MATHEMCAITCA – REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION

MATHEMATICA, Tome 19 (42), N° 2, 1977, pp. 183–187

NORM PRESERVING EXTENSION OF STARSHAPED LIPSCHITZ FUNCTIONS

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1. Let X be a normed linear space over the field of real numbers and Y a nonvoid subset of X. The set Y is called *starshaped* (with respect to 0) if $\alpha y \in Y$ for all $\alpha \in [0, 1]$ and $y \in Y$. If Y is a starshaped set, then a function $f: Y \to \mathbf{R}$ is called *starshaped on* Y if

$$f(\alpha y) \leqslant \alpha f(y), \tag{1}$$

for all $\alpha \in [0, 1]$ and $y \in Y$.

If (X, d) is a metric space and x_0 a fixed element in X, we denote by $\mathrm{Lip}_0\, X$ the space of all real-valued Lipschitz functions on X which vanish at x_0. Endowed with the norm

$$\|f\|_X = \sup \{|f(x) - f(y)|/d(x, y) : x, y \in X,\ x \neq y\}, \tag{2}$$

for $f \in \mathrm{Lip}_0\, X$, this space is a Banach space (even a conjugate Banach space, see [3]).

If Y is a subset of X such that $x_0 \in Y$, then for every $f \in \mathrm{Lip}_0\, Y$ there exists $F \in \mathrm{Lip}_0\, X$ such that $F|_Y = f$ and $\|F\|_X = \|f\|_Y$ ([4]). We call the function F, an extension of f.

2. Suppose now $(X, \|\ \ \|)$ to be a normed linear space and Y a nonvoid starshaped subset of X. If $f \in \mathrm{Lip}_0\, Y$ is starshaped, then one can ask the following question: *has f an extension $F \in \mathrm{Lip}_0\, X$ which is starshaped on X?* (Here the fixed element at which all the functions vanish is 0).

The answer to this question is affirmative as follows from the following theorem:

THEOREM 1. *Let Y be a non-void starshaped subset of a normed linear space X and f a starshaped function in $\mathrm{Lip}_0\, Y$. Then there exists a starshaped function F in $\mathrm{Lip}_0\, X$ such that:*

a) $F|_Y = f$

and

b) $\|F\|_X = \|f\|_Y$.

Proof. The function

(3) $F(x) = \inf \{[f(y) + \|f\|_Y \|x - y\|] : y \in Y\}, \ x \in X,$

is in $\mathrm{Lip}_0 X$ and verifies the conditions *a)* and *b)* ([4]). We shal show that F is also starshaped. Let $x \in X$ and $\alpha \in [0, 1]$. Then for $y \in Y$ we have

$$F(\alpha x) \leqslant f(\alpha y) + \|f\|_Y \cdot \|\alpha x - \alpha y\| \leqslant \alpha[f(y) + \|f\|_Y \cdot \|x - y\|],$$

and taking the infimum with respect to $y \in Y$ one obtains

$$F(\alpha x) \leqslant \alpha F(x),$$

which shows that F is starshaped on X.

For a starshaped functions f in $\mathrm{Lip}_0 Y$, denote

(4) $P_Y^{st}(f) = \{F : F \in \mathrm{Lip}_0 X$ *and* F *is a starshaped extension of* $f\}$.

Lemma 1. *The set* $P_Y^{st}(f)$ *is downward directed with respect to the pointwise ordering.*

Proof. For $F_1, F_2 \in P_Y^{st}(f)$ put

(5) $F_0(x) = \min (F_1(x), F_2(x)), \ x \in X.$

For every $\alpha \in [0, 1]$ we have

$$F_0(\alpha x) \leqslant \alpha \min (F_1(x), F_2(x)) = \alpha F_0(x),$$

which shows that the function F_0 is starshaped on X.

Since $F_0|_Y = F_1|_Y = F_2|_Y$, we have

$$\|F_0\|_X \geqslant \|f\|_Y.$$

To prove the converse inequality, let $x, y \in X$, $x \neq y$. Then

(i) $F_0(x) = F_k(x)$ and $F_0(y) = F\ (y)$ for $k \in \{1, 2\}$,

(ii) $F_0(x) = F_1(x)$ and $F_0(y) = F_2(y)$,

or

(iii) $F_0(x) = F_2(x)$ and $F_0(y) = F_1(y)$.

In the case (i)

$$\frac{|F_0(x) - F_0(y)|}{\|x - y\|} = \frac{|F_k(x) - F_k(y)|}{\|x - y\|} \leqslant \|F_k\|_X, \quad k \in \{1, 2\}.$$

In the case (ii), we have

$$\frac{|F_0(x) - F_0(y)|}{\|x - y\|} = \frac{F_0(x) - F_0(y)}{\|x - y\|} \leqslant \frac{F_2(x) - F_2(y)}{\|x - y\|} \leqslant \|F_2\|_X$$

or

$$\frac{|F_0(x) - F_0(y)|}{\|x - y\|} = \frac{F_0(y) - F_0(x)}{\|x - y\|} \leqslant \frac{F_1(y) - F_1(x)}{\|x - y\|} \leqslant \|F_1\|_X.$$

Case (iii) is similar to the case (ii).

In all the cases it follows

$$\|F_0\|_X \leqslant \|F_1\|_X = \|f\|_Y$$

or

$$\|F_0\|_X \leqslant \|F_2\|_X = \|f\|_Y,$$

so that

$$\|F_0\|_X = \|f\|_Y$$

which ends the proof of the lemma 1.

THEOREM 2. *For a starshaped function* f *in* $\mathrm{Lip}_0 Y$ *there exist two functions* F_1 *and* F_2 *in* $P_Y^{st}(f)$ *such that*

(6) $F_1(x) \leqslant F(x) \leqslant F_2(x),$

for all $x \in X$ *and* $F \in P_Y^{st}(f)$.

Proof. Let

(7) $F_1(x) = \inf \{F(x) : F \in P_Y^{st}(f)\}, \quad x \in X.$

We want to show that $F_1 \in P_Y^{st}(f)$. Then, for $\alpha \in [0, 1]$, $F \in P_Y^{st}(f)$ and $x \in X$, we have

$$F_1(\alpha x) \leqslant F(\alpha x) \leqslant \alpha F(x),$$

and taking infimum with respect to $x \in X$, one gets

$$F_1(\alpha x) \leqslant \alpha F_1(x),$$

which shows that the function F_1 is starshaped.

Obviously, $F_1|_Y = f$. Prove now that $\|F_1\|_Y = \|f\|_Y$. Evidently $\|F_1\|_X \geqslant \|f\|_Y$. Suppose that $\|F_1\|_X > \|f\|_Y$. Then, there exists $\delta > 0$ such that $\|F_1\|_X = \|f\|_Y + \delta$ and by definition of the norm in $\mathrm{Lip}_0 X$, there exist $x, y \in X$, $x \neq y$ such that

$$\frac{F_1(y) - F_1(x)}{\|x - y\|} \geqslant \|f\|_Y + \varepsilon,$$

where $0 < \varepsilon < \delta$. By the definition of F_1, for $0 < \eta < \varepsilon\|x - y\|$, there exist $G_1, G_2 \in P_Y^{st}(f)$ such that $G_1(x) < F_1(x) + \eta$ and $G_2(y) < F_1(y) + \eta$. The set $P_Y^{st}(f)$, being downward directed (Lemma 1) there exists $G_3 \in P_Y^{st}(f)$ such that $G_3 \leqslant G_1$ and $G_3 \leqslant G_2$. Consequently

$$F_1(x) \leqslant G_3(x) < F_1(x) + \eta,$$
$$F_1(y) \leqslant G_3(y) < F_1(y) + \eta,$$

or equivalently

$$0 \leqslant G_3(x) - F_1(x) < \eta,$$
$$0 \leqslant G_3(y) - F_1(y) < \eta.$$

From these inequalities it follows

$$G_3(y) - G_3(x) > F_1(y) - F_1(x) - \eta,$$

and

$$\frac{G_3(y) - G_3(x)}{\|y - x\|} > \frac{F_1(y) - F_1(x)}{\|y - x\|} - \frac{\eta}{\|y - x\|} \geqslant \|f\|_Y + \varepsilon - \frac{\eta}{\|y - x\|} > \|f\|_Y.$$

But then $\|G_3\|_X > \|f\|_Y$, contradicting the fact that G_3 is in $P_Y^{st}(f)$.

Let F_2 be the function defined by (3). By the proof of Theorem 1, F_2 is in $P_Y^{st}(f)$. It was proved in [5] that the function F_2 verifies

(8) $$F(x) \leqslant F_2(x), \qquad x \in X,$$

for every extension F of f. The inequality will be true, a fortiori, for every function F in $P_Y^{st}(f)$. Theorem 2 is proved.

3. Let X and Y be as in section 2, $\mathcal{S}_Y$ and $\mathcal{S}_X$ be the cones of starshaped functions in $\mathrm{Lip}_0\, Y$ and $\mathrm{Lip}_0\, X$, respectively. By Theorem 1, the cone $\mathcal{S}_Y$ is a *P-cone* (see Definition 2 in [6]).

If

(10) $$X_S = \mathcal{S}_X - \mathcal{S}_X,$$

is the space generated by the cone $\mathcal{S}_X$ and

(11) $$Y_S^{\perp} = \{f \in X_S,\ f|_Y = 0\},$$

is the null space of the starshaped set Y in X_S then, for $f \in \mathcal{S}_X$, the distance from f to $Y_S^{\perp}$ may be exprimed in the following way:

(12) $$d(f,\ Y_S^{\perp}) = \|f|_Y\|_Y,$$

and all the elements $g \in Y_S^{\perp}$ of the best approximation of f is exactly of the form $g = f - F$, where $F \in P_Y^{st}(f|_Y)$ (see Theorem 1 in [6]).

The set $P_Y^{st}(f)$, $f \in \mathcal{S}_Y$, being nonvoid, the extension of f is unique if and only if $Y_S^{\perp}$ is $\mathcal{S}_X$-*Chebyshevian* (see Theorem 2 (b) in [6]).

Let $f \in \mathcal{S}_X$ and $G(f)$ be the set of elements of best approximation of f by elements of $Y_S^{\perp}$, in the norm (2). Suppose $Y_S^{\perp}$ is not $\mathcal{S}_X$-Chebyshevian. Let Z be a bounded nonvoid subset of X and denote by $\mathrm{Lip}_0\, X|_Z$ the space of all restrictions of functions $f \in \mathrm{Lip}_0\, X$, to the set Z. Then, in $\mathrm{Lip}_0\, X|_Z$ may be defined the uniform norm $\|\ \|_u : \mathrm{Lip}_0\, X|_Z \to \mathbf{R}_+$:

(13) $$\|h\|_u = \sup_{z \in Z} |h(z)|, \quad h \in \mathrm{Lip}_0\, X|_Z.$$

Problem (A). For $f \in \mathcal{S}_X$, find two elements g_* and g^* in $G(f)$ such that

(14) $$\|f|_Z - g_*|_Z\|_u = \inf\ \{\|f|_Z - g|_Z\|_u : g \in G(f)\},$$

and

(15) $$\|f|_Z - g^*|_Z\|_u = \sup\ \{\|f|_Z - g|_Z\|_u : g \in G(f)\}.$$

THEOREM 3. *The problem* (A) *has a solution for all* $f \in \mathcal{S}_X$.

Proof. By Theorem 2 *b)*, for every $f \in \mathcal{S}_X$, there exist the extensions $F_1,\ F_2 \in P_Y^{st}(f|_Y)$ such that

$$F_1(x) \leqslant F(x) \leqslant F_2 x), \ x \in X,\ F \in P_Y^{st}(f|_Y),$$

i.e.

$$f(x) - g_1(x) \leqslant f(x) - g(x) \leqslant f(x) - g_2(x),$$

for $x \in X$, where $g_i = f - F_i$, $i = 1, 2$.

Consequently

$$\min\ (\|f|_Z - g_1|_Z\|_u,\quad \|f|_Z - g_2|_Z\|_u) \leqslant \|f|_Z - g|_Z\|_u \leqslant$$
$$\leqslant \max\ (\|f|_Z - g_1|_Z\|_u,\ \|f|_Z - g_2|_Z\|_u).$$

It follows that a solution of the Problem (A) is given by $g_* = g_i$ and $g^* = g_j$, where $i, j \in \{1, 2\}$ are such that

$$\|f|_Z - g_i|_Z\|_u = \min\ (\|f|_Z - g_1|_Z\|_u,\ \|f|_Z - g_2|_Z\|_u)$$

and

$$\|f|_Z - g_j|_Z\|_u = \max\ (\|f|_Z - g_1|_Z\|_u,\ \ \|f|_Z - g_2|_Z\|_u).$$

Remark. The Problem (A) is solved analogously in the case when the function f is in K_X — the cone of convex functions in $\mathrm{Lip}_0\, X$ — (see [1]).

If Y is a convex nonvoid subset of X, such that $0 \in Y$, then the cone K_Y of convex functions in $\mathrm{Lip}_0\, Y$, is a subcone of $\mathcal{S}_Y$. If $X_K = K_X - K_X$ is the space generated by the cone of convex functions in $\mathrm{Lip}_0\, X$ and $Y_C^{\perp}$ is the null space of convex sets Y in X_K, then, for $f \in K_X$ it follows that

$$d(f,\ Y_C^{\perp}) = d(f,\ Y_S^{\perp}).$$

(see, for exemple [5]).

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Received, 19. X. 1977