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C O N T E N T S

A NORMED SPACE ADMITTING COUNTABLE MULTI-VALUED METRIC PROJECTIONS

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Let X be a normed vector space and M an arbitrary subset of X. The <u>metric projection</u> on M is the mapping $P_M : X \longrightarrow 2^M$, defined by:

$$P_M(x) = \{m \in M : \|x - m\| = d(x,M)\} ,$$

where $d(x,M)$ is the distance from x to M. If card $P_M(x) \geq 2$, for all $x \in X \setminus M$, we say that the metric projection is <u>totally multi-valued</u> and in tho special case when card $P_M(x) = \aleph_0$, for all $x \in X \setminus M$, we say that the metric projection is <u>countably multi-valued</u>. The set M is called <u>proximinal</u> if $P_M(x) \neq \emptyset$ for all $x \in X \setminus M$. If P_M is a totally multi-valued metric projection, then the set M will be called <u>strongly proximinal</u>.

It is clear that every proximinal set is closed and every strongly proximinal set is proximinal, hence closed.

Some results about the existence of bounded or compact strongly proximinal sets in concrete Banach spaces was obtained by S. V. KONJAGIN [2]. He calls a strongly proximinal set, a set with the anti-uniqueness property.

In this note we shall give a normed space X containing a bounded strongly proximinal set M with P_M countable multi-valued. First, concerning strongly proximinal sets we have:

PROPOSITION. <u>If M is a strongly proximinal set of the Banach space X, then:</u>

$$\text{card } P_M(x) \geq c,$$

<u>for all</u> $x \in X \setminus M$.

<u>Proof</u>. We denote by $B(x,r)$ the closed ball of center x and radius r. Let M be a strongly proximinal set of the Banach space X. Then M is a closed set and $P_M(x) = M \cap B(x,d(x,M))$ is a closed set too, as an intersection of two closed sets. We will show that if $x \in X \setminus M$, then $P_M(x)$ does not contain isolated points.

We suppose, on the contrary, that $m_0 \in P_M(x)$ is an isolated point of $P_M(x)$, for a given $x \in X \setminus M$. Then there exists an $\varepsilon \in (0,1)$ such that $B(m_0, \varepsilon d(x,M)) \cap P_M(x) = \{m_0\}$.

Let $x_0 = (\varepsilon/3)\, x + (1 - \varepsilon/3)\, m_0$. We have :

$$\|x - x_0\| = \|x - (\varepsilon/3)\, x - (1 - (\varepsilon/3))\, m_0\| = (1 - (\varepsilon/3))\, \|x - m_0\| =$$
$$= (1 - (\varepsilon/3))\, d(x,M) < d(x,M).$$

It follows that $x_0 \in X \setminus M$. On the other hand

$$\|x_0 - m_0\| = \|(\varepsilon/3)\, x + (1 - (\varepsilon/3))\, m_0 - m_0\| = (\varepsilon/3)\, \|x - m_0\| =$$
$$= (\varepsilon/3)\, d(x,M).$$

From this it follows $d(x_0,M) \leq (\varepsilon/3)\, d(x,M)$. Let $m \in M$. If $m \notin P_M(x)$ we have :

$$\|x_0 - m\| \geq \big|\, \|x - m\| - \|x_0 - x\| \,\big| = \|x - m\| - \|x_0 - x\| >$$
$$> d(x,M) - \|x_0 - x\| = d(x,M) - (1 - (\varepsilon/3))\, d(x,M) = (\varepsilon/3)\, d(x,M).$$

It is clear now that $m \notin P_M(x_0)$. If $m \in P_M(x) \setminus m_0$ we have :

$$\|x_0 - m\| \geq \big|\, \|m_0 - m\| - \|m_0 - x_0\| \,\big| \geq \varepsilon\, d(x,M) - (\varepsilon/3)\, d(x,M) =$$
$= (2\varepsilon/3)\, d(x,M)$ and $m \notin P_M(x_0)$.

Then, for all $m \neq m_0$, $m \in M$, we have $m \notin P_M(x_0)$ and it follows that $P_M(x_0) = \{m_0\}$ and this contradicts the fact that M is a strongly proximinal set. It follows that $P_M(x)$ is a closed set dense in itself, i.e. a perfect set in X for all $x \in X \setminus M$. But every perfect subset of a complete metric space has the cardinality at least c (theorem 6.65, p. 72, [1]). Hence card $P_M(x) \geq c$, for all $x \in X \setminus M$.

In the precedent proposition, the condition " X is a Banach space" cannot be improved by " X is a normed space".

EXAMPLE. Let X be the space of all real sequences $x = (x_n)_{n=1}^{\infty}$ having only a finite number of nonzero therms. With the norm :

$$\|x\| = \max_{n \in N} \{|x_n|\} ,$$

X is a noncomplete normed vector space. Let M be the set

$$M = \{x \in X : x_n \in \{0, 1/2^n\}\}.$$

We will show that P_M is a countably multi-valued metric projection, i.e. card $P_M(x) = \aleph_0$ for all $x \in X \setminus M$. Let be $x \in X \setminus M$. Then the therms of the sequence $x = (x_n)_{n=1}^{\infty}$ will be of the form :

$$x_n = 1/2^n, \text{ if } n = n_1, n_2, \dots, n_k$$
$$x_n \in R \setminus \{0, 1/2^n\} \text{ if } n = n_{k+1}, \dots, n_r$$
$$x_n = 0 \text{ in rest.}$$

Let $m^o = (m_n^o)_{n=1}^{\infty}$ the element of M defined by

$$m_n^o = \begin{cases} 1/2^n & \text{if } n = n_1, n_2, \dots, n_k \\ & \text{or if } n \in \{n_{k+1}, \dots, n_r\} \text{ and } |x_n| > |x_n - 1/2^n|, \\ 0 & \text{in rest.} \end{cases}$$

For every $m = (m_n)_{n=1}^{\infty} \in M$ we have :

$$\|x - m\| = \max_{n \in N} \{|x_n - m_n|\} \geq \max_{n \in \{n_{k+1}, \dots, n_r\}} \{|x_n - m_n|\} \geq$$

$$\geq \max_{n \in \{n_{k+1}, \dots, n_r\}} \{\min(|x_n|, |x_n - 1/2^n|)\} =$$

$$= \max_{n \in \{n_{k+1}, \dots, n_r\}} \{|x_n - m_n^o|\} = \max_{n \in \{n_1, \dots, n_r\}} \{|x_n - m_n^o|\} =$$

$$= \max_{n \in N} \{|x_n - m_n^o|\} = \|x - m^o\| .$$

This implies that $m^o \in P_M(x)$ and then $d(x,M) = \|x - m^o\| > 0$.

Let $N_0 \in N$ be such that $1/2^n < \|x - m^0\|$, for all $n > N_0$. Let $N_1 = \max\{n_1, n_2, \dots, n_r\}$ and $N_2 = \max\{N_0 + 1, N_1 + 1\}$. Let

$$M_1 = \{m^0 + (1/2^n)\, e_n\}_{n \geq N_2}$$

where $e_n = (\underbrace{0, \dots, 0,}_{n} 1, 0, \dots)$.

It is clear that M_1 is a countable subset of M and if $m^1 = (m_n^1)_{n=1}^{\infty} \in M_1$, then :

$$\|x - m^1\| = \max_{n \in N} \{|x_n - m_n^1|\} = \max\{\max_{n \geq N_2} |x_n - m_n^1|, \max_{n < N_2} |x_n - m_n^1|\} \leq$$

$$\leq \max\{1/2^{N_2}, \|x - m^0\|\} = \|x - m^0\|.$$

We have that $\|x - m^1\| \leq \|x - m^0\|$ and since $m^0 \in P_M(x)$, it follows that $m^1 \in P_M(x)$ for all $m^1 \in M_1$.

Finally, if $x \in X \setminus M$ we have proved that card $P_M(x) \geq$ card $M_1 = \aleph_0$ and card $P_M(x) \leq$ card $M = \aleph_0$. This implies that P_M is a countable multi-valued metric projection.

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