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PROPERTIES OF THE GALERKIN-CRANK- NICOLSON UPWINDING
 SCHEME FOR AN UNSTEADY CONVECTION-DIFFUSION PROBLEM

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1. The Galerkin-Crank-Nicolson Discretisation.

a) The one-dimensional convection-diffusion equation. The parabolic equation with constant coefficients is taken with the initial condition [3]

$$(1.1) \quad \frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}$$

$$(1.2) \quad u(x,0) = u_0 e^{ikx} ; (v, \varepsilon, k \text{ are constants, } \varepsilon > 0, t > 0)$$

The problem (1.1)-(1.2) has a solution of the form

$$(1.3) \quad u(x,t) = e^{-(D+i\omega)t} X(x) \quad \text{with } D = \varepsilon k^2, \omega = vk$$

where D is the exact damping parameter and ω is the (exact) frequency.

Regarding the points $(x_j, (n-1)\Delta t)$, $(x_j, n\Delta t)$ the following relation appears

$$u(x_j, n\Delta t) = e^{-(D+i\omega)\Delta t} u(x_j, (n-1)\Delta t)$$

showing that all harmonics damp in time.

In this paper, this solution will be compared with a numerical one.

b) The Galerkin equation discretized on finite elements with combined interpolation functions. The grid is chosen on the Ox real axis

$$\Delta_h = \{x_j \mid x_j = jh, j = 0, 1, 2, \dots, N; x_j - x_{j-1} = h \text{ (const)}\}$$

A real function $\varphi_j : R \rightarrow R$ given by [1] will be associated to each x_j point

$$(1.4) \quad \varphi_j(x) = \begin{cases} 1 - \frac{x-x}{h} \exp(r(1 - \frac{x-x}{h})) & , x \in [x_{j-1}, x_j] \\ (1 - \frac{x-x_j}{h}) \exp(r \frac{x-x_j}{h}) & , x \in [x_j, x_{j+1}] \\ 0 & , x \notin (x_{j-1}, x_{j+1}) \end{cases}$$

where r is a real variable parameter.

The functions $\varphi_j(x)$ are piecewise defined and continuous on R , having continuous piecewise derivatives and compact support (equal to $2h$) on R .

Let the exact solution of the equation (1.1) be approximated by the interpolation (piecewise) function

$$U(x, t) = u_N(x, t) = \sum_{j=0}^N U_j(t) \varphi_j(x)$$

where $U_j(t)$ are the unknown functions and φ_j are the trial functions (1.4). In order to determine the $U_j(t)$ functions, the generalized Galerkin method (Petrov-Galerkin method) will be applied using a test function system noted by $\{\psi_j(x)\}$. It consists of the association to (1.1)-(1.2) of the equation

$$(1.5) \quad \frac{\partial}{\partial t} (U, \psi_j)(t) + v(U', \psi_j)(t) + (U'', \psi_j')(t) = 0$$

where $U' = \partial U / \partial x$, $\psi_j' = d\psi_j / dx$ and $(\cdot, \cdot)$ represents the scalar product referring to the x coordinate.

The L_2 scalar products are calculated like in the paper [2].

For example

$$(U', \psi'_j) = -\frac{1}{h} (\beta_3 \mu + \frac{1}{2} \alpha_3 \delta^2) U_j$$

where

$$\alpha_3 = c_1 + c_2, \quad \beta_3 = -(c_1 - c_2),$$

$$c_1 = \int_{-1}^0 (1 + rs) e^{r(1+s)} \psi'(s) ds; \quad (s = \frac{x-x_j}{h})$$

$$c_2 = \int_0^1 (-1 + r - rs) e^{rs} \psi'(s) ds.$$

If $\psi_j = \varphi_j$ (Galerkin procedure), then $\alpha_3 = \alpha'_{3G}$,

$$(1.6) \quad \alpha'_{3G} = \frac{1}{2h} [-1 + 2r(1-r) + e^{2r}]$$

In the case of the generalized Galerkin method (Petrov-Galerkin) we impose the following conditions for the test functions

$$\int_{-1}^1 \psi(s) ds = 1; \quad \int_{-1}^1 \psi'(s) \psi'(s) \operatorname{sgn}(s) ds = 0;$$

$$\int_{-1}^1 \psi'(s) \psi'(s) ds = \alpha'_{3G}; \quad \int_{-1}^1 \psi'(s) \psi(s) \operatorname{sgn}(s) ds = -1$$

and we find ($\beta_3 = -1, \alpha_3 = \alpha'_{3G}$)

$$(U', \psi'_j) = -\frac{1}{2h} \alpha'_{3G} \delta^2 U_j$$

$$(U', \psi_j) = (\mu - \frac{1}{2} \alpha \delta^2) U_j, \quad \alpha \text{ arbitrary,}$$

where

$$\mu U_j = \frac{1}{2} (U_{j+1} - U_{j-1})$$

$$\delta^2 U_j = U_{j+1} - 2 U_j + U_{j-1}$$

From (1.5) the finite differences equation (referring to t) is obtained at the x_j point

$$(1.7) \quad h(1 + \beta_1 \mu + \frac{1}{2} \alpha_1 \delta^2) U_j(t) + v(\mu - \frac{1}{2} \alpha \delta^2) U_j(t) - \frac{\varepsilon}{2h} \alpha'_{3G} \delta^2 U_j(t) = 0$$

The equality (1.7), representing a three parameter family $\alpha_1, \beta_1,$

$\mathcal{L}$ of differential equations in terms of t (discretized in terms of x), can be written in the form (semi-discrete equations)

$$(1.8) \quad (\alpha_1 - \beta_1) \ddot{u}_{j-1} - 2(\alpha_1 - 1) \ddot{u}_j + (\alpha_1 + \beta_1) \ddot{u}_{j+1} - \\ - \left(\frac{v}{h} + \frac{v}{h} \alpha + \frac{\varepsilon}{h^2} \alpha_{3G} \right) u_{j-1} + 2 \left(\frac{v}{h} \alpha + \frac{\varepsilon}{h^2} \alpha_{3G} \right) u_j + \\ + \left(\frac{v}{h} - \frac{v}{h} \alpha - \frac{\varepsilon}{h^2} \alpha_{3G} \right) u_{j+1} = 0$$

c) Crank-Nicolson discrete equation (t variable discretization)

On the t variable axis, ($t \geq 0$), we consider the grid

$$\Delta_t = \{ t_n \mid t_n = n \Delta t, n = 0, 1, 2, \dots \}$$

In the Oxt plane we have the grid

$$\Omega_h = \Delta_h \times \Delta_t = \{ (x_j, t_n) \mid x_j = jh, t_n = n \Delta t \}$$

characterized by the h and Δt steps.

Let a scheme with finite differences of the Crank-Nicolson type be associated to the ordinary differential equation (1.8), on the grid Δ_t . On this purpose, the equation (1.8) is applied at the $(x_j, t_{n-1/2})$ point, and a second order approximation in terms of t is used, according to the formulas

$$\left(\frac{dU_k}{dt} \right)_{n-1/2} = \frac{U_{k,n} - U_{k,n-1}}{\Delta t} + o(\Delta t^2) \\ (U_k)_{n-1/2} = \frac{1}{2} (U_{k,n} + U_{k,n-1}) + o(\Delta t^2)$$

where $k = j-1$; j ; $j+1$.

A finite differences equation of Galerkin-Crank-Nicolson type is obtained with these formulas from (1.8), for the given parabolic equation (1.1)

$$(1.9) \quad (\alpha_1 - \beta_1 - B + \frac{C}{2}) u_{j-1,n} + 2(1 - \alpha_1 + B) u_{j,n} + \\ + (\alpha_1 + \beta_1 - B + \frac{C}{2}) u_{j+1,n} = (\alpha_1 - \beta_1 + B + \frac{C}{2}) u_{j-1,n-1} + \\ + 2(1 - \alpha_1 - B) u_{j,n-1} + (\alpha_1 + \beta_1 + B - \frac{C}{2}) u_{j+1,n-1}$$

where (C - Courant number)

$$(1.10) \quad C = v \frac{\Delta t}{h}, \quad H = \varepsilon \frac{\Delta t}{h^2}, \quad B = \frac{1}{2} (C \alpha + H \alpha_{3G})$$

In the finite differences equation (1.9) α_1 , β_1 and α are parameters that do not depend on j and n subscripts, α_{3G} is a function of r, defined by (1.6), and C and H are parameters of the Ω_h grid.

2. Properties of the Solution of the Discrete Equation (1.9).

Calculation of the Damping and Frequency Coefficient.

a) Solution of the Fourier type. Amplification factor. Let us consider the linear finite differences equation (1.9) and a solution with separable indices of the complex Fourier series form

$$(2.1) \quad U_{j,n} = \sum_{k=-\infty}^{\infty} V_n(k) e^{ikjh}$$

and observe the evolution of a single harmonic (with the wave number k). If the equation (1.9) is verified with this solution, the following relation will be found

$$(2.2) \quad V_n(k) = \frac{2(\alpha_1 + B) \cos kh - 2(\alpha_1 - 1 + B) + i(2\beta_1 - C) \sin kh}{2(\alpha_1 - B) \cos kh - 2(\alpha_1 - 1 - B) + i(2\beta_1 + C) \sin kh} V_{n-1}(k)$$

The amplification factor (a transfer operator) of this relation, that performs the transition from n-1 to the n step, has the form

$$(2.3) \quad R(h, \Delta t) = \frac{2 - 2(\alpha_1 + B)(1 - \cos kh) + i(2\beta_1 - C) \sin kh}{2 - 2(\alpha_1 - B)(1 - \cos kh) + i(2\beta_1 + C) \sin kh}$$

b) Approximation. By noting $y = kh$ and using the series expansion for the sin and cos functions (having low h and a given k), we will find

$$R(h, \Delta t) = \frac{2 - (\alpha_1 + B)y^2 + i(2\beta_1 - C)(y - \frac{1}{6}y^3) + o(y^4)}{2 - (\alpha_1 - B)y^2 + i(2\beta_1 + C)(y - \frac{1}{6}y^3) + o(y^4)}$$

Hence the following expansion is obtained

$$(2.4) \quad R(h, \Delta t) = 1 - iCy - (B + \frac{C^2}{2} + \beta_1 C)y^2 + \frac{i}{4}[-2C\alpha_1 + 4(B + C^2)\beta_1 + 4C\beta_1^2 + (\frac{2}{3} + 4B + C^2)C]y^3 + o(y^4)$$

When passing from the $n-1$ to the n level (for the harmonic of k order amplitude), in the Δt time interval, the amplification factor in the exact solution will expand as follows

$$(2.5) \quad e^{-(\varepsilon k^2 + i\nu k)\Delta t} = e^{-(Hy^2 + iCy)} = 1 - iCy - (\frac{C^2}{2} + H)y^2 + i(\frac{1}{6}C^3 + CH)y^3 + o(y^4)$$

From (2.4) and (2.5) we find the difference (error)

$$E = R(h, \Delta t) - e^{-(Hy + iC)y} = (H - B - C\beta_1)y^2 + \frac{i}{4}[-2C\alpha_1 + 4B\beta_1 + 4C\beta_1(C + \beta_1) + \frac{2}{3}C + 4C(B - H) + \frac{1}{3}C^3]y^3 + o(y^4)$$

The β_1 parameter will be chosen so that

$$(2.6) \quad H - B - C\beta_1 = 0 \\ \beta_1 = -\frac{\alpha}{2} + \frac{H}{2C} (2 - \alpha_{3G})$$

The E difference is reduced to

$$E = \frac{i}{4}y^3 \left\{ (-2\alpha_1 + \frac{2}{3} + \frac{1}{3}C^2)C + 4\beta_1[B + C(C + \beta_1)] + 4C(B - H) \right\} + o(y^4)$$

The α_1 parameter will be chosen so that

$$(2.7) \quad -2\alpha_1 + \frac{2}{3} + \frac{1}{3}C^2 = 0 \quad \alpha_1 = \frac{1}{6}(2 + C^2)$$

By these determinations of the α_1 and β_1 parameters, the E difference is reduced to

$$E = i \left\{ \beta_1[B + C(C + \beta_1)] + C(B - H) \right\} (kh)^3 + o(k^4h^4)$$

Hence, having $B = H - C\beta_1$, in terms of C and H we get

$$(2.8) \quad E = \frac{i}{2} \left[-H\alpha + \frac{H^2}{C} (2 - \alpha_{3G}) \right] (kh)^3 + o(k^4 h^4)$$

If the grid Reynolds (Péclet) number is introduced $P = \frac{vh}{\varepsilon}$, since $C = v \frac{\Delta t}{h}$ and $H = \varepsilon \frac{\Delta t}{h^2}$, the following relation will be found

$$H = \frac{1}{2} \frac{C}{P}$$

For the E error, the following estimation is obtained in terms of C and P

$$(2.9) \quad E = -\frac{iC}{4} \left[\frac{\alpha}{P} + \frac{1}{2P^2} (\alpha_{3G} - 2) \right] (kh)^3 + o(k^4 h^4)$$

This result indicates that in the numerical study of the convection-diffusion modeled by equation (1.1), the E error decreases with the increase of the P Reynolds number in the case of dominant convection, if α_1 and β_1 parameters are chosen according to (2.7) and (2.6).

Since the equation $\alpha_{3G}(r) = 2$, according to (1.6) or to the proposition below, is verified only by $r = 0$, if $\alpha_{3G} = 2$ is chosen we are in the case of piecewise linear approximation. For $\alpha_{3G} = 2$ the formula (2.8) is reduced to the one given in [4].

PROPOSITION. The expression $F(r) = \frac{\alpha}{P} + \frac{\alpha_{3G}(h) - 2}{2P^2}$ defines a positive function on $R - \{0\}$.

Proof. Let us consider the expression (1.6) of α_{3G} . We take the function

$$f(r) = \alpha_{3G}(r) - 2, \quad f(r) = \frac{1}{2h} [-1 + 2r(1 - 2r) + e^{2r}] - 2, \\ r \in R - \{0\}.$$

We have $\lim_{h \rightarrow 0} \alpha_{3G}(r) = 2$ and $\lim_{h \rightarrow 0} f(r) = 0$.

The α_{3G} function may be extended by continuity in $r = 0$, by noting

$$\overline{\alpha}_{3G}(r) = \begin{cases} \alpha_{3G}(r), & r \neq 0 \\ 2, & r = 0 \end{cases}; \quad \overline{f}(r) = \begin{cases} f(r), & r \neq 0 \\ 0, & r = 0 \end{cases}$$

The sign of the $\bar{f}$ function on r is to be determined. We write

$$f(r) = \frac{1}{2h} [e^{2r} - (2r^2 + 2r + 1)]$$

and we introduce the function $g: \mathbb{R} \rightarrow \mathbb{R}$, $g(r) = e^{2r} - (2r^2 + 2r + 1)$. We have $g'(r) = 2[e^{2r} - (2r + 1)]$, $g''(r) = 4(e^{2r} - 1)$ and $g(0) = g'(0) = g''(0) = 0$, $g''(r) < 0$ for $r < 0$, $g''(r) > 0$ for $r > 0$. Therefore $g'(r) \geq 0$ for $r \in \mathbb{R}$ (the equality occurs only for $r = 0$). Hence it results that $g(r) < 0$ for $r < 0$ and $g(r) > 0$ for $r > 0$. Since $f(r) = \frac{1}{2h} g(r)$ and $\bar{f}(r)$ is the extension of the f function in $r = 0$, it is implied that $\bar{f}(r) \geq 0$ for any $r \in \mathbb{R}$ and therefore $\alpha_{3G} \geq 2$ for any $r \in \mathbb{R}$. Consequently, the equation $\alpha_{3G}(r) = 2$ has only a single root $r = 0$.

The parameters α and P have the same sign, namely $\alpha/P \geq 0$. Thus, with $\alpha_{3G}(r) \geq 2$ for any $r \in \mathbb{R}$, we find that P , the $(kh)^3$ coefficient from (2.9), is non-negative. This is zero only in the case of simultaneous presence of $\alpha = 0$ and $\alpha_{3G} = 2$, which implies a discrete Galerkin method with piecewise linear approximation.

c) Calculation of the discrete damping parameter and of discrete frequency ($\tilde{D}$ and $\tilde{\omega}$). The solving of the Galerkin-Crank-Nicolson equation will be continued with the study of the equation (2.2), for which a solution of the form

$$(2.10) \quad v_n = e^{-\tilde{\lambda} n \Delta t}, \quad \tilde{\lambda} = \tilde{D} + i\tilde{\omega}$$

is chosen, where $\tilde{\omega}$ is the discrete frequency and $\tilde{D}$, the discrete damping coefficient. By verifying the equation (2.2) with this solution, the following identity is obtained

$$(2.11) \quad R(h, \Delta t) e^{(\tilde{D} + i\tilde{\omega}) \Delta t} \equiv 1$$

which, after replacing the $R(h, \Delta t)$ amplification factor by its expression (2.3), leads to the identity

$$(2.12) \quad e^{-\tilde{D}\Delta t} [2 - 2(\alpha_1 + B)(1 - \cos kh) + i(2\beta_1 + C)\sin kh] = \\ = [2 - 2(\alpha_1 + B)(1 - \cos kh) \cos \tilde{\omega}\Delta t - (2\beta_1 - C)\sin kh \sin \tilde{\omega}\Delta t + \\ + i\{[2 - 2(\alpha_1 - B)(1 - \cos kh) \sin \tilde{\omega}\Delta t + (2\beta_1 - C)\sin kh \cos \tilde{\omega}\Delta t\}.$$

From this identity the $\tilde{D}$ and $\tilde{\omega}$ parameters are obtained by using the formulas

$$(2.13) \quad \operatorname{tg} \tilde{\omega}\Delta t = \frac{E_1}{E_2}, \quad \tilde{D}\Delta t = \frac{\sqrt{E_1^2 + E_2^2}}{E_3}$$

$$(2.14) \quad \begin{aligned} E_1 &= 4[C - (\alpha_1 C + 2\beta_1 B)(1 - \cos kh)] \sin kh; \\ E_2 &= (4\beta_1^2 - C^2)\sin^2 kh + 4[1 - 2\alpha_1(1 - \cos kh) + \\ &\quad + (\alpha_1^2 - B^2)(1 - \cos kh)^2]; \\ E_3 &= (2\beta_1 - C)^2 \sin^2 kh + 4[1 - (\alpha_1 + B)(1 - \cos kh)]^2 \end{aligned}$$

where the C and B coefficients are given in (1.10), and α_1 and β_1 are chosen according to (2.7) and (2.6). The problem contains only one parameter, α (from β_1 and B).

3. Galerkin Procedure with Combined Interpolation Functions

This procedure is characterized by

$$(3.1) \quad \alpha = 0, \alpha_1 = \frac{1}{3}, \beta_1 = 0, \alpha_{3G} \neq 0; B = \frac{\alpha_{3G}}{2} H$$

The equation (1.9) takes the form

$$(3.2) \quad \begin{aligned} &(\frac{1}{3} - \frac{1}{2}\alpha_{3G}H - \frac{1}{2}C)U_{j-1,n} + 2(\frac{2}{3} + \frac{1}{2}\alpha_{3G}H)U_{j,n} + \\ &+ (\frac{1}{3} - \frac{1}{2}\alpha_{3G}H + \frac{1}{2}C)U_{j+1,n} = \\ &= (\frac{1}{3} + \frac{1}{2}\alpha_{3G}H + \frac{1}{2}C)U_{j-1,n-1} + 2(\frac{2}{3} - \frac{1}{2}\alpha_{3G}H)U_{j,n-1} + \\ &+ (\frac{1}{3} + \frac{1}{2}\alpha_{3G}H - \frac{1}{2}C)U_{j+1,n-1} \end{aligned}$$

$$(C = \frac{\nu\Delta t}{h} = \frac{\omega\Delta t}{kh}; \quad H = \frac{\varepsilon\Delta t}{h^2} = \frac{D\Delta t}{k^2h^2})$$

The numerical solution of the equation (3.2) is

$$U_{j,n} = v_n(k) e^{ikjh} = e^{-(\tilde{D} + i\tilde{\omega})n\Delta t + ikjh}$$

with

$$\operatorname{tg}(\tilde{\omega} \Delta t) = \frac{E_1}{E_2}, \quad \tilde{D} \Delta t = \frac{\sqrt{E_1^2 + E_2^2}}{E_3}$$

where, according to formulas (2.14), E_1 , E_2 and E_3 have the expressions

$$\begin{aligned} E_1 &= \frac{4}{3} (2 + \cos kh) \frac{\sin kh}{kh} \omega \Delta t \\ (3.3) \quad E_2 &= \frac{4}{9} (2 + \cos kh)^2 - \left[\left(\frac{\sin kh}{kh} \right)^2 + \left(\alpha_{3G} \frac{1 - \cos kh}{k^2 h^2} \frac{D}{\omega} \right)^2 \right] (\omega \Delta t)^2 \\ E_3 &= \frac{4}{9} (2 + \cos kh - 3 \frac{\alpha_{3G}}{2} \frac{1 - \cos kh}{k^2 h^2} D \Delta t)^2 + \left(\frac{\sin kh}{kh} \omega \Delta t \right)^2 \end{aligned}$$

Let us study, first, the $\tilde{\omega}$ discrete frequency, with formula

$$\tilde{\omega} \Delta t = \arctg \frac{E_1}{E_2}$$

The $\tilde{\omega}$ discrete frequency can be expressed by the formula

$$(3.4) \quad \frac{\tilde{\omega}}{\omega} = \frac{1}{\tilde{\omega} \Delta t} \arctg \frac{\tilde{\omega} \Delta t}{1 - [1 + a(kh)]^2 \left(\frac{\tilde{\omega} \Delta t}{2} \right)^2}$$

if we choose $\alpha_{3G} = \frac{\omega}{D}$ and we note

$$(3.5) \quad \bar{\omega} = 3 \frac{\sin kh}{kh (2 + \cos kh)} \omega$$

$$(3.6) \quad a(kh) = \frac{1 - \cos kh}{k^2 h^2} \frac{kh}{\sin kh}$$

Remarks. Interpretation of $\bar{\omega}$. Let us appeal again to the Galerkin equation (1.8) - with respect to discretization only in the x variable - for the case here considered, $\alpha = 0$, $\alpha_1 = \frac{1}{3}$, $\beta_1 = 0$:

$$\begin{aligned} (3.7) \quad \ddot{U}_{j-1}(t) + 4 \ddot{U}_j(t) + \ddot{U}_{j+1}(t) - \frac{3}{h} \left[\left(v + \frac{\varepsilon}{h} \alpha_{3G} \right) U_{j-1}(t) - \right. \\ \left. - 2 \frac{\varepsilon}{h} \alpha_{3G} U_j(t) - \left(v - \frac{\varepsilon}{h} \alpha_{3G} \right) U_{j+1}(t) \right] = 0 \end{aligned}$$

When using a solution of the form $U_j = V_j T(t)$, where $V_j = V(x_j)$, the Galerkin semi-discrete equation (3.7) becomes

$$(3.8) \quad \frac{3}{h} \frac{(1 + \frac{\varepsilon}{h} \alpha_{3G}) V_{j-1} - 2 \frac{\varepsilon}{h} \alpha_{3G} V_j - (1 - \frac{\varepsilon}{h} \alpha_{3G}) V_{j+1}}{V_{j-1} + 4 V_j + V_{j+1}} = \frac{\dot{T}}{T} = -\bar{\lambda}$$

where $\bar{\lambda}$ is a constant (independent of x and t).

Taking into account the initial condition, it may be written that

$$V_j T(0) = u_0 e^{ikx_j} = u_0 e^{ikjh}$$

Therefore, if we note $\bar{\lambda} = \bar{D} + i\bar{\omega}$, the following formula will be obtained from (3.8)

$$-\bar{\lambda} = -(\bar{D} + i\bar{\omega}) = \frac{3}{h} \frac{\frac{\varepsilon}{h} \alpha_{3G} (\cos kh - 1) - i \nu \sin kh}{2 + \cos kh}$$

So, $\bar{\omega}$ introduced by (3.5) represents the semi-discrete harmonic frequency of the Galerkin equation (a discrete equation only in the x variable).

- Now, the expressions (2.7)-(2.6) for the α_1 and β_1 parameters and respectively (1.10) for C , H and B are used. Let us introduce the D and ω exact parameters. We have the formulas

$$C = \frac{\Delta t}{kh} \omega, \quad H = \frac{\Delta t}{k^2 h^2} D, \quad B = \frac{\Delta t}{2kh} \left(\alpha + \frac{\alpha_{3G}}{kh} \frac{D}{\omega} \right) \omega$$

$$\alpha_1 = \frac{1}{3} + \frac{1}{6} \left(\frac{\Delta t}{kh} \omega \right)^2, \quad \beta_1 = -\frac{1}{2} \left(\alpha + \frac{\alpha_{3G} - 2}{kh} \frac{D}{\omega} \right)$$

By using these relations, the following expressions will be obtained from (2.14)

$$E_1 = 2 \frac{\Delta t}{kh} \left\{ 2\omega - \left[\frac{2}{3} \left(1 + \frac{1}{2} \frac{\Delta t^2}{k^2 h^2} \omega^2 \right) \omega - \left(\alpha + \frac{\alpha_{3G} - 2}{kh} \frac{D}{\omega} \right) \cdot \omega \left(\alpha + \frac{\alpha_{3G}}{kh} \frac{D}{\omega} \right) \right] (1 - \cos kh) \right\} \sin kh;$$

$$E_2 = \left[\left(\alpha + \frac{\alpha_{3G}^{-2}}{k h} \frac{D}{\omega} \right)^2 - \frac{\Delta t^2}{k^2 h^2} \omega^2 \right] \sin^2 kh + 4 \left\{ 1 - \frac{2}{3} \left(1 + \frac{\Delta t^2}{2 k^2 h^2} \omega^2 \right) (1 - \cos kh) + \left[\frac{1}{9} \left(1 + \frac{\Delta t^2}{2 k^2 h^2} \omega^2 \right)^2 - \frac{\Delta t^2}{4 k^2 h^2} (\alpha \omega + \frac{\alpha_{3G}}{k h} D)^2 \right] (1 - \cos kh)^2 \right\} ;$$

$$E_3 = \left(\alpha + \frac{\Delta t}{k h} \omega + \frac{\alpha_{3G}^{-2}}{k h} \frac{D}{\omega} \right)^2 \sin^2 kh + 4 \left\{ 1 - \left[\frac{1}{3} + \frac{\Delta t}{2 k h} \left(\frac{1}{3} \frac{\Delta t}{k h} \omega^2 + \alpha \omega + \frac{\alpha_{3G}}{k h} D \right) \right] (1 - \cos kh) \right\}^2$$

If the $o(\Delta t^3)$ terms are removed, the expressions of E_1 , E_2 and E_3 are reduced to the form

$$E_1 = \frac{4}{3} \frac{\sin kh}{k h} \omega \Delta t \left\{ 2 + \cos kh + \frac{3}{2} \alpha_{3G} (\alpha_{3G}^{-2} - 2) \frac{1 - \cos kh}{k^2 h^2} \frac{D^2}{\omega^2} + \alpha kh \left[\alpha kh + (\alpha_{3G} + \alpha_{3G}^{-2}) \frac{D}{\omega} \right] \frac{1 - \cos kh}{k^2 h^2} \right\} ;$$

$$E_2 = E_2' - \left[\frac{\sin^2 kh}{k^2 h^2} + \frac{4(2 + \cos kh)}{9} \frac{1 - \cos kh}{k^2 h^2} + (\alpha kh + \alpha_{3G} \frac{D}{\omega})^2 \left(\frac{1 - \cos kh}{k^2 h^2} \right)^2 \right] (\omega \Delta t)^2$$

$$E_3 = E_3' - \left[\frac{4}{3} \alpha_{3G} (2 + \cos kh) \frac{1 - \cos kh}{k^2 h^2} - 2(\alpha_{3G}^{-2} - 2) \frac{\sin^2 kh}{k^2 h^2} \right] D \Delta t + \alpha \left[2 \frac{\sin^2 kh}{k h} - \frac{4}{3} (2 + \cos kh) \frac{1 - \cos kh}{k h} \right] \omega \Delta t + (\omega \Delta t)^2 \left[\frac{\sin kh}{k^2 h^2} - \frac{4(2 + \cos kh)}{9} \frac{1 - \cos kh}{k^2 h^2} + (\alpha kh + \alpha_{3G} \frac{D}{\omega})^2 \left(\frac{1 - \cos kh}{k^2 h^2} \right)^2 \right]$$

where

$$E_2' = \frac{4}{9} (2 + \cos kh)^2 + \left[(\alpha_{3G} - 2) \frac{\sin kh}{k h} \frac{D}{\omega} \right]^2 + \alpha kh \left[\alpha kh + 2(\alpha_{3G} - 2) \frac{D}{\omega} \right] \frac{\sin^2 kh}{k^2 h^2}$$

The discrete frequency and the discrete damping parameters will

be calculated with the formulas (2.13).

4. The Method of Separation of Variables

Let us use, for the numerical study of the convection-diffusion equation, the method of separation of variables. This method will be applied to the equation (1.7). Thus, a solution of the form

$$U_j(t) = U(x_j, t) = V(x_j) T(t) = V_j T(t)$$

By replacing this solution into (1.7) the equations we obtain

$$(4.1) \quad \frac{1}{2} (\bar{\alpha} \alpha_1 - \frac{v}{h} \alpha - \frac{2}{h^2} \alpha_{3G}) \delta^2 V_j + (\bar{\alpha} \beta_1 + \frac{v}{h}) \mu V_j + \bar{\alpha} V_j = 0$$

$$(4.2) \quad \frac{dT}{dt} - \bar{\alpha} T = 0$$

where $\bar{\alpha}$ is a constant value, independent of x and t , that is to be determined. The equation (4.1) is a finite difference equation (μ and δ^2 are operators of the finite differences), while (4.2) is an ordinary differential equation in terms of the T function. Thus, the solution of the problem is of the form

$$U_j(t) = e^{\bar{\alpha} t} V_j$$

Using the initial conditions (1.2), we have

$$V_j T(0) = u_0 e^{ikx_j} = u_0 e^{ikjh}$$

$$(4.3) \quad \mu V_j = \frac{1}{2} (V_{j+1} - V_{j-1}) = i V_j \sin kh$$

$$\delta^2 V_j = V_{j+1} - 2 V_j + V_{j-1} = 2 V_j (\cos kh - 1)$$

From the equation (4.1) we obtain the parameter

$$(4.4) \quad \bar{\lambda} = -(\bar{D} + i\bar{\omega})$$

where

$$(4.5) \quad \bar{D} = \frac{(\omega \alpha k h + D \alpha_{3G}) \frac{1 - \cos kh}{k^2 h^2} [1 - \alpha_1 (1 - \cos kh)] + \omega \beta_1 \frac{\sin^2 kh}{k h}}{[1 - \alpha_1 (1 - \cos kh)]^2 + \beta_1^2 \sin^2 kh}$$

$$(4.6) \quad \bar{\omega} = \frac{-(\omega \alpha k h + D \alpha_{3G}) \frac{1 - \cos kh}{k^2 h^2} \beta_1 \sin kh + \omega [1 - \alpha_1 (1 - \cos kh)] \frac{\sin kh}{k h}}{[1 - \alpha_1 (1 - \cos kh)]^2 + \beta_1^2 \sin^2 kh}$$

Here $\bar{D}$ is the semi-discrete damping coefficient and $\bar{\omega}$ is the semi-discrete frequency. These values correspond to the discretization of the x variable only.

The discretization of the t variable. We take the differential equation (4.2), where the independent variable is the time t , and the unknown function is $T(t)$, $t \in [0, \infty)$. The interval $[0, \infty)$ - that is the time axis - will be discretized at the points $t_n = n \Delta t$, $n = 0, 1, 2, \dots$. A finite differences equation of Crank-Nicolson type will be associated to the equation (4.2), so that

$$\left(\frac{dT}{dt} \right)_{n-\frac{1}{2}} - \bar{\lambda} T_{n-\frac{1}{2}} = 0, \quad n = 1, 2, 3, \dots$$

where

$$\left(\frac{dT}{dt} \right)_{n-\frac{1}{2}} = \frac{T_n - T_{n-1}}{\Delta t} + o(\Delta t^2)$$

$$T_{n-\frac{1}{2}} = \frac{1}{2} (T_n + T_{n-1}) + o(\Delta t^2)$$

The following finite differences equation is obtained

$$(4.7) \quad \left(1 + \frac{\bar{D} + i\bar{\omega}}{2} \Delta t \right) T_{n+1} = \left(1 - \frac{\bar{D} + i\bar{\omega}}{2} \Delta t \right) T_n$$

having a second order approximation (truncation error of the Δt^2)

order). For this equation a solution will be considered ($i = \sqrt{-1}$) in the form

$$(4.8) \quad T_n = T_0 e^{\tilde{\lambda}_1 t_n} = T_0 e^{\tilde{\lambda}_1 n \Delta t},$$

where

$$\tilde{\lambda}_1 = -(\tilde{D}_1 + i\tilde{\omega}_1)$$

We verify (4.7) with the solution (4.8) and we obtain the identity

$$\begin{aligned} & \left(1 + \frac{\bar{D} + i\bar{\omega}}{2} \Delta t\right) e^{-\tilde{D}_1 \Delta t} (\cos \tilde{\omega}_1 \Delta t - i \sin \tilde{\omega}_1 \Delta t) = \\ & = 1 - \frac{\bar{D} + i\bar{\omega}}{2} \Delta t \end{aligned}$$

Hence, the following system results for $\tilde{D}_1$ and $\tilde{\omega}_1$

$$\begin{aligned} 2 \cos \tilde{\omega}_1 \Delta t + \bar{D} \Delta t \cos \tilde{\omega}_1 \Delta t + \bar{\omega} \Delta t \sin \tilde{\omega}_1 \Delta t &= \\ &= (2 - \bar{D} \Delta t) e^{\tilde{D}_1 \Delta t} \\ 2 \sin \tilde{\omega}_1 \Delta t + \bar{D} \Delta t \sin \tilde{\omega}_1 \Delta t - \bar{\omega} \Delta t \cos \tilde{\omega}_1 \Delta t &= \\ &= \bar{\omega} \Delta t e^{\tilde{D}_1 \Delta t} \end{aligned}$$

From this system the formulas

$$(4.9) \quad \operatorname{tg}(\tilde{\omega}_1 \Delta t) = \frac{4 \bar{\omega} \Delta t}{4 - (\bar{D}^2 + \bar{\omega}^2) \Delta t^2}$$

$$(4.10) \quad e^{-\tilde{D}_1 \Delta t} = \frac{\sqrt{(4 \bar{\omega} \Delta t)^2 + [4 - (\bar{D}^2 + \bar{\omega}^2) \Delta t^2]^2}}{(2 + \bar{D} \Delta t)^2 + (\bar{\omega} \Delta t)^2}$$

are obtained by means of which, depending on $\bar{\omega}$ and $\bar{D}$ the discrete frequency $\tilde{\omega}_1$ and the discrete damping coefficient $\tilde{D}_1$ will be calculated. In their turn, $\bar{\omega}$ and $\bar{D}$ are expressed by the formulas (4.6) and (4.5).

Pure convection. Let us suppose that $D = 0$, ($\bar{D} = 0$), which corresponds to the pure convection case, and take $\alpha = 0$, $\alpha_1 = \frac{1}{3}$, $\beta_1 = 0$ ($\alpha_{3G} \neq 0$). Then, the expression of $\bar{\omega}$ from (4.6) is reduced to

$$(4.11) \quad \frac{\bar{\omega}}{\omega} = \frac{3 \sin kh}{kh(2 + \cos kh)}$$

which is identical with (3.5). The formula (4.9) becomes ($\tilde{\omega}_1 \Delta t > 0$)

$$\operatorname{tg}(\tilde{\omega}_1 \Delta t) = \frac{4 \bar{\omega} \Delta t}{4 - (\bar{\omega} \Delta t)^2} = \frac{\bar{\omega} \Delta t}{1 - (\frac{\bar{\omega} \Delta t}{2})^2}$$

For reasons of simplification, we will note $\bar{\omega} \Delta t = 2x$ and from the previous formula we obtain ($\bar{\omega} \Delta t > 0$)

$$\frac{\tilde{\omega}_1}{\bar{\omega}}(x) = \begin{cases} \frac{1}{2x} \operatorname{arc} \operatorname{tg} \frac{2x}{1-x^2}, & 0 < x < 1 \\ \frac{1}{2x} (\operatorname{arc} \operatorname{tg} \frac{2x}{1-x^2} + \pi), & x > 1 \end{cases}$$

The function $\tilde{\omega}_1 / \bar{\omega}$ defined on $I = (0, \infty) - \{1\}$ has the following properties

$$1^\circ \quad \lim_{x \rightarrow 0} \frac{\tilde{\omega}_1}{\bar{\omega}}(x) = 1 \quad ; \quad 2^\circ \quad \lim_{x \rightarrow 1} \frac{\tilde{\omega}_1}{\bar{\omega}}(x) = \frac{\pi}{4} \quad ;$$

$$3^\circ \quad \text{The } \left(\frac{\tilde{\omega}_1}{\bar{\omega}} \right)' \text{ derivative is defined on } I = (0, \infty) - \{1\}$$

and

$$a) \quad \lim_{x \rightarrow 0} \left(\frac{\tilde{\omega}_1}{\bar{\omega}}(x) \right)' = 0 \quad ; \quad b) \quad \lim_{x \rightarrow 1} \left(\frac{\tilde{\omega}_1}{\bar{\omega}}(x) \right)' = -\frac{\pi-2}{4} \quad ;$$

$$c) \quad \left(\frac{\tilde{\omega}_1}{\bar{\omega}}(x) \right)' < 0 \quad \text{for any } x \in I.$$

Thus, the $\tilde{\omega}_1 / \bar{\omega}$ function is strictly decreasing on I .

Similarly, the $(\bar{\omega} / \omega)(kh)$ function is strictly decreasing, and the $\bar{\omega} / \omega$ and $\tilde{\omega}_1 / \omega$ functions are positive.

Taking into account the results above, we can conclude, from the equality $\tilde{\omega}_1 / \omega = (\tilde{\omega}_1 / \bar{\omega})(\bar{\omega} / \omega)$, that the Galerkin-Crank-Nicolson scheme defines a convenient procedure for the numerical solving of the evolution problems considered.

The upwind type procedure, described in this paper, can also be applied to the dominant convection problem - in which case, as it is well-known, the Galerkin classical scheme becomes unstable even if combined trial functions or linear function are used.

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