[25], [26], [27]

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C O N T E N T S

APPROXIMATION SOLUTIONS FOR THE NONSTATIONARY HEAT TRANSFER WITHIN THE MOVEMENT OF A FLUID THROUGH CIRCULAR CYLINDRICAL DUCTS

by

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The exact solution of the nonstationary energy equation for the movement of a viscous incompressible fluid through circular cylindrical ducts is given in [1] by Bessel functions and Green function.

The following lines will be devoted to the establishment approximation formulas for the heat transfer taking into consideration the same suppositions as those in the work [1] : nonstationary and axisymmetrical movement with thermal conductivity.

We shall consider the nonhomogeneous energy equation corresponding to the dissipative case with homogeneous boundary conditions

$$A(\theta_1) \equiv \sigma \frac{\partial \theta_1}{\partial \tau} - \frac{1}{y}\frac{\partial}{\partial y}\left(y \frac{\partial \theta_1}{\partial y}\right) = \sigma m \left(\frac{\partial U}{\partial y}\right)^2 \qquad (1)$$

$$\theta_1(y,o) = o, \quad \theta_1(1,\tau) = o, \quad (\theta_1(o,\tau) = \text{finite})$$

where m and U (the velocity in the fluid) are known.

For this boundary problem we shall determine approximating solutions using the weighted residual method.

1. First order approximation. The solution of the stationary energy equation is known to have the form

$$\theta_1(y) = \frac{\sigma m}{4}(1 - y^4)$$

On the first approximation we shall choose the solution of problem (1) in the form

$$\theta_1^{(1)}(y,\tau) = \frac{\sigma m}{4}\left[1 - F_1(\tau)\right](1 - y^4) \qquad (2)$$

where $F_1(\tau)$ is an unknown function which is 1 for $\tau = o$ according to the condition $\theta_1(y,o) = o$.

In order to determine $F_1(\tau)$ we shall use a weighted residual method (moments), which at first consists in imposing the orthogonality condition to weight function $w_1(y) = 1$:

$$\int_0^1 \left[A(\theta_1^{(1)}) - \sigma g\right] dy = o \qquad (3)$$

where

$$A(\theta_1^{(1)}(y,\tau)) = \sigma \frac{\partial \theta_1^{(1)}}{\partial \tau} - \frac{1}{y}\frac{\partial \theta_1^{(1)}}{\partial y} - \frac{\partial \theta_1^{(1)}}{\partial y^2}$$

$$g(y,\tau) = m\left(\frac{\partial U}{\partial y}\right)^2$$

The dissipative term is calculated by first order approximation of velocity, [2], which has the form

$$U_1(y,\tau) = (1-y^2)(1-e^{-6\tau})$$

The condition (3) thus turns into the following ordinary differential equation, which is nonhomogeneous and has constant coefficients

$$3\sigma F_1'(\tau) + 2o\, F_1(\tau) = 2o\, e^{-6\tau}(2-e^{-6\tau}) \qquad (4)$$

The general solution to this equation has the form

$$F_1(\tau) = C\, e^{-\frac{20}{3\sigma}\tau} + \frac{20}{10-9\sigma}e^{-6\tau} - \frac{5}{5-9\sigma}e^{-12\tau} \qquad (5)$$

and C is determined, using the initial condition $F_1(o)=1$, in the form

$$C = \frac{81\sigma^2}{(10-9\sigma)(5-9\sigma)} \qquad (6)$$

2. Second order approximation. We shall consider an approximating solution which has the form

$$\theta_1^{(2)}(y,\tau) = \frac{\sigma m}{4}\left\{\left[1-F_1(\tau)\right](1-y^4) + \sum_{k=1}^{2} y^{2k} F_{2k}(\tau)\right\}$$

This solution has to fulfil the boundary condition $\theta_1^{(2)}(1,\tau) = o$.

Thus we can eliminate one of the functions, for instance $F_4(\tau)$:

$$\theta_1^{(2)}(y,\tau) = \frac{\sigma m}{4}\left\{\left[1-F_1(\tau)\right](1-y^4) + y^2(1-y^2)F_2(\tau)\right\} \tag{7}$$

The approximation functions $\phi_1(y)=1-y^4$ and $\phi_2(y)=y^2(1-y^2)$ are linear independent, they belong to a complete system and verify all the boundary conditions as well as the symmetry condition. Therefore, these functions fulfil the conditions of the residual methods.

In order to determine the unknown functions F_1 and F_2 we shall now impose the residuum orthogonality condition to the functions $w_1(y)=1$ and $w_2(y)=y$:

$$\int_0^1 \left(A(\theta_1^{(2)}) - \sigma g\right) y^i \, dy = 0 \;, \quad i=0, i=1 \tag{8}$$

In this way we are lead to the ordinary differential equations

$$\begin{aligned} &2\sigma F_1'(\tau) - 8F_1(\tau) + 14F_2(\tau) = -8e^{-6\tau}(2-e^{-6\tau}) \\ &\sigma F_2'(\tau) - 64F_1(\tau) + 52F_2(\tau) = -64e^{-6\tau}(2-e^{-6\tau}) \end{aligned} \tag{9}$$

which has the general solution

$$\begin{aligned} &F_1(\tau) = (7+\sqrt{21})C_1 e^{\lambda_1\tau} + (7-\sqrt{21})C_2 e^{\lambda_2\tau} + ae^{-6\tau} + be^{-12\tau} \\ &F_2(\tau) = 16C_1 e^{\lambda_1\tau} + 16C_2 e^{\lambda_2\tau} + ce^{-6\tau} + de^{-12\tau} \end{aligned} \tag{10}$$

where

$$\lambda_1 = \frac{4}{\sigma}(-6+\sqrt{21}) \;, \quad \lambda_2 = \frac{4}{\sigma}(-6-\sqrt{21}) \tag{11}$$

$$a = \frac{4(\sigma+10)}{3\sigma^2-24\sigma+20} \;; \quad b = -\frac{\sigma+5}{3\sigma^2-12\sigma+5} \;; \quad c = \frac{64\sigma}{3\sigma^2-24\sigma+20} \;; \quad d = \frac{-16\sigma}{3\sigma^2-12\sigma+5}$$

From the conditions $F_1(0)=1$ and $F_2(0)=0$ we obtain the expressions for C_1 and C_2 in the form

$$C_1 = \frac{\sqrt{21}\,\sigma^2\left[(3\sigma^2-39\sigma+94)-(7-\sqrt{21})(-3\sigma+8)\right]}{14(3\sigma^2-24\sigma+20)(3\sigma^2-12\sigma+5)}$$

$$C_2 = \frac{\sqrt{21}\,\sigma^2\left[(7+\sqrt{21})(-3\sigma+8)-(3\sigma^2-39\sigma+94)\right]}{14(3\sigma^2-24\sigma+20)(3\sigma^2-12\sigma+5)} \qquad (12)$$

3. Approximation formulas for heat transfer. The rate of heat transfer at the wall of the duct dQ_w per dS area during dt (nonstationary heat transfer) is given by Fourier's formula

$$dQ_w(t) = -\lambda\left(\frac{\partial T}{\partial r}\right)_{r=R} dS\, dt$$

Making the following transformations (for temperature T and time t)

$$T = (T_w - T_o)\theta + T_o\,, \quad t = \frac{\rho R^2}{\mu}\tau\,, \quad (dS = 2\pi R\, dz)$$

where $\theta\,(y,\tau)$ is the temperature distribution (function unknown), the heat quantity which goes through the duct surface $S = 2\pi RL$ during t is calculated with the formula

$$Q_w(\tau) = -2\pi L\lambda(T_w - T_o)\frac{\rho R^2}{\mu}\int_0^{\tau}\left[\frac{\partial\theta(y,\tau)}{\partial y}\right]_{y=1} d\tau$$

We introduce

$$q_w(\tau) = \frac{Q_w(\tau)}{S\rho R^2/\mu}$$

that is, the variation of the heat flow per unit area at the wall with time.

In nondimensional form, the heat transfer coefficient, or Nusselt number Nu is

$$Nu(\tau) = \frac{q_w(\tau)}{\lambda\,\frac{T_w - T_o}{R}} = -\int_0^{\tau}\left[\frac{\partial\theta(y,\tau)}{\partial y}\right]_{y=1} d\tau \qquad (13)$$

In this formula, $\theta\,(y,\tau)$ is the solution of the dissipative thermal problem under nonhomogeneous conditions which is exactly determined in [1], in the form

$$\theta\,(y,\tau) = \theta_0(y,\tau) + \theta_1(y,\tau) \qquad (14)$$

where

$$\theta_o(y,\tau) = 1 - 2\sum_{k=1}^{\infty} \frac{J_o(\alpha_k y)}{\alpha_k J_1(\alpha_k)} e^{-\frac{1}{\sigma}\alpha_k^2 \tau} \tag{15}$$

represent the exact solution of the energy equation, in the nondissipative case, under nonhomogeneous conditions, which is expressed by means of Bessel functions.

If in (14), $\theta_1(y,\tau)$ is replaced by the approximate solutions $\theta_1^{(i)} = \theta_1^{(i)}(y,\tau)$, i=1;2, given by the residual method, we find from (13) for Nu the expressions

$$Nu^{(1)}(\tau) = \sigma m\Big[\tau + \frac{3\sigma}{20} C e^{-\frac{20}{3\sigma}\tau} + \frac{10}{3(10-9\sigma)} e^{-6\tau} - \frac{5}{12(5-9\sigma)} e^{-12\tau}\Big] + 2\sigma \sum_{n=1}^{\infty} \alpha_n^{-2} e^{-\alpha_n^2 \tau/\sigma}; \tag{16}$$

$$Nu^{(2)}(\tau) = \sigma m\Big[\tau + \frac{3+\sqrt{21}}{12} C_1 e^{\lambda_1\tau} + \frac{3-\sqrt{21}}{12} C_2 e^{\lambda_2\tau} + \frac{-14\sigma+20}{3(3\sigma^2-24\sigma+20)} e^{-6\tau} + \frac{7\sigma-5}{12(3\sigma^2-12\sigma+5)} e^{-12\tau}\Big] + 2\sigma\sum_{n=1}^{\infty} \alpha_n^{-2} e^{-\frac{\alpha_n^2}{\sigma}\tau} \tag{17}$$

In the particular case m=1, σ =1 respectively σ =o,7, the values of Nusselt number calculated with the formulas (16)-(17) are given in the Table 1 and are graphically represented in Fig.1.

Remarks 1°. There are values of Prandtl number σ, for which the coefficients from (5) and (10) are senseless, although those values have hydro and thermodynamic signification. This situation is caused by the approximation method we used. Except for these singular values, the approximation method can be applied to all the other values of σ hydrodynamically admittable.

2°. In the formulas (16) and (17) the first positive roots α_i, $i=\overline{1,4}$, of the equation $J_o(\alpha) = o$ are

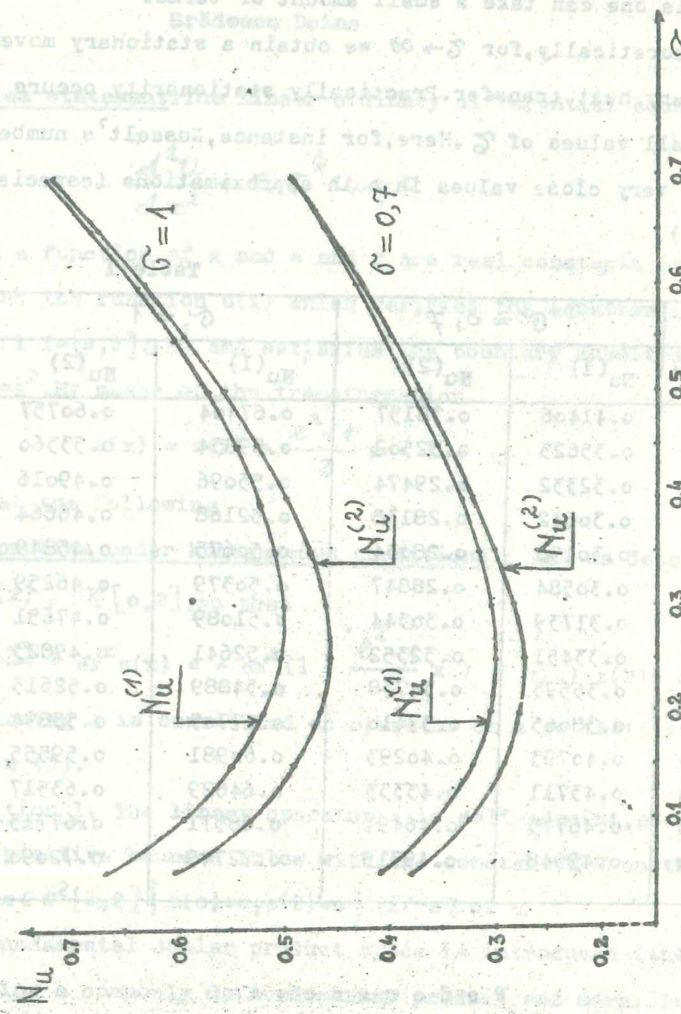

Fig. 1

$\alpha_1 = 2,4048$; $\alpha_2 = 5,5201$; $\alpha_3 = 8,6537$; $\alpha_4 = 11,7915$

Consequently,in numerical applications,in the series from Nusselt number formula one can take a small amount of terms.

3°.Theoretically,for $\tau \to \infty$ we obtain a stationary movement and a stationary heat transfer.Practically stationarity occurs for relatively small values of τ.Here,for instance,Nusselt's number for σ = o,7 has very close values in both approximations (especially for σ = o,7).

Table 1

σ	$\sigma = o,7$		$\sigma = 1$	
τ \ Nu	$Nu^{(1)}$	$Nu^{(2)}$	$Nu^{(1)}$	$Nu^{(2)}$
o,o5	o.414o6	o.38157	o.674o4	o.6o757
o.1o	o.35623	o.325o2	o.59834	o.5336o
o.15	o.32332	o.29474	o.55o96	o.49o16
o.2o	o.3o642	o.28138	o.52168	o.46664
o.25	o.3o15o	o.28o34	o.5o675	o.45849
o.3o	o.3o584	o.28847	o.5o379	o.46259
o.35	o.31739	o.3o344	o.51o89	o.47651
o.4o	o.33451	o.32352	o.52641	o.49823
o.45	o.35593	o.34738	o.54889	o.52615
o.5o	o.38o65	o.3741o	o.577o5	o.55894
o.55	o.4o793	o.4o293	o.6o981	o.59555
o.6o	o.43711	o.43333	o.64629	o.63517
o.65	o.46773	o.46491	o.68571	o.67713
o.7o	o.49948	o.49715	o.72749	o.72o9o

R e f e r e n c e s

1.Brădeanu D.,The Heat Transfer in a Viscous Unsteady Flow through Circular Ducts (in this Preprint)

2.Brădeanu P.,The Method of Weighted Int.Relations for Unsteady Poiseuille Flow (in this Preprint)

3.Finlayson B.A., The Method of Weighted Residuals and Variational Principles,Acad.Press,1972.