75

"BABES - BOLYAI" University
Faculty of Mathematics and Physics
Research Seminars
Seminar on Functional Analysis and Numerical Methods
Preprint Nr. 1, 1987, pp.

BEST BOUND FOR THE ARGUMENT OF CERTAIN ANALYTIC FUNCTIONS WITH POSITIVE REAL PART (II)

Petru T. Mocanu, Mihai Popovici and Dumitru Ripeanu

1. Let $\gamma > 0$ and denote by P_γ the class of functions p, with $p(0) = 1$, which are analytic in the unit disc $U = \{z ; |z| < 1\}$ and satisfy

$$\text{(1)} \qquad \mathrm{Re}\left[p(z) + \frac{1}{\gamma} z p'(z)\right] > 0 \quad , \qquad z \in U .$$

It is well-known that all functions in P_γ have positive real part in U and a natural problem is to find

$$\text{(2)} \qquad \varphi_\gamma = \sup\left\{ |\arg p(z)| \ ; \ z \in U, \ p \in P_\gamma \right\}$$

In [2] we showed that

$$\varphi_1 = 0.91162190\ldots \quad (52^\circ.23\ldots)$$

and in this paper we shall show that

$$\varphi_2 = 1.15579\ldots \quad (66^\circ.15\ldots)$$

where φ_2 is given by a rather complicated equation.

2. According to a well-known result due to Hallenbeck and Rusheweyh [1] the inequality (1) implies the sharp inclusion $p(U) \subset q_\gamma(U)$, where

(3) $$q_\gamma(z) = \frac{\gamma}{z^\gamma}\int_0^z \frac{1-s}{1+s}\, s^{\gamma-1}\, ds\ , \quad z \in U\ .$$

The domain $q_\gamma(U)$, which lies in the right half-plane, is convex and symmetric with respect to the real axis and φ_γ defined by (2) is given by

$$\varphi_\gamma = \max_{0\le\theta\le\pi} |\arg q_\gamma(e^{i\theta})|$$

For $\gamma = 2$ from (3) we deduce $q_2(z) = 2Q_2(z)$, where

(4) $$Q_2(z) = -\frac{1}{2} + \frac{2}{z} - \frac{2}{z^2}\log(1+z)\ , \quad z \in U$$

and

(5) $$\varphi_2 = \max_{0\le\theta\le\pi} |\arg Q_2(e^{i\theta})|\ .$$

3. PROPOSITION 1. Let

(6) $$\varphi(\theta) = \arg Q_2(e^{i\theta})\ , \quad 0 \le \theta \le \pi\ ,$$

where Q_2 is given by (4). The behaviour of φ is given by the following table

θ	0		θ_0		π
$\varphi(\theta)$	0	$\searrow$	$-\varphi_2$	$\nearrow$	0

Table 1.

where $\varphi_2 = \mathrm{arctg}(-\Phi(\theta_0))$, $(0 < \varphi_2 < \frac{\pi}{2})$, with *)

(7) $$\Phi(\theta) = \frac{-2\sin\theta - \theta\cos 2\theta + 2\sin 2\theta\log(2\cos\frac{\theta}{2})}{-\frac{1}{2} + 2\cos\theta - \theta\sin 2\theta - 2\cos 2\theta\log(2\cos\frac{\theta}{2})}$$

and $\theta_0 \in (\frac{\pi}{2}, \pi)$ is the unique root in the interval $(0, \pi)$ of the function

(8) $$H_0(\theta) = 13 - 4\cos\theta - 2\theta\,\mathrm{tg}\frac{\theta}{2}(7 + 4\cos\theta - 2\cos^2\theta) + 4\theta^2 - 8(1 + 3\cos\theta - \cos^2\theta)\log(2\cos\frac{\theta}{2}) + 16\log^2(2\cos\frac{\theta}{2})$$

or, equivalently, of the function

*) where log is the natural logarithm and log1 = 0

(9) $$H_1(\theta) = 1 + 3\cos\theta - \cos^2\theta + \sqrt{R(\theta)} - 4\log(2\cos\frac{\theta}{2})$$

with

(10) $$R(\theta) = \left[-2\theta + \mathrm{tg}\frac{\theta}{2}(4 + 2\cos\theta - \cos^2\theta)\right]\left[2\theta - \sin\theta(3 - \cos\theta)\right]$$

Proof. From (4) we deduce

$$Q_2(e^{i\theta}) = u(\theta) + i\, v(\theta)\ ,$$

where

(11) $$\begin{aligned} u(\theta) &= -\frac{1}{2} + 2\cos\theta - \theta\sin 2\theta - 2\cos 2\theta\log(2\cos\frac{\theta}{2}) \\ v(\theta) &= -2\sin\theta - \theta\cos 2\theta + 2\sin 2\theta\log(2\cos\frac{\theta}{2}) \end{aligned}$$

and

$$\varphi(\theta) = \mathrm{arctg}\,\Phi(\theta)\ ,$$

where $\Phi(\theta) = v(\theta)/u(\theta)$ is given by (7).

We have

$$\Phi'(\theta) = -\frac{H_0(\theta)}{2u^2(\theta)}\ ,$$

where H_0 is given by (8). We also can write

(12) $$\begin{cases} \Phi'(\theta) = \dfrac{1}{2u^2(\theta)}\left[R(\theta) - (1 + 3\cos\theta - \cos^2\theta - 4\log(2\cos\frac{\theta}{2}))^2\right] = \\ \qquad = -\dfrac{H_1(\theta)H_2(\theta)}{2u^2(\theta)} \\ H_2(\theta) = 1 + 3\cos\theta - \cos^2\theta - \sqrt{R(\theta)} - 4\log(2\cos\frac{\theta}{2})\ , \end{cases}$$

where H_1 and R are given by (9) and (10) respectively.

We can write

(13) $$R(\theta) = H_3(\theta)\, H_4(\theta)\ ,$$

with

(14) $$\begin{aligned} H_3(\theta) &= \mathrm{tg}\frac{\theta}{2}(4 + 2\cos\theta - \cos^2\frac{\theta}{2}) - 2\theta \\ H_4(\theta) &= 2\theta - \sin\theta(3 - \cos\theta)\ . \end{aligned}$$

Since

$$H_3'(\theta) = \frac{\cos\theta}{1+\cos\theta}\left[2 + \cos\theta - 2\cos^2\theta\right]\ ,$$

in Table 2 we have $0 < \theta_1 = \arccos\frac{-1+\sqrt{17}}{4} < \frac{\pi}{2}$.

θ	0		$\frac{\pi}{2}$		$\pi-\theta_1$		π
$H_3'(\theta)$		$+$	0	$-$	0	$+$	
$H_3(\theta)$	0	$\nearrow$	$4-\pi$	$\searrow$	$H_3(\pi-\theta_1)$	$\nearrow$	∞

Table 2

Since $\cos\frac{5\pi}{24} = \frac{1}{2}\sqrt{\frac{1}{\sqrt{2}}(-1+2\sqrt{2}+\sqrt{3})} > \frac{-1+\sqrt{17}}{4}$

we have $\theta_1 > \frac{5\pi}{24}$ and $-16(\pi-\theta_1) > -\frac{38}{3}\pi > -39.7936$.

Hence we easily obtain

$$H_3(\pi-\theta_1) = \frac{1}{8}\left[3\sqrt{2(43+13\sqrt{17})} - 16(\pi-\theta_1)\right] > 0$$

and from Table 2 we deduce $H_3(\theta) > 0$, for $0 < \theta \le \pi$, so that from (13) we obtain

(15) $\quad \operatorname{sg} R(\theta) = \operatorname{sg} H_4(\theta) \qquad (0 < \theta \le \pi)$.

From (14) we deduce $H_4'(\theta) = (1-\cos\theta)(1-2\cos\theta)$, which gives Table 3.

θ	0		$\frac{\pi}{3}$		θ_2		$\frac{\pi}{2}$		π
$H_4'(\theta)$		$-$	0	$+$					
$H_4(\theta)$	0	$\searrow$	$H_4(\frac{\pi}{3})$	$\nearrow$	0	$\nearrow$	$\pi-3$	$\nearrow$	2π

Table 3

From (12), (15) and Table 3 we deduce

(16) $\quad \Phi'(\theta) < 0 \quad \text{for} \quad 0 \le \theta \le \theta_2$.

On the other hand, we have

(17) $$\begin{cases} H_1'(\theta) = \dfrac{1}{(1+\cos\theta)\sqrt{R(\theta)}}\Big[\sin\theta(2-8\cos\theta+7\cos^3\theta-2\cos^4\theta) - \\ \quad -\theta(1-4\cos\theta-2\cos^2\theta+4\cos^3\theta) - \sin\theta(1-\cos\theta)(1+2\cos\theta)\sqrt{R(\theta)}\Big] \\ H_2'(\theta) = -\dfrac{1}{(1+\cos\theta)\sqrt{R(\theta)}}\Big[\sin\theta(2-8\cos\theta+7\cos^3\theta-2\cos^4\theta) - \\ \quad -\theta(1-4\cos\theta-2\cos^2\theta+4\cos^3\theta) + \sin\theta(1-\cos\theta)(1+2\cos\theta)\sqrt{R(\theta)}\Big] . \end{cases}$$

We can write

(18) $$\left[\sin\theta(2-8\cos\theta+7\cos^3\theta-2\cos^4\theta) - \theta(1-4\cos\theta-2\cos^2\theta+4\cos^3\theta)\right]^2 - \\ -\sin^2\theta(1-\cos\theta)^2(1+2\cos\theta)^2 R(\theta) = (5-4\cos^2\theta)\, H_5(\theta)\, H_6(\theta) ,$$

where

(19) $$H_5(\theta) = \theta - \sin\theta(2-\cos\theta)$$
$$H_6(\theta) = \theta - \sin\theta\,\frac{8-5\cos\theta-2\cos^2\theta}{5-4\cos^2\theta} .$$

Since $H_5'(\theta) = -2\cos\theta(1-\cos\theta)$, we obtain Table 4.

θ	0		$\frac{\pi}{2}$		θ_3		π
$H_5'(\theta)$		$-$	0	$+$			
$H_5(\theta)$	0	$\searrow$	$-\frac{4-\pi}{2}$	$\nearrow$	0	$\nearrow$	π

Table 4

We also have

$$H_6'(\theta) = \frac{2\cos\theta(1-\cos\theta)(1-2\cos\theta)(2+\cos\theta-2\cos^2\theta)}{(5-4\cos^2\theta)^2} ,$$

which gives Table 5, with θ_1 given by Table 2.

θ	0		$\frac{\pi}{3}$		$\frac{\pi}{2}$		$\pi-\theta_1$		θ_4		π
$H_6'(\theta)$		$-$	0	$+$	0	$-$	0	$+$			
$H_6(\theta)$	0	$\searrow$	$H_6(\frac{\pi}{3})$	$\nearrow$	$\frac{5\pi-16}{10} < 0$	$\searrow$	$H_6(\pi-\theta_1)$	$\nearrow$	0	$\nearrow$	π

Table 5

Since Table 4 and Table 5 give $H_5(\theta_3) = H_6(\theta_4) = 0$, i.e

(20) $\quad \theta_3 = \sin\theta_3(2-\cos\theta_3) \;,\quad \theta_4 = \sin\theta_4\,\dfrac{8-5\cos\theta_4-2\cos^2\theta_4}{5-4\cos^2\theta_4}$,

from (17) we deduce

(21) $\quad H_1'(\theta_3) = \dfrac{\sin 2\theta_3}{(1+\cos\theta_3)\sqrt{R(\theta_3)}}(1+2\cos\theta_3)(1-\cos\theta)^2 \ne 0$

(since by (19) we have $H_5(\frac{2\pi}{3}) \neq 0$, hence $\theta_3 \neq \frac{2\pi}{3}$) and

$$(22) \qquad H_2'(\theta_3) = 0 .$$

We also have

$$(23) \qquad H_1'(\theta_4) = \frac{2\sin\theta_4(1-\cos\theta_4)^2(1-2\cos\theta_4)(1+2\cos\theta_4)(2+\cos\theta_4-2\cos^2\theta_4)}{(1+\cos\theta_4)(5-4\cos^2\theta_4)\sqrt{R(\theta_4)}} \neq 0$$

(since in Table 5 $\theta_4 \neq \pi - \theta_1$, hence $2+\cos\theta_4 - 2\cos^2\theta_4 \neq 0$ and in (19) we have $H_6(\frac{2\pi}{3}) \neq 0$, hence $\theta_4 \neq \frac{2\pi}{3}$) and

$$(24) \qquad H_2'(\theta_4) = 0 .$$

In order to be more explicit, we mention that (10) and (20) yield

$$R(\theta_4) = \left[\frac{1-\cos\theta_4}{5-4\cos^2\theta_4}(1-2\cos\theta_4)(2+\cos\theta_4-2\cos^2\theta_4)\right]^2 .$$

Since Table 5 gives $\cos\theta_4 < \cos(\pi-\theta_1) = (-1+\sqrt{17})/4$, we have $2+\cos\theta_4 - 2\cos^2\theta_4 < 0$, so that

$$\sqrt{R(\theta_4)} = -\frac{1-\cos\theta_4}{5-4\cos^2\theta_4}(1-2\cos\theta_4)(2+\cos\theta_4-2\cos^2\theta_4)$$

and we can deduce the values of $H_1'(\theta_4)$ and $H_2'(\theta_4)$.

From (17) and (18) we deduce that the possible roots of $H_1'(\theta)$ and $H_2'(\theta)$ in the interval $(0,\pi)$ are among the roots of $H_5(\theta)$ and $H_6(\theta)$, i.e. the numbers θ_3 and θ_4 .Hence from (21) and (23) we deduce

$$(25) \qquad H_1'(\theta) \neq 0 \qquad (\theta_2 < \theta \leq \pi) .$$

From (14) we obtain $\lim_{\theta \nearrow \pi} \sin\theta\, H_3(\theta) = 2$ and $\lim_{\theta \nearrow \pi} H_4(\theta) = 2\pi$. Therefore (13) yields

$\lim_{\theta \nearrow \pi}\left[\sin\theta\sqrt{R(\theta)}\right] = 0$ and from (17) we deduce

$$\lim_{\theta \nearrow \pi}\left[(1+\cos\theta)\sqrt{R(\theta)}\, H_1'(\theta)\right] = \pi .$$

This shows that in (25) we have $H_1'(\theta) > 0 \quad (\theta_2 < \theta \leq \pi)$ and we obtain the following table.