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C O N T E N T S

NORMED SPACES WITH BOUNDED OR COMPACT STRONGLY PROXIMINAL SETS

by

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In this paper we denote by X a real normed vector space. If M is an arbitrary subset of X, then, as usual, the <u>metric projection</u> on M is the mapping $P_M : X \longrightarrow 2^M$ defined by

$$P_M(x) = \{ m \in M : \|x - m\| = d(x,M) \},$$

where $d(x,M)$ is the distance from x to M. A set $M \subset X$ is called <u>proximinal</u> if $P_M(x) \neq \emptyset$ for all $x \in X$. We say that the set M is <u>strongly proximinal</u> if card $P_M(x) \geq 2$ for all $x \in X \setminus M$. It is clear that every strongly proximinal set is proximinal. As it was shown in [4], if X is a Banach space and $M \subset X$ is a strongly proximinal set, then card $P_M(x) \geq c$ for all $x \in X \setminus M$ and this property is not true in a general normed space. By a result of S. B. STEČKIN [5], if M is a subset of a strictly convex normed space then we have card $P_M(x) \leq 1$, for x in a dense subset of X. On the other hand, if X does not be a strictly convex normed space, then there exists a hyperplane H in X with card $P_H(x) \geq c$, for all $x \in X \setminus H$.

Accordingly, the normed space X contains a strongly proximinal subset M, if and only if, X is not strictly convex.

S. V. KONJAGIN in [3] posed the problem of finding the Banach spaces X which contain bounded or compact strongly proximinal sets. He calls a strongly proximinal set, a set with the anti-uniqueness property. S. V. Konjagin has observed that in a finite dimensional space there exist no such sets. For some concrete spaces he have obtained the following results :

a) If A is a complete metric space and A_0 is a closed nowhere dense subset of A, then the Banach space of continuous bounded functions f on A with $f|_{A_0} = 0$, endowed with the sup norm,

1) contains a bounded strongly proximinal set if and only if card $A \geq \aleph_0$.

2) contains a compact strongly proximinal set if and only if card $A \geq \aleph_0$ and $A_0 \neq \emptyset$.

b) If (A, Σ, μ) is a positive measure space and $L^1(A, \Sigma, \mu)$ is the Banach space of integrable (classes of integrable) functions on the space (A, Σ, μ) then $L^1(A, \Sigma, \mu)$ contains a bounded strongly proximinal set if and only if the measure μ is non-atomic.

We shall give in the present paper necessary and respectively sufficient conditions in order to a normed space contain compact and respectively bounded strongly proximinal sets.

Let X^* be the set of all continuous linear functionals on X. We say that $x^* \in X^*$ is a support functional (exposing functional) for a given set $M \subset X$, if there exists an $m_0 \in M$ so as to have either $x^*(m_0) \geq x^*(m)$ (respectively $x^*(m_0) > x^*(m)$) for every $m \in M \setminus \{m_0\}$, or $x^*(m_0) \leq x^*(m)$ (respectively $x^*(m_0) < x^*(m)$) for every $m \in M \setminus \{m_0\}$.

If $x^* \in X^*$ is a non-zero support functional (exposing functional) for $M \subset X$, it is clear that λx^* $(\lambda \neq 0)$ is a support functional (exposing functional) for M. In the sequel by support functional (exposing functional) we understand such a functional of norm one.

The set of support functionals, respectively exposing functionals (of norm one) for the set M will be denoted by $\mathcal{S}(M)$ respectively by $\mathcal{E}(M)$. If for some $m_0 \in M$ there exists a functional $x^* \in X^*$ such that $x^*(m_0) \geq x^*(m)$ (respectively $x^*(m_0) > x^*(m)$), then m_0 is called a support (respectively an exposed) point of M. We denote by $U(X)$ the closed unit ball of X.

LEMMA 1. <u>If</u> X <u>is a normed space and</u> M <u>is a strongly proximinal subset of</u> X , <u>then</u> :

$$[\mathcal{S}(U(X)) \cap \mathcal{E}(M)] \cup [\mathcal{S}(M) \cap \mathcal{E}(U(X))] = \emptyset .$$

<u>Proof</u>. Suppose that M is a strongly proximinal subset of X , and $[\mathcal{S}(U(X)) \cap \mathcal{E}(M)] \cup [\mathcal{S}(M) \cap \mathcal{E}(U(X))] \neq \emptyset$. Then either

a) there exists $x^* \in \mathcal{S}(U(X)) \cap \mathcal{E}(M)$ or

b) there exists $x^* \in \mathcal{S}(M) \cap \mathcal{E}(U(X))$.

First, we consider the case a). From a) it follows that there exists $m_0 \in M$ such that either c) $x^*(m_0) > x^*(m)$ for every $m \in M$, $m \neq m_0$ or d) $x^*(m_0) < x^*(m)$ for every $m \in M$, $m \neq m_0$. It is enough to discuss only the case c). It follows from a), that there exists $x_0 \in U(X)$ such that $x^*(x_0) \leq x^*(x)$, for all $x \in U(X)$.(If we suppose that $x^*(x_0) \geq x^*(x)$ for all $x \in U(X)$, then $x^*(-x_0) \leq x^*(x)$, for all $x \in U(X)$, and $-x_0$ will be the element which we need).

Consequently

$$- x^*(x_0) = x^*(- x_0) = \sup_{x \in U(X)} x^*(x) = \|x^*\| = 1, \text{ so that } x^*(x_0) = -1.$$

But from $1 = |x^*(x_0)| \leq \|x^*\| \cdot \|x_0\| = \|x_0\| \leq 1$, it follows $\|x_0\| = 1$

Let now : $x_1 = m_0 - x_0$. We have :

$$\|x_1 - m\| \geq |x^*(x_1 - m)| = |x^*(m_0 - m) - x^*(x_0)| = x^*(m_0 - m) + 1 >$$
$$> 1 = \|- x_0\| = \|x_1 - m_0\| ,$$

for every $m \in M$, $m \neq m_0$. Then $\|x_1 - m_0\| < \|x_1 - m\|$, for every $m \in M$ $m \neq m_0$. It follows that $P_M(x_1) = \{m_0\}$ and by hypothesis this implies that $x_1 = m_0$, i.e. $x_0 = 0$, in contradiction with $\|x_0\| = 1$.

We consider now the case b). There exists $m_0 \in M$ such that c') $x^*(m_0) \geq x^*(m)$ for all $m \in M$. We have as in the preceding case $x^*(x_0) = -1$ and $\|x_0\| = 1$, if $x_0 \in U(X)$ and $x^*(x_0) < x^*(x)$, $\forall x \in U(X)$.

Let $x_1 = m_0 - x_0$ and $m \in M$, $m \neq m_0$. If $x^*(m_0) > x^*(m)$ then with the same proof as in the case a) we have

$$\| x_1 - m \| > \| x_1 - m_0 \| .$$

If $x^*(m) = x^*(m_0)$ then, $x^*(x_1 - m) = x^*(x_1 - m_0) = 1$, and from $x_1 - m \neq x_1 - m_0 = - x_0$, it follows that $x_1 - m \notin U(X)$, since, if contrary, $x^*(x_1 - m) < 1$.

Hence $\|x_1 - m\| > 1 = \|- x_0\| = \|x_1 - m_0\|$, $m \in M$, $m \neq m_0$. Then for every $m \in M$, $m \neq m_0$, if $x^*(m_0) \geq x^*(m)$ we have

$$\| x_1 - m \| > \| x_1 - m_0 \| ,$$

hence $P_M(x_1) = \{m_0\}$. It follows that $x_1 = m_0$ i.e. $x_0 = 0$, in contradiction with $\|x_0\| = 1$. Then, in both cases a) and b) it follows that $[\mathcal{Y}(M) \cap \mathcal{E}(U(X))] \cup [\mathcal{Y}(U(X)) \cap \mathcal{E}(M)] = \emptyset$.

Remarks. 1) If X is a finite dimensional Banach space then X does not contain bounded strongly proximinal sets. Indeed, if M is a bounded strongly proximinal subset of X, M is a closed set (since it is proximinal) so that it is a compact set. Then $\mathcal{Y}(M) = U(X^*)$. But, from the Krein-Milman property $\mathcal{E}(U(X)) \neq \emptyset$ and $\mathcal{Y}(M) \cap \mathcal{E}(U(X)) = U(X^*) \cap \mathcal{E}(U(X)) = \mathcal{E}(U(X)) \neq \emptyset$, relation contradicting the condition of our lemma.

2) If in a normed space X, there exists a compact strongly proximinal subset M, then $U(X)$ contains no exposed points. If contrary, then $\mathcal{Y}(M) \cap \mathcal{E}(U(X)) = U(X^*) \cap \mathcal{E}(U(X)) = \mathcal{E}(U(X)) \neq \emptyset$ and by lemma, M is not a strongly proximinal set.

As usual, m is an extremal point of M means that m is not the midpoint of any segment of positive length contained in M.

We shall give in the next proposition a sufficient condition in order to exist a bounded strongly proximinal set in a normed space.

PROPOSITION 2. *If* X *is a normed space and* $U(X)$ *hasn't extremal points then* $U(X)$ *is a strongly proximinal set in* X.

Proof. Let $x_0 \in X \setminus U(X)$ and let $y_0 = x_0/\|x_0\|$. It is clear that $\|x_0 - x\| \geq \|x_0 - y_0\| = \|x_0\| - 1 = \lambda > 0$, for all $x \in U(X)$. Since $U(X)$ hasn't extremal points, it follows that there exist $y_1, y_2 \in U(X)$, $y_1 \neq y_2$, such that $y_0 = (y_1 + y_2)/2$ and $\|y_1\| = \|y_2\| = \|y_0\| = 1$.

Denote by $\alpha = \|y_1 - y_2\|/2$. Then $\alpha = \|y_1 - y_2\|/2 \leq (\|y_1\| + \|y_2\|)/2 \leq 1$.

Let be $\beta = \alpha(\min\{1, \lambda\})$ and $z_0 = y_0 + \beta(y_1 - y_2)/2$. We have $0 < \beta \leq 1$ and $\beta \leq \alpha\lambda \leq \lambda$.

But

$$\|z_0\| = \|y_0 + \beta(y_1 - y_2)/2\| = \|(y_1 + y_2)/2 + \beta(y_1 - y_2)/2\| =$$
$$= \|(1 + \beta)\, y_1/2 + (1 - \beta)\, y_2/2\| \leq 1 \text{ , hence } z_0 \in U(X).$$

On the other hand we have :

$$\|x_0 - z_0\| = \|x_0 - y_0 - \beta(y_1 - y_2)/2\| = \|\lambda y_0 - \beta(y_1 - y_2)/2\| =$$
$$= \|\lambda(y_1 + y_2)/2 - \beta(y_1 - y_2)/2\| = \|(\lambda - \beta)\, y_1/2 + (\lambda + \beta)\, y_2/2\| =$$
$$= \lambda\, \|(\lambda - \beta)\, y_1/2\lambda + (\lambda + \beta)\, y_2/2\lambda\| \leq \lambda \text{ .}$$

From this, it follows that $z_0 = y_0 + \beta(y_1 - y_2)/2 \in P_M(x_0)$. But $\beta \neq 0$, $y_1 - y_2 \neq 0$ implies $z_0 \neq y_0$. Then $P_M(x_0) \supset \{y_0, z_0\}$ and since x_0 was an arbitrary element of $X \setminus U(X)$, it follows that $U(X)$ is a bounded strongly proximinal set in X.

In particular, if we denote by (A, Σ, μ) a positive measure space it is well known (see for instance R. B. HOLMES [2]) that $U(L^1(A, \Sigma, \mu))$ contains an extremal point if and only if Σ contains at least an atom. If A_0 is an atom of Σ then $f = \pm \chi_{A_0}/\mu(A_0)$, where χ_{A_0} is the characteristic function of A_0, is an extremal point of $U(L^1(A,\Sigma,\mu))$. Then if Σ does not contain atoms, then $U(L^1(A,\Sigma,\mu))$ contains no extremal points and hence by the preceding proposition $U(L^1(A,\Sigma,\mu))$ will be a convex, bounded strongly proximinal set.

We shall give in the sequel a generalization of the notion of exposed points.

The point $x_o \in M$ is called a k-exposed point of the set M if there exist the linear independent functionals $x_1^*, x_2^*, \dots, x_k^* \in U(X^*)$ such that :

$$
\begin{array}{ll}
x_1^*(x_o) \geq x_1^*(x) & \text{for all } x \in M \\
x_2^*(x_o) \geq x_2^*(x) & \text{for all } x \in M \cap H_1 \\
\dots\dots\dots\dots\dots\dots & \\
x_{k-1}^*(x_o) \geq x_{k-1}^*(x) & \text{for all } x \in M \cap H_1 \cap \dots \cap H_{k-2} \\
x_k^*(x_o) > x_k^*(x) & \text{for all } x \in M \cap H_1 \cap \dots \cap H_{k-1} \setminus \{x_o\},
\end{array}
$$

where by H_i we understand the hyperplanes of equations $x_i^*(x) = x_i^*(x_o)$, $i = 1, 2, \dots, k-1$. If $k = 1$, then $x_o \in M$ is 1-exposed point if and only if it is an exposed point. If the dimension of X is at least $k+1$, then a point $x_o \in M$ which is a k-exposed point is also a $k+1$ exposed point.

On the other hand there exist k-exposed points, which are not $k-1$ -exposed points. We will give an example of a 2-exposed point which is not an exposed point.

If we consider in R^3 the convex body obtained from a cylinder completed with two semi-spheres it is easy to see that a point p of the circle of contact of the cylinder with a semi-sphere is not an exposed point but it is a 2-exposed point. Here H_1 is the unique supporting plane in the point p, to this convex body and H_2 $(x_2^*(x) = x_2^*(p))$ is for instance, the plane containing the circle of contact passing through p.

Concerning k-exposed points and compact strongly proximinal sets we have :

PROPOSITION 3. <u>If the normed space</u> X <u>contains a compact strongly proximinal set, then</u> $U(X)$ <u>contains no</u> k-<u>exposed points, for</u> $k \in N$.

<u>Proof</u>. Let M be a compact strongly proximinal set of X. We suppose that $x_0 \in U(X)$ is a k-exposed point of $U(X)$, for a given $k \in N$. Then, there exist the functionals $x_i^* \in U(X^*)$, $i = 1, \ldots ,k$, such that

$$
\begin{array}{ll}
x_1^*(x_0) \geq x_1^*(x) & \text{for all } x \in U(X) \\
x_2^*(x_0) \geq x_2^*(x) & \text{for all } x \in U(X) \cap H_1^x \\
\cdots\cdots\cdots & \\
x_k^*(x_0) > x_k^*(x) & \text{for all } x \in U(X) \cap H_1^x \cap \ldots \cap H_{k-1}^x \setminus \{x_0\},
\end{array}
$$

where H_i^x are the hyperplanes $x_i^*(x) = x_i^*(x_0)$, $i = 1, \ldots ,k-1$. From the proof of lemma we have $\|x_0\| = x_1^*(x_0) = 1$. Since M is a compact set it follows that the sets defined inductively by

$$
\begin{aligned}
M_1 &= \{m \in M : x_1^*(m) = \inf_{x \in M} x_1^*(x)\} , \\
M_2 &= \{m \in M_1 : x_2^*(m) = \inf_{x \in M_1} x_2^*(x)\} , \\
&\cdots\cdots\cdots \\
M_k &= \{m \in M_{k-1} : x_k^*(m) = \inf_{x \in M_{k-1}} x_k^*(x)\} ,
\end{aligned}
$$

are all nonvoid and compact.

Let be $m_0 \in M_k$. We have :

$$
\begin{array}{ll}
x_1^*(m_0) \leq x_1^*(m) & \text{for all } m \in M \\
x_2^*(m_0) \leq x_2^*(m) & \text{for all } m \in M \cap H_1^m \\
\cdots\cdots\cdots & \\
x_k^*(m_0) \leq x_k^*(m) & \text{for all } m \in M \cap H_1^m \cap \ldots \cap H_{k-1}^m .
\end{array}
$$

Let $x_1 = m_0 - x_0$ and $m \in M$, $m \neq m_0$. If $x_1^*(m_0) < x_1^*(m)$, then

$x_1^*(m - x_1) = x_1^*(m - m_0 + x_0) > x_1^*(x_0) = 1$. Hence $\|m - x_1\| \geq$ $\geq |x_1^*(m - x_1)| > 1 = \|x_0\| = \|m_0 - x_1\|$. In this case it is clear that $m \notin P_M(x_1)$. If $x_1^*(m_0) = x_1^*(m)$, then $x_1^*(m - x_1) = x_1^*(m_0 - x_1) =$ $= x_1^*(x_0)$ and it follows that $m - x_1 \in H_1^x$. If in what follows we suppose that $x_2^*(m_0) < x_2^*(m)$ for all $m \in M \cap H_1^m$; we have $x_2^*(m - x_1) >$ $> x_2^*(m_0 - x_1) = x_2^*(x_0)$ and since $m - x_1 \in H_1^x$, we have again $m - x_1 \notin U(X)$ so that $m \notin P_M(x_1)$.

Suppose now that $x_2^*(m_0) = x_2^*(m)$. We obtain inductively that $m \notin P_M(x_1)$ if at least one inequality $x_i^*(m_0) \leq x_i^*(m)$ is sharp. Suppose that $x_i^*(m_0) = x_i^*(m)$ for $i = 1, 2, \ldots, k$. This implies that $x_i^*(m - x_1) = x_i^*(m_0 - x_1) = x_i^*(x_0)$, $i = 1, \ldots, k$ i.e. $m - x_1 \in$ $\in H_1^x \cap H_2^x \cap \ldots \cap H_k^x$. Now, since $m \neq m_0$, it follows that $m - x_1 \neq$ $\neq x_0$. But $x_k^*(m - x_1) = x_k^*(m_0 - x_1) = x_k^*(x_0) > x_k^*(x)$, for all $x \in U(X) \cap H_1^x \cap \ldots \cap H_{k-1}^x \setminus \{x_0\}$. Hence $m - x_1 \notin U(X)$, i.e. $m \notin P_M(x_1)$ Therefore in all of the cases $m \neq m_0$ implies $m \notin P_M(x_1)$. Then $P_M(x_1) = \{m_0\}$ and this contradiction shows that $U(X)$ does not contain k-exposed points.

<u>Remark</u>. If A is a separable metric space and $x_1, x_2, \ldots$ is a dense sequence of points in A, if $C(A)$ is the Banach space of bounded continuous functions defined on A, normed with the sup norm, then it is clear that the continuous linear functional on $C(A)$ given by

$$x^*(f) = \sum_{i=1}^{\infty} f(x_i)/2^i ,$$

has the property that

$$x^*(e) > x^*(f) ,$$

for all $f \neq e$, with $\|f\| \leq 1$. (Here e stands for the function identically 1). This implies that e is an exposed point of $C(A)$ and by the preceding proposition $C(A)$ does not contain compact strongly proximinal sets.

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