bar{p}((1 - \alpha)^{-1}x_i - (1 - \alpha)^{-1}(y_0 - x'') - (1 - \alpha)^{-1}x) \leq (1 - \alpha)^{-1}\bar{p}(x_i - x) + \alpha(1 - \alpha)^{-1} \bar{p}(y_0 - x'') < r/2 + r/2 = r$, (by the definition of x'' we have $x = \alpha x' + (1 - \alpha)x''$).

Put $b = \max(0, \alpha f(y_0) + (1 - \alpha)a)$ and $c_i = f(x_i) + |f(x_i)|$ Then, by (3) and the convexity of f we have:

$f(x_i) = f(\alpha y_0 + (1 - \alpha)z_i) \leq \alpha f(y_0) + (1 - \alpha)f(z_i) \leq \alpha f(y_0) + (1 - \alpha)a \leq b$, for all $i \geq i_0$. Therefore, $0 \leq c_i \leq 2b$, and the net $\{c_i\}$ contains a subnet $\{c_{i_j}\}$ converging to a number c. But $f(x_{i_j}) \leq c_{i_j}$, i.e. $(x_{i_j}, c_{i_j}) \in \operatorname{epi} f$, so that $(c, x) \in \overline{\operatorname{epi} f} = E$, which shows that $c \in E_x$.

Let prove now that $\inf E_x > -\infty$. Suppose, on the contrary, that $\inf E_x = -\infty$ for a point $x \in \operatorname{int} \bar{C}$. Let y_0 be an arbitrary point in C, $p \in \mathfrak{P}$, $r > 0$, $V = \{y \in Y : p(y - y_0) \leq r\} \subset C$ — a neighborhood of y_0, and b be a real number. Put

$$U = \{y \in X : \bar{p}(y - x) \leq r\}, \quad W = Y \cap U, \quad \alpha = r(r + \bar{p}(x - y_0))^{-1}$$

and $c = (b - (1 - \alpha)f(y_0))\alpha^{-1}$. Then, there exists $x' \in W$ such that

$$f(x') < c. \tag{4}$$

Indeed, from $\inf E_x = -\infty$, it follows that there exists a real number a such that $(x, a) \in E$ and $a < c$. But $U \times]-\infty, c[$ is a neighborhood of (x, a) in $X \times R$, and since $E = \overline{\operatorname{epi} f}$, there exists $(x', a') \in \operatorname{epi} f$ such that $x' \in U \cap Y = W$ and $a' < c$, so that $f(x') \leq a' < c$.

Observe now, that $x'' = (1 - \alpha)y_0 + \alpha x'$ belongs to V since

$$p(x'' - y_0) = \alpha \bar{p}(x' - y_0) \leq \alpha(\bar{p}(x' - x) + \bar{p}(x - y_0)) \leq$$
$$\leq \alpha(r + p(x - y_0)) = r.$$

By (4), $f(x'') \leq (1 - \alpha)f(y_0) + \alpha f(x') < (1 - \alpha)f(y_0) + \alpha c = b$, which shows that f is not bounded from below on the neighborhood V of y_0, and by 2.1, this contradicts the continuity of f on C.

c) Put $G = \{(x, a) : (x, a) \in E$ and $x \in \operatorname{int} \bar{C}\}$. Since the sets E and $\operatorname{int} \bar{C}$ are convex, the set G is also convex. We shall show that epi $F = G$, which will imply the convexity of F. If $(x, a) \in G$, then

$x \in \operatorname{int} \bar{C}$ and $(x, a) \in E$. By (2), $F(x) \le a$, or equivalently $(x, a) \in \operatorname{epi} F$ and $G \subset \operatorname{epi} F$. Conversely, if $x \in \operatorname{int} \bar{C}$ then, by (2), there exists a sequence $\{a_n\}$ in R, such that $(x, a_n) \in E$ and $a_n \to F(x)$. The set $E = \overline{\operatorname{epi} f}$, being closed, from $(x, F(x)) = \lim (x, a_n)$ it follows that $(x, F(x)) \in E$. By a), $(x, a) \in E$ for all $a \ge F(x)$. Since $x \in \operatorname{int} \bar{C}$, we have $(x, a) \in G$ which shows that $\operatorname{epi} F \subset G$. Therefore, $\operatorname{epi} F = G$.

We have to prove the continuity of F on $\operatorname{int} \bar{C}$. By 2.1 it is sufficient to prove that F is locally bounded from above at some point in $\operatorname{int} \bar{C}$. Choose $x_0 \in C \subset \operatorname{int} \bar{C}$. By the continuity of f and 2.1, there exist $p \in \mathfrak{P}$, $r > 0$ and $a \in R$ such that $f(y) \le a$ (or, equivalently $(y, a) \in \operatorname{epi} f$) for all $y \in V = \{y \in Y : p(y - x_0) \le r\} \subset C$. Put $W = \{x \in X : \bar{p}(x - x_0) \le r\}$ and observe that $U = \operatorname{int} W = \{x \in X : \bar{p}(x - x_0) < r\}$ is a neighborhood of x_0 in X and $U = \operatorname{int} \bar{V} \subset \operatorname{int} \bar{C}$. If x is an arbitrary point in $U \subset \bar{U} = \bar{V}$, there exists a net $\{x_i\}$ in V which converges to x. Then $(x_i, a) \to (x, a) \in \overline{\operatorname{epi} f} = E$. By (2), $F(x) \le a$, which shows that F is locally bounded from above at x_0.

d) Let $x \in C$. Then $x \in \operatorname{int} \bar{C}$ and $(x, f(x)) \in \operatorname{epi} f \subset E$ so that, by (2), $F(x) \leqslant f(x)$. If $F(x) < f(x)$, then it would exist a number $a \in R$ such that $(x, a) \in E$ and $a < f(x)$. If $\{(x_j, a_j)\}$ is a net in $\operatorname{epi} f$ which converges to (x, a), then the continuity of f on C, implies $f(x) \le a$, which contradicts the relation $a < f(x)$. We have proved that $F(x) = f(x)$ for $x \in C$, i.e. $F|_C = f$ which ends the proof of the lemma 3.2.

4. Locally Lipschitz convex functions

Let X be a locally convex space and $\mathfrak{P}$ a directed family of semi-norms generating the topology of X. Let f be a real-valued function defined on a non-void subset C of X. The function f is called *Lipschitz on* $M \subset C$ if there exist $p \in \mathfrak{P}$ and $L > 0$ such that

(5) $$|f(x) - f(y)| \le Lp(x - y) \text{ for all } x, y \in M.$$

It is easily seen that this property does not depend on the directed family of semi-norms which generate the topology of X. If C is an open subset of X the function $f : C \to R$ is called *locally Lipschitz* on C if for every $x \in C$ there exists a neighborhood $V \subset C$, such that f is Lipschitz on V.

Evidently the locally Lipschitz functions are continuous. The following theorem shows that a continuous convex function is locally Lipschitz.

4.1. THEOREM *Every convex continuous function f defined on a non-void convex open subset C of a locally convex space X is locally Lipschitz on C.*

Proof. Let $\mathfrak{P}$ be a directed family of semi-norms generating the topology of X. If $x_0 \in C$, then by 2.1, there exist a neighborhood $U \subset C$ of x_0 and a number $a > 0$ such that $|f(x)| \le a$ for all $x \in U$. Let $U = \{x \in X : p(x - x_0) \le r\}$ for $p \in \mathfrak{P}$ and $r > 0$. Then the set $V = \{x \in X : p(x - x_0) \le r/2\}$ is a neighborhood of x_0, contained in C. Put $L = 4a/r$ and let $x, y \in V$. For $\varepsilon > 0$ put $\alpha = \varepsilon + p(x - y) > 0$ and $z = y + r(2\alpha)^{-1}(y - x) \in U$. Then $y = 2\alpha(r + 2\alpha)^{-1}z + r(r + 2\alpha)^{-1}x$ and, by the convexity of f, $f(y) \le 2\alpha(r + 2\alpha)^{-1}f(z) + r(r + 2\alpha)^{-1}f(x)$ so that

$$f(y) - f(x) \le 2\alpha(r + 2\alpha)^{-1}(f(z) - f(x)) \le 4a\alpha(r + 2\alpha)^{-1} < L\varepsilon + Lp(x - y).$$

Since $\varepsilon > 0$ is arbitrary choosen, it follows that $f(y) - f(x) \le Lp(x - y)$ and changing the roles of x and y, we obtain $|f(x) - f(y)| \le Lp(x - y)$, which shows that f is Lipschitz on V. Theorem 4.1 is proved.

From 4.1 follows the well known characterization of the continuity of linear applications between two locally convex spaces:

4.2 Proposition. *Let $(X, \mathfrak{P})$ and $(Y, \mathfrak{Q})$ be two locally convex spaces whose topologies are generated by the directed families of semi-norms $\mathfrak{P}$ and $\mathfrak{Q}$, respectively. A linear application $A : X \to Y$ is continuous if and only if for every $q \in \mathfrak{Q}$ there exist $p \in \mathfrak{P}$ and $L > 0$ such that $q(Ax) \le Lp(x)$ for all $x \in X$.*

Proof. Take $C = X$ and $f(x) = q(Ax)$ in **4.1.**

Now we state the main result of this paper:

4.3 THEOREM *If f is a continuous convex function defined on a non-void convex open subset C of a locally convex Hunsdorff space X, then f is Lipschitz on every compact subset of C.*

Proof. Firstly suppose that the space X is complete, and let K be a compact set contained in C. Then the closed convex hull $M = \overline{\operatorname{conv}}\ (K)$ of K is compact ([10], Ch. II. §4). Furthermore, we have $M \subset C$. This is evidently true if $C = X$. If $C \neq X$, then the boundary $\operatorname{fr} C$ of C is non-void. Fix an element $x_0 \in K$ and define the function $\gamma : \operatorname{fr} C \to R$ by

$$\gamma(x) = \sup \{\alpha \in [0, 1] : (1 - \alpha)x_0 + \alpha x \in K\}, \qquad x \in \operatorname{fr} C.$$

Then $0 \le \gamma(x) < 1$ for all $x \in \operatorname{fr} C$, since $K \cap \operatorname{fr} C = \emptyset$. We have further

$$\Gamma = \sup \{\gamma(x) : x \in \operatorname{fr} C\} < 1.$$

Indeed, suppose $\Gamma = 1$ and choose two sequences, $\{x_n\}$ in $\operatorname{fr} C$ and $\{\alpha_n\}$ in $[0, 1]$, such that $\alpha_n \to 1$ and $z_n = (1 - \alpha_n)x_0 + \alpha_n x_n \in K$. By the compactness of K there exists a subnet $\{z_{n_i}\}$ of $\{z_n\}$ which converges to an element $z_0 \in K$. But, then the net $x_{n_i} = (\alpha_{n_i})^{-1}(z_{n_i} - (1 - \alpha_{n_i})x_0 \in \operatorname{fr} C$, converges also to z_0. Since $\operatorname{fr} C$ is closed, we have $z_0 \in \operatorname{fr} C$ and $z_0 \in K \cap \operatorname{fr} C$ in contradiction to $K \cap \operatorname{fr} C = \emptyset$. Consequently $\Gamma < 1$.

Choose now α such that $\Gamma < \alpha < 1$. Then $D = (1 - \alpha)x_0 + \alpha\bar{C}$ is a closed convex set contained in C (see [10], Ch. II, § 1, 1.1). We have also $K \subset D$. Indeed, let $x \in K$ and let $\overrightarrow{x_0x} = \{x_0 + \beta(x - x_0) : \beta \ge 0\}$. If $\overrightarrow{x_0x} \subset C$ then $u = x_0 + \alpha^{-1}(x - x_0) \in \overrightarrow{x_0x} \subset C$, and $x = (1 - \alpha)x_0 + \alpha u \in$

$\in D$. If $\overrightarrow{x_0x} \not\subset C$, let z be the only point of the intersection $\overrightarrow{x_0x} \cap \mathrm{fr}\, C$. Then $x = (1-\beta)\, x_0 + \beta z$ for a $\beta > 0$. But then $\beta \leq \gamma(z) \leq \Gamma < \alpha < 1$, and $u = \alpha^{-1}[(\alpha - \beta)\, x_0 + \beta z] \in \overline{C}$, so that $x = (1-\alpha)x_0 + \alpha u \in D$. Therefore $K \subset D$, and the set D being closed and convex it follows that $M = \overline{\mathrm{conv}(K)} \subset D \subset C$. By 4.1, for every $x \in M$ there exist an open convex neighborhood V_x of x, such that $\overline{V}_x \subset C$, a semi-norm $p_x \in \mathfrak{P}$ and a number $L_x > 0$ such that

$$|f(y) - f(z)| \leq L_x p_x(p - z), \text{ for all } y,z \in V_x. \tag{6}$$

Let $\{V_{x_1}, \ldots, V_{x_n}\}$ be a finite subcovering of the open covering $\{V_x : x \in M\}$ of M. Denote $V_i = V_{x_i}$, $p_i = p_{x_i}$, $L_i = L_{x_i}$, $i = 1, 2, \ldots, n$, $L = \max\{L_1, \ldots, L_n\}$ and choose $p \in \mathfrak{P}$ such that $p_i \leq p$, $i = 1, 2, \ldots, n$. We shall show that

$$|f(x) - f(y)| \leq Lp(x - y) \text{ for all } x, y \in M. \tag{7}$$

Let $x, y \in M$ and let $\overrightarrow{xy} = \{x + \lambda(y - x) : \lambda \geq 0\}$. We shall construct a finite sequence $V_{i_0}, \ldots, V_{i_k}$ of sets belonging to the family $\{V_1, \ldots, V_n\}$ and a finite sequence $x = z_0, z_1, \ldots, z_k = y$ of elements in M such that $z_j \in \overrightarrow{xy}$, $j = 0, \ldots, k$ and $z_j, z_{j+1} \in V_{i_j}$, $j = 0, \ldots, k-1$. To this end observe that, for $x \in M \subset \bigcup_{i=1}^{n} V_i$, there exists $i_0 \in \{i_1, \ldots, i_n\}$ such that $x \in V_{i_0}$. If $y \in \overline{V}_{i_0}$ put $z_1 = y$, and stop. If $y \notin \overline{V}_{i_0}$ let z_1 be the only point in the interesection $\overrightarrow{xy} \cap \mathrm{fr}\, V_{i_0}$ and choose $i_1 \in \{1, \ldots, n\} \setminus \{i_0\}$ such that $z_1 \in V_{i_1}$. If $y \in \overline{V}_{i_1}$ put $z_2 = y$ and stop the construction. If $y \notin \overline{V}_{i_1}$ we continue as above, and after a finite number of steps we obtain the desired sequences $\{z_j\}$ and $\{V_{i_j}\}$.

Then $z_j = x + \lambda_j(y - x)$, $j = 0, \ldots, k$ with $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_k = 1$, so that

$$y - x = \sum_{j=0}^{k-1} (z_{j+1} - z_j) \tag{8}$$

and

$$p(y - x) = \sum_{j=0}^{k-1} p(z_{j+1} - z_j). \tag{9}$$

As $z_j, z_{j+1} \in \overline{V}_{i_j}$, for $j = 0, 1, \ldots, k-1$, by (6) and (9) we get

$$|f(x) - f(y)| \leq \sum_{j=0}^{k-1} |f(z_{j+1}) - f(z_j)| \leq \sum_{j=0}^{k-1} L_{i_j} p_{i_j}(z_{j+1} - z_j)$$
$$\leq L \sum_{j=0}^{k-1} p(z_{j+1} - z_j) = Lp(y - x).$$

In particular, the inequality (7) holds for all $x, y \in K \subset M$, and the theorem is proved in the case when X is complete.

If X is not complete let $\tilde{X}$ be a completion of X and identify X to a dense subspace of $\tilde{X}$. The topology of $\tilde{X}$ is generated by the family $\overline{\mathfrak{P}}$ of semi-norms $\overline{p}$ which are uniformly continuous extensions of the semi-norms $p \in \mathfrak{P}$. By 3.1, f has a unique continuous convex extension $F : \mathrm{int}\, \overline{C} \to R$. By the first part of the proof, F is Lipschitz on the compact set $K \subset C \subset \mathrm{int}\, \overline{C}$, i.e there exist $\overline{p} \in \overline{\mathfrak{P}}$ and $L > 0$ such that $|F(x) - F(y)| \leq L\overline{p}(x - y)$ for all $x,y \in K$. Since $x,y \in K \subset C \subset X$, this is just the relation $|f(x) - f(y)| \leq Lp(x) - y)$ for all $x, y \in K$. Theorem 4.1 is completely proved.

5. Locally Lipschitz convex functions on metrizable topological vector spaces

Let X be a topological vector space whose topology is given by a metric d. A function f defined on an open subset C of X is called *locally Lipschitz* (with respect to d) on C if for every $x \in C$ there exist a neighborhood $V \subset C$ of x and a number $L > 0$ such that

$$|f(y) - f(z)| \leq Ld(y, z), \text{ for all } y, z \in V. \tag{10}$$

We give some results analogous to theorem 4.1 in the case of some concrete metrizable topological vector spaces. For $0 < p < 1$ denote by l^p the space of all sequences $x = (x_1, x_2, \ldots)$ of real numbers such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$. Then $d(x, y) = \sum_{n=1}^{\infty} |x_n - y_n|^p$ is a metric on l^p and (l^p, d) is a metric topological vector space.

5.1. Proposition. *Every continuous convex function f defined on a convex open subset C of l^p, $0 < p < 1$, is locally Lipschitz on C with respect to the metric d.*

Proof. Let $x_0 \in C$. By 2.1 there exist a neighborhood $U \subset C$ of x_0 and a number $a > 0$ such that $|f(x)| \leq a$ for all $x \in U$. We can suppose that $U = \{x \in l^p : d(x_0, x) \leq r\}$ for $r > 0$. The set $V = \{x \in l^p : d(x_0, x) \leqslant r/4\} \subset C$, is also a neighborhood of x_0. Denoting $L = 4a/r$ we have

$$d\left(\frac{r}{2d(x,y)}(y - x), 0\right) = \left[\frac{r}{2d(x,y)}\right]^p d(y - x, 0) =$$
$$= \left[\frac{r}{2d(x,y)}\right]^p d(x, y) = \left(\frac{r}{2}\right)^p (d(x, y))^{1-p} \leq r/2$$

for all $x, y \in V$, $x \neq y$. Consequently, the element $z = y + r(2d(x, y))^{-1}(y - x)$ belongs to U since $d(z - x_0, 0) \leq d(y - x_0, 0) + d(r(2d(x, y))^{-1}$

$(y - x), 0) \leq r/4 + r/2 < r$. Reasoning now as at the end of the proof of 4.1 we obtain the inequality (10).

Remark. Proposition 5.1 remains valid, with the same proof, in p-homogeneous metric topological vector spaces, i.e. such that $d(\alpha x, 0) = |\alpha|^p d(x, 0)$ (see [9]). Some extensions of 5.1 are given in [3].

Let now X be a metrizable locally convex space and let $\mathfrak{P} = \{p_n : n \in N\}$ be a countable directed family of semi-norms which generate the topology of X. It is well known (see [10], Ch, I, §6) that the topology of X is generated by the metric

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x-y)}{1+p_n(x-y)}, \quad x, y \in X. \tag{11}$$

We have the following result:

5.2 Proposition. *Let X be a metrizable locally convex space. If f is a continuous convex function defined on a non-void convex open subset C of X then f is locally Lipschitz on C with respect to the metric* (11).

Proof. Let $x_0 \in C$. By 4.1 there exist a neighborhood $U \subset C$ of x_0, a semi-norm $p_{n_0} \in \mathfrak{P}$ and a number $L_0 > 0$ such that

$$|f(x) - f(y)| \leq L_0 p_{n_0}(x - y) \text{ for all } x,y \in U. \tag{12}$$

Let $r > 0$ be such that the neighborhood $W = \{x \in X : d(x_0, x) \leq r\}$ is contained in $\{x \in X : p_{n_0}(x - x_0) \leq 1\} \cap U$. Put $L = 3.2^{n_0}.L_0$. If $x,y \in W$ then by (12) we obtain $|f(x) - f(y)| \leq L_0 p_{n_0}(x - y) = 2^{n_0} L_0 (1 + p_{n_0}(x - y)) \cdot \frac{1}{2^{n_0}} \frac{p_{n_0}(x-y)}{1 + p_{n_0}(x-y)} \leq 3 \cdot 2^{n_0} \cdot L_0 \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x-y)}{1+p_n(x-y)} = Ld(x, y)$, which shows that f is locally Lipschitz on C with respect to the metric (1).

Remark. The metrics in 5.1 and 5.2 are translation invariant, i.e. $d(x - y, 0) = d(x, y)$ for all $x, y \in X$. The following example shows that this hypothesis is essential. Consider on the real axis R the metric $d(x, y) = |x^3 - y^3|$ which generates the usual topology on R. The function $f(x) = x$ for $x \in R$ is continuous and linear, but it is not locally Lipschitz on R, as follows from the following relations

$$|f(x) - f(y)| = |x - y| = \frac{1}{|x^2 + xy + y^2|} |x^3 - y^3| = |x^2 + xy + y^2|^{-1} \cdot d(x, y)$$

if we let x and y to tend to 0.

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