DESCRIPTION of the FINITE ELEMENT METHOD with SPLINE FUNCTIONS on a SIMPLE BILOCAL PROBLEM

by

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1. Problem statement. The linear ordinary differential equation is given

$$-\frac{d^2u}{dx^2}+ax^k u=0 \qquad (1)$$

where u is a function of x and a and k are real constants ($k\geq 0$). We have to find the function u(x) which verifies the equation (1) inside the interval $I=[0,2]\in R^1$ and satisfies the boundary conditions : $u(0)=1$ and $u(2)=e^2$. By means of the transformation

$$u(x)=z(x)+\frac{e^2-1}{2}x+1 \qquad (2)$$

we arrive at the following

Bilocal problem under homogeneous conditions : let us determine the function z(x) , $x\in[0,2]$, so that

$$\frac{d^2z(x)}{dx^2}+ax^k z(x)=-ax^k\left(1+\frac{e^2-1}{2}x\right),\quad z(0)=z(2)=0 \qquad (3)$$

In the following A is considered an operator on z so that Az is the left side in (3).

Proposition 1. The linear operator A is self-adjoint, positive definite and positive bounded below with the constant 1/2 on the linear space $Z=\{z\in C^2[0,2]\mid z(0)=0, z(2)=0\}$, if $a\geq 0$.

Proof. A fundamental scalar product space is introduced (the space $L_2[0,2]$ having a commonly defined scalar product and norm). It is known that $C^2[0,2]$ is dense in the space $L_2[0,2]$. If $z,v\in Z$ and we calculate the L_2 scalar product

$$(Az,v)=\int_0^2 (z'v'+ax^k zv)\,dx=\int_0^2 z(-v''+ax^k v)\,dx=(z,A^* v) \qquad (4)$$

we deduce from $A^{*}v = -v'' + ax^{k}v$, A^{*} being the adjoint of A, that $A^{*} = A$ for all $z, v \in Z$. Consequently, the operator A is self-adjoint on Z.

If $z=v$, from (4) we obtain

$$(Az,z) = \int_{o}^{2} z'^{2}\,dx + a\int_{o}^{2} x^{k}z^{2}\,dx \geqslant o \quad , \qquad \text{if } a \geqslant o \qquad (5)$$

The equality is valid if and only if $z(x) \equiv o$. Indeed, $z'(x)=o$ implies $z(x)=c$ (const.); but $z(o) = o$ so that $c = o$. The inequality (5) proves that the operator is positive definite.

Let us prove now that A is a positive bounded below operator, i.e there exists a constant $\alpha > 0$ so that

$$(Az,z) \geqslant \alpha^{2}(z,z) \qquad (6)$$

In order to demonstrate this let us notice that we may write

$$z(x) = \int_{o}^{x} z'(s)\,ds \quad , \quad \text{with } z(o) = o$$

By squering and applying the Cauchy-Schwartz inequality, we get

$$z^{2}(x) \leq \int_{o}^{x} ds \int_{o}^{x} z'^{2}\,ds = x\int_{o}^{x} z'^{2}\,ds$$

Hence, by extending the interval (o,x) to $(o,2)$ and by integration we obtain

$$\int_{o}^{2} z^{2}(x)\,dx \leq 4\int_{o}^{2} z'^{2}\,ds,$$

hence, if $a \geqslant o$, we deduce the inequality

$$\frac{1}{4}\int_{o}^{2} z^{2}(x)\,dx \leq \int_{o}^{2} z'^{2}(x)\,dx + a\int_{o}^{2} z^{2}(x)x^{k}\,dx$$

which, after (4) can be written in the form $(Az,z) \geqslant (z,z)/4$. Consequently, considering $\alpha = 1/2$ the inequality (6) is proved.

Remarks 1°. The above calculations, also show that the bilinear functional $a(z,v)=(Az,v)$ given on (4) is an energy scalar product noted by $(z,v)_{A}$; we have

$$(z,v)_A = (Az,v) = \int_0^2 (z'v' + ax^k zv)\,dx$$

2^o. The inequality (6) can also be directly demonstrated by means of Friedricks' inequality

$$\| u \|_2 \leq (b-a) \| u \|_A$$

where $\| \cdot \|_2$ is the norm in the space $L_2[a,b]$ and $\| \cdot \|_A$ is the energy norm of the operator A. Concerning the problem under disscution we shall have ;

$$\sqrt{(z,z)} \leq 2\sqrt{(Az,z)} \quad \text{and} \quad (Az,z) \geq \frac{1}{4}(z,z).$$

Proposition 1 and the well known existence and unicity theorem for the operatorial equation $Az = f$ lead to

<u>Proposition</u> 2. In space Z the bilocal problem (3) has a unique solution ; let it be $z_o \in Z$.

The following proposition also takes place from the Ritz fundamental theorem,

<u>Proposition</u> 3. The quadratic functional (the energy functional) $F : Z \to R^1$ defined by

$$F(z) = (Az,z) - 2(f,z) =$$

$$= \int_0^2 (z'^2 + ax^k z^2)\,dx + 2a\int_0^2 x^k\left(1 + \frac{\ell^2 - 1}{2} z\right) dx \qquad (7)$$

has an absolute minimal value for $z = z_o$, i.e

$$F(z_o) = \inf \{ F(z) \mid z \in Z \}$$

and reciprocally : if $z_o \in Z$ fulfills a minimal value for the energy functional $F(z)$ then z_o is the solution of the differential problem (3).

As known, in this case we say that the energy functional generates an equivalent variational minimum formulation (minimal variational principle) for the differential problem (3).

To solve the variational problem we apply the Rayleigh-Ritz finite element approximate method, in a finite dimension space (of dimension N), with the shape (interpolation, basis) functions provided by splines.

2. The Rayleigh-Ritz procedure. The solution approximation by means of cubic splines. One considers the problem (3) in the case $a=1$; $k=o$ for which we have

$$A(z)\equiv -\frac{d^2z}{dx^2}+z(x)=-\left(1+\frac{e^2-1}{2}x\right), \quad x\in I=(o,2) \tag{8}$$
$$z(o)=o, \quad z(2)=o$$

In order to determine the approximate solution instead of the space Z we choose an N-dimensional subspace Z_N, characterized by a smoothness degree equal with that of the space Z. For a given N a basis of functions $\{\phi_i\}$, $i=\overline{1,N}$ in Z_N is chosen. These functions fulfil the following conditions : ϕ_i represent a complete set of linear independent functions (finite elements, subintervals), belong to the space $C^2[o,2]$, have compact support and verify the homogeneous boundary conditions ($\phi_i=o$ when $x=o$ and $x=2$). Such a subspace of Z is provided by the set

$$Z_N=\left\{z_N\in\tilde{S}_3(\pi)\ \middle|\ z_N(o)=z_N(2)=o\right\}$$

where $\tilde{S}_3(\pi)$ is the linear space of the cubic splines on partition

$$\pi:\ o=x_o<x_1<x_2<x_3<x_4=2$$

having the spacing $h=1/2$, of $N=5$ dimension. The basis of this space consists in the cubic splines $\phi_i=\tilde{B}_i$, $i=\overline{o,4}$ which fulfil the above conditions. Therefore, the subspace Z_N of Z has the form

$$Z_N=\operatorname{span}\left\{\tilde{B}_o,\tilde{B}_1,\tilde{B}_2,\tilde{B}_3,\tilde{B}_4\right\}$$

In order to construct the basis $\{\tilde{B}_i\}$, $i=\overline{o,4}$, we use the well known basis $\{B_i\}$, $i=\overline{-1,5}$ of the cubic splines space $S_3(\pi)=\operatorname{span}\{B_{-1},B_o,B_1,B_2,B_3,B_4,B_5\}$, [3], as follows

$$\tilde{B}_0 = B_0-4B_{-1}\,,\ \tilde{B}_1=B_0-4B_1\,,\ \tilde{B}_2=B_2\,,\ \tilde{B}_3=B_4-4B_3\,,\ \tilde{B}_4=B_4-4B_5$$

The following formulae are obtained :

$$\tilde{B}_0(x)=-8\begin{cases} x(-9x+3+7x^2), & x\in[0,1/2] \\ (1-x), & x\in[1/2,1] \\ 0 & x\geq 1 \end{cases}$$

$$\tilde{B}_1(x)=8\begin{cases} -3x(1+3x-5x^2), & x\in[0,1/2] \\ -\frac{1}{2}-3(1-x)-6(1-x)^2+13(1-x)^3, & x\in[1/2.1] \\ -4(\frac{3}{2}-x)^3, & x\in[1,3/2] \\ 0 & x\geq 3/2 \end{cases}$$

$$\tilde{B}_2(x)=8\begin{cases} x^3, & x\in[0,1/2] \\ \frac{1}{8}+\frac{3}{4}(x-\frac{1}{2})+\frac{3}{2}(x-\frac{1}{2})^2-3(x-\frac{1}{2})^3, & x\in[1/2,1] \\ \frac{1}{8}+\frac{3}{4}(\frac{3}{2}-x)+\frac{3}{2}(\frac{3}{2}-x)^2-3(\frac{3}{2}-x)^3, & x\in[1,3/2] \\ (2-x)^3, & x\in[3/2,2] \end{cases}$$

$$\tilde{B}_3(x)=8\begin{cases} 0 & x\leq 1/2 \\ -4(x-\frac{1}{2})^3, & x\in[1/2,1] \\ -\frac{1}{2}-3(x-1)-6(x-1)^2+13(x-1)^3, & x\in[1,3/2] \\ 3(1-x)(5x^2-17x+13), & x\in[3/2,2] \end{cases}$$

$$\tilde{B}_4(x)=8\begin{cases} 0, & x\leq 1 \\ (x-1)^3, & x\in[1,3/2] \\ \frac{1}{8}+\frac{3}{4}(x-\frac{3}{2})+\frac{3}{2}(x-\frac{3}{2})^2-7(x-\frac{3}{2})^3, & x\in[3/2,2] \end{cases}$$

Figure 1 shows the diagrams of the functions $\tilde{B}_i(x)$.

In order to effectively solve the differential problem(8) we seek a Rayleigh-Ritz approximation for the equivalent variational problem for the functional $F(z)$.An approximate solution in the cubic splines form is attempted

$$z_N(x)=\sum_{k=0}^{4} c_k\tilde{B}_k(x)\,,\ x\in[0,2] \tag{9}$$

where c_k are unknown constant coefficients.These coefficients can b[e] determined by solving the Rayleigh-Ritz system

$$\sum_{k=0}^{4}(A\tilde{B}_i,\tilde{B}_k)c_k=(f,\tilde{B}_i)\ ,\ i=\overline{0,4} \qquad (10)$$

where ($'$ = d/dx)

$$b_i=(f,\tilde{B}_i)=\int_0^2 f\tilde{B}_i\,dx\ ,\ i=\overline{0,4}\ ; \qquad (11)$$

$$a_{ik}=(A\tilde{B}_i,\tilde{B}_k)=\int_0^2(-\tilde{B}_i''+\tilde{B}_i)\tilde{B}_k\,dx=$$

$$=\int_0^2(\tilde{B}_i\tilde{B}_k+\tilde{B}_i'\tilde{B}_k')\,dx\ ,\quad i=0,4\ ;\ k=\overline{0,4} \qquad (12)$$

$$a_{ik}=\begin{cases}a_{ki} & ;\ i,k=0,4\\ 0\ , & \text{if}\quad i-k=4\end{cases}$$

The algebraic system (10) is written in the matrix form

$$[\tilde{A}]\{c\}=\{b\} \qquad (13)$$

$$[\tilde{A}]=[a_{ik}],\ \{c\}=(c_0\ldots c_4)^T\ ,\ \{b\}=(b_0\ \ldots\ b_4\ )^T$$

In order to find the values of the coefficients a_{ik} we shall previous compute the derivatives $\tilde{B}_i'(x)$ of the splines $\tilde{B}_i(x)$ according to the common derivation rules.If we set

$$\alpha_{ik}=\tilde{B}_i'\tilde{B}_k'+\tilde{B}_i\tilde{B}_k$$

and if symmetry is taken into account,we only have to calculate the coefficients

$$a_{i(i+m)}=\sum_{k=0}^{3-m}\int_{x_{i-k+1}}^{x_{i-k+2}}\alpha_{i(i+m)}(x)\,dx\ ;\ m=\overline{0,3}\ ;\ i=\overline{0,4-m} \qquad (14)$$

In these formulae,we admit that the terms for which the integration limit index is smaller than zero or higher than four should not be considered. Moreover, the calculation amount can be reduced both due to the symmetry of operator A and that of the functions $\tilde{B}_i$ and $\tilde{B}_i'$, $i = \overline{0,4}$. This induces a symmetry of the stiffness matrix $[\tilde{A}]$ to both diagonals. Therefore, in the formulae (14) we have

$$a_{ii} = a_{(4-i)(4-i)},\ i=0,2\ ;\ a_{i(i+1)} = a_{(3-i)(4-i)},\ i = 0;1\ ;$$

$$a_{i(i+2)} = a_{(2-i)(4-i)},\ i=0;1\ ;\ a_{i(i+3)} = a_{(1-i)(4-i)},\ i=0.$$

Consequently, it is only necessary to compute by means of the formulae (14) but eight elements of the matrix $[\tilde{A}]$; the other elements are obtained by symmetry to the diagonals:

$$[\tilde{A}] = \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} & 0 \\ \cdot & a_{11} & a_{12} & a_{13} & \cdot \\ \cdot & \cdot & a_{22} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

The right-side terms b_i are calculated by means of the formulae (11) which can be written as

$$b_i = -\int_0^2 \tilde{B}_i(x)\,dx - a_0\int_0^2 x\tilde{B}_i(x)\,dx \quad ; \quad a_0 = \frac{e^2-1}{2} = 3{,}194528 \qquad (15)$$

The numerical values b_i computed by means of (15) are written in the system (16).

The Rayleigh-Ritz algebraic system becomes (Eq.(13))

$$\begin{bmatrix} 3484 & -4513 & -811 & 167 & 0 \\ \cdot & 66562 & 351 & -15314 & \cdot \\ \cdot & \cdot & 3964 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{Bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = \begin{Bmatrix} -1{,}958358 \\ 28{,}208810 \\ -12{,}583584 \\ 55{,}681751 \\ -6{,}430698 \end{Bmatrix} \qquad (16)$$

This system has been solved with the Gauss elimination method. The following solutions have been found:

$c_0=-0,049061$; $c_1=0,042327$; $c_2=-0,264463$; $c_3=0,062205$; $c_4=-0,112217$

These values together with the relation (9) give the approximate solution of problem (8) : the approximate values of the solution of problem (8) in each point x of the interval $[0,2]$ can thus be obtained.

3. Error evaluation. The following evaluation [3]

$$\| z - z_N \|_{\infty} \leq K h^3 \tag{17}$$

referring to the error introduced by $z_N(x)$ is given. In (17) $z(x)$ is the exact solution, $z_N(x)$ is Ritz approximation of cubic splines type, h is the spacing of uniform partition on $[0,2]$ (the lenght of the rectilinear finite element) and K is a positive number independent of N. The mathematical theory of the Ritz method, [1] [2], which is also valid in the case of discretisation by one-dimensional finite elements as well as the evaluation (17) provide the convergence of the method in the problem under consideration.

An approximate solution by piecewise Lagrange linear polynomials (an approximation of class C^0) was applied to problem (1) in paper [2] The resulting numerical data appear in Table 1 where we give the nodal values for the exact solution $u(x)$, the Lagrange type solution $u_L(x)$ and the cubic splines solution $u_s(x)$ - of class C^2. A high degree of accuracy is obtained when cubic splines are used (Table 1)

Table 1

x	0,5	1	1,5
$u(x)$	1,648721	2,718281	4,481689
$u_s(x)$	1,648835	2,718546	4,482029
$u_L(x)$	1,634821	2,696116	4,460750

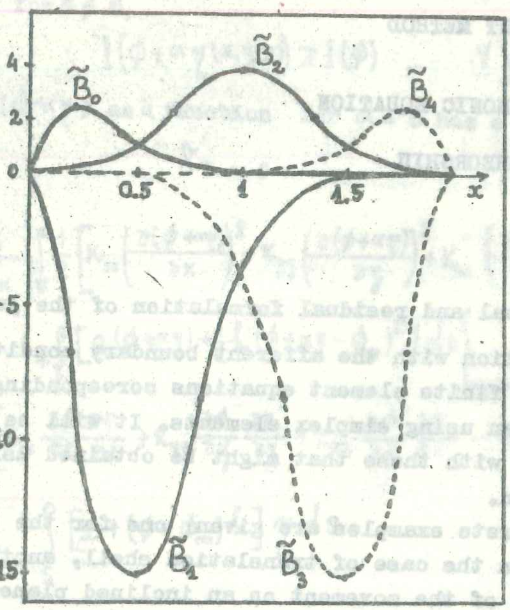

Fig.1

References

1. Mikhlin S.G., Variational Methods in Math.Physics, Pergamon, 1964
2. Norrie D.N., de Vries G., The Finite Element Method, Acad.Press, 1973
3. Prenter P.M., Splines and Variational Methods, J.Wiley, 1975

Summary

In this paper, a variational method on one-dimensional finite elements was outlined, within the general context of Ritz mathematical approximation, for a simple bilocal differential problem (1). This problem is also considered and studied by piecewise linear polynomials in [2]. Another approximate solution is suggested here, (9), given by means of the spline functions. This trial solution, (9), is then determined by computing the $\tilde{B}_i(x)$ splines and by calculating the coefficients c_k, (16), by using the Ritz procedure. The trial solution $u_s(x)$, from (12), is compared to the exact solution $u(x)$, which can be found analytically, and to the linear solution $u_L(x)$ (Lagrange). The spline solution $u_s(x)$ is better than the linear solution u_L (Table 1)