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ON THE EXACTITY OF THE ERROR EVALUATION IN THE APPROXIMATION BY CUBIC SPLINE OF INTERPOLATION

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In the present paper we complete and extend the results obtained in [1] concerning the evaluation of error in the approximation by cubic spline of interpolation.

Let $f : [a,b] \to R$ be a Lipschitz function on $[a,b]$, i.e.

(1) $$|f(x) - f(y)| \leq \|f\|_L \cdot |x - y| \quad , \quad x,y \in [a,b]$$

where $\|f\|_L$ is the Lipschitz constant given by

(2) $$\|f\|_L = \sup \{ |f(x) - f(y)| / |x - y| \ : \ x,y \in [a,b], \ x \neq y \}.$$

The number $\|f\|_L$ is the smallest Lipschitz constant for f and is called <u>the Lipschitz norm of</u> f .

Let $\Delta : a = x_0 < x_1 < \dots < x_n = b$ be a fixed division of the interval $[a,b]$ and let $f_i = f(x_i)$, $i = 0,1,\dots,n$ the values of the function f on the knots of the division Δ .

The following cubic spline of interpolation

(3) $$s(x) = \frac{M_i - M_{i-1}}{6\, h_i}(x-x_{i-1})^3 + \frac{M_{i-1}}{2}(x-x_{i-1})^2 + m_{i-1}(x-x_{i-1}) + f_{i-1}$$

$x \in [x_{i-1}, x_i]$, $i=1,2,\dots,n$, where $h_i = x_i - x_{i-1}$, $f_i = f(x_i)$

$i = 1,2,\ldots,n$, $m_i = s'(x_i)$, $M_i = s''(x_i)$, $i = 0,1,\ldots,n$, was considered in [1] where it was proved that this spline s is uniquely determined by the conditions :

(i) $s(x_i) = f_i$, $i = 1,2,\ldots,n$,

(ii) $s'(x_i) = m_i$, $i = 1,2,\ldots,n$,

(iii) $m_0 = p$, $M_0 = q$, p,q given real numbers .

The problem we consider is the following :

If $f : [a,b] \to R$ is a given Lipschitz function with norm $\|f\|_L$ then knowing the values $f_i = f(x_i)$, $x_i \in \Delta$, $i = 0,1,\ldots,n$, find evaluations from below and from above for the uniform norm

(4) $$\|s - f\| = \sup\left\{|s(x) - f(x)| \;:\; x \in [a,b]\right\},$$

and, if $f \in C^1[a,b]$ find also evaluations from below and from above for

(5) $$\|s' - f'\| = \sup\left\{|s'(x) - f'(x)| \;:\; x \in [a,b]\right\}.$$

In [1] it was proved also that the evaluations given for the norms (4) are exact in the class of Lipschitz functions having Lipschitz norm $\leq \|f\|_L$ and passing through the points $(x_i , f(x_i))$, $i = 0,1,\ldots,n$.

This problem is considered in many papers, but under more restrictive conditions of f and for cubic splines of interpolation different from (3) . For instance in [2] , [3] , one suppose $f \in W^4_\infty [a,b]$, i.e. f''' is absolutely continuous on $[a,b]$ and $f^{(IV)} \in L_\infty [a,b]$, or more generally $f \in C^2 W^4_\Delta [a,b]$, i.e. $f \in C^2[a,b] \cap W^4_\infty [x_i, x_{i+1}]$, $i = 0,1,\ldots,n-1$.

The evaluations of error are given in terms of the norm of $f^{(IV)}$ and of the norm of the division Δ (the greatest of the distances between two consecutive points in Δ) .

In [1], the delimitations obtained for (4) and (5) are

$$(6) \qquad \min\left\{\|s - F_1\| ; \|s - F_2\|\right\} \le \|s - f\| \le \max\left\{\|s - F_1\| ; \|s - F_2\|\right\}$$

respectively

$$(7) \qquad \min\left\{\|s' + \|f\|_L\| ; \|s' - \|f\|_L\|\right\} \le \|s' - f'\| \le$$

$$\le \max\left\{\|s' + \|f\|_L\| ; \|s' - \|f\|_L\|\right\}$$

if $f \in C^1[a,b]$.

The function F_1, F_2 in (6) are given by

$$(8) \qquad \begin{aligned} F_1(x) &= \sup\left\{ f(x_k) - \|f\|_L \cdot |x - x_k| \ : \ k = 0,1,\dots,n\right\} \\ F_2(x) &= \inf\left\{ f(x_k) + \|f\|_L \cdot |x - x_k| \ : \ k = 0,1,\dots,n\right\} \end{aligned}$$

In the case $f \in C^1[a,b]$ the Lipschitz norm $\|f\|_L$ of f is given by

$$(9) \qquad \|f\|_L = \max\left\{ |f'(x)| \ : \ x \in [a,b]\right\}.$$

2. In the following we shall show that the delimitations (7) are not attained and we shall motivate why the delimitations (6) are attained.

Let $f : [a,b] \to R$ be a Lipschitz function on $[a,b]$ with Lipschitz norm $\|f\|_L$ and let $f_i = f(x_i)$ be the values of f on the knots of the division $\Delta : a = x_0 < x_1 < \dots < x_n = b$. By a result of Mc SHANE [5] there exists at least one function F, F Lipschitz on $[a,b]$, verifying the conditions :

$$F|_\Delta = f|_\Delta \quad \text{and} \quad \|F\|_L = \|f\|_L .$$

Let

$$(10) \qquad \mathcal{E}(f|_\Delta ; [a,b]) = \left\{ F : F \text{ is Lipschitz on } [a,b],\ F|_\Delta = f|_\Delta,\ \|F\|_L \le \|f\|_L \right\}.$$

the set of all Lipschitz extensions of f to $[a,b]$ with Lipschitz norm $\le \|f\|_L$.

Denote by Lip $[a,b]$ the set of all real valued Lipschitz functions defined on $[a,b]$. Then Lip $[a,b]$ is a linear subspace of the Banach space $C[a,b]$ of all real-valued contnuous functions on $[a,b]$ with the uniform norm :

$$(11) \qquad \|f\| = \max\{|f(x)| : x \in [a,b]\} \quad , \quad f \in C[a,b] .$$

PROPOSITION 1. <u>Let</u> $f \in \mathrm{Lip}\,[a,b]$ <u>having the Lipschitz norm</u> $\|f\|_L$ <u>and let</u> $\Delta : a = x_0 < x_1 < \dots < x_n = b$ <u>be a division of</u> $[a,b]$. <u>Then the set</u>

$$\mathcal{E}(f|_\Delta ; [a,b]) = \{F : F \in \mathrm{Lip}[a,b] \ , \ F|_\Delta = f|_\Delta \ , \ \|F\|_L \le \|f\|_L\}$$

<u>has the following properties</u> :

(a) $F_1, F_2 \in \mathcal{E}(f|_\Delta ; [a,b])$, <u>where</u> F_1, F_2 <u>are given by</u> (8) ;

(b) $\mathcal{E}(f|_\Delta ; [a,b])$ <u>is a convex set in</u> Lip$[a,b]$;

(c) <u>The inequalities</u>

$$F_1(x) \le F(x) \le F_2(x) \quad , \quad x \in [a,b]$$

<u>hold for all</u> $F \in \mathcal{E}(f|_\Delta ; [a,b])$;

(d) $\mathcal{E}(f|_\Delta ; [a,b])$ <u>is bounded with respect to the uniform norm</u> (11) ;

(e) $\mathcal{E}(f|_\Delta ; [a,b])$ <u>is equicontinuous</u> ;

(f) $\mathcal{E}(f|_\Delta ; [a,b])$ <u>is closed with respect to the uniform norm</u>;

(g) $\mathcal{E}(f|_\Delta ; [a,b])$ <u>is compact with respect to the uniform norm</u> .

<u>Proof</u>.

(a) . Is obvious since $F_1|_\Delta = F_2|_\Delta = f|_\Delta$ and $\|F_1\|_L = \|F_2\|_L = \|f\|_L$.

(b) . If $F, G \in \mathcal{E}(f|_\Delta ; [a,b])$ and $\lambda \in [0,1]$ then

$$\lambda F + (1-\lambda)G\big|_\Delta = \lambda F\big|_\Delta + (1-\lambda)G\big|_\Delta = \lambda f\big|_\Delta + (1-\lambda)f\big|_\Delta = f\big|_\Delta .$$

and

$$\| \lambda F + (1-\lambda)G \|_L \leq \lambda \|F\|_L + (1-\lambda)\|G\|_L \leq \|f\|_L \quad .$$

(c) . Suppose there exists $u \in [a,b] \setminus \Delta$ such that $F_2(u) < F(u)$. Then there exists $x_i \in \Delta$ such that

$$f(x_i) + \|f\|_L \, |x_i - u| < F(u) \quad .$$

Therefore

$$\frac{f(x_i) - F(u)}{|x_i - u|} < -\|f\|_L$$

or

$$\frac{F(x_i) - F(u)}{|x_i - u|} < -\|f\|_L \quad ,$$

as $F(x_i) = f(x_i)$.

On the other hand

$$\|F\|_L = \sup \{ |F(x) - F(y)| / |x-y| \ : \ x,y \in [a,b] \ , \ x \neq y \} \geq$$

$$\geq \frac{F(u) - F(x_i)}{|x_i - u|} \quad .$$

It follows that

$$-\|F\|_L \leq \frac{F(x_i) - F(u)}{|x_i - u|} < -\|f\|_L$$

in contradiction to the hypothesis $F \in \mathcal{E}(f|_\Delta; [a,b])$. Consequently $F_2(u) \geq F(u)$, for all $u \in [a,b]$.

Similarly , $F_1(u) \leq F(u)$ for all $u \in [a,b]$.

Since on the points in Δ all the functions F_1, F_2, F agree with f , it follows that the inequalities (c) hold for all $x \in [a,b]$.

(d) . By the inequalities (c)

$$\|F\| \leq \max \{ \|F_1\| \ , \ \|F_2\| \} \quad ,$$

for all $F \in \mathcal{E}(f|_\Delta; [a,b])$, so that $\mathcal{E}(f|_\Delta; [a,b])$ is bounded in $C[a,b]$.

(e). For $\varepsilon>0$ take $\delta = \varepsilon/(\|f\|_L + 1)$. Then

$$|F(x) - F(y)| \leq \|f\|_L \cdot |x - y| \quad ,$$

for all $x,y \in [a,b]$ with $|x - y| < \delta$ and all $F \in \mathcal{E}(f|_\Delta ; [a,b])$ which shows that $\mathcal{E}(f|_\Delta ; [a,b])$ is an equicontinuous subset of $C[a,b]$.

(f). Let $(F_n)_{n\geq 1}$ be a sequence in $\mathcal{E}(f|_\Delta ; [a,b])$ converging uniformly to F. Then for all $n \in \mathbb{N}$ and all $x,y \in [a,b]$ one has $|F(x) - F(y)| \leq |F(x) - F_n(x)| + |F_n(x) - F_n(y)| + |F_n(y) - F(y)| \leq$ $2\|F - F_n\| + \|f\|_L \cdot |x - y|$.

Taking $n \to \infty$ one obtains

$|F(x) - F(y)| \leq \|f\|_L \cdot |x - y|$,

so that $\|F\|_L \leq \|f\|_L$. Since $F_n(x_i) = f(x_i)$ it follows $F(x_i) =$ $= f(x_i)$, for all $x_i \in \Delta$, so that $F \in \mathcal{E}(f|_\Delta ; [a,b])$. Therefore $\mathcal{E}(f|_\Delta ; [a,b])$ is closed in $C[a,b]$.

(g). Follows by (d), (e), (f) and Arzelà - Ascoli theorem. Proposition 1 is proved.

THEOREM 1. <u>Let</u> $f \in \mathrm{Lip}[a,b]$ <u>with Lipschitz norm</u> $\|f\|_L$. <u>Then there exist two function</u> $\bar{f}$ <u>and</u> $\underline{f}$ <u>in</u> $\mathcal{E}(f|_\Delta ; [a,b])$ <u>such that</u>

$$\text{(12)} \qquad \|\bar{f} - s\| \leq \|f - s\| \leq \|\underline{f} - s\| \quad ,$$

<u>for all</u> $f \in \mathcal{E}(f|_\Delta ; [a,b])$.

<u>Proof</u>. Follows by the compactness of the set $\mathcal{E}(f|_\Delta ; [a,b])$.

<u>Remarks</u>. a) Using the functions F_1 , F_2 given by (8) , in [1] it was constructed efectively two functions $\bar{f}$, $\underline{f}$ in $\mathcal{E}(f|_\Delta ; [a,b])$ verifying (12).

b) For all $F \in \mathcal{E}(f|_\Delta ; [a,b])$ $\|f|_\Delta\|_L \leq \|F\|_L \leq \|f\|_L$. If $\|f|_\Delta\|_L = \|f\|_L$, then there exists $x_{i_0} \in \Delta$ such that all the functions in $\mathcal{E}(f|_\Delta ; [a,b])$ agree on $[x_{i_0}, x_{i_0+1}]$ with the line passing through $(x_{i_o}, f(x_{i_o}))$ and $(x_{i_o+1}, f(x_{i_o+1}))$.

3 . For $f \in C^1[a,b]$,

$$\|f\|_L = \max \{ |f'(x)| : x \in [a,b] \} .$$

Let $\Delta : a = x_o < x_1 < \dots < x_n = b$ be a fixed division of the interval $[a,b]$ and let

$$\mathcal{E}^1(f|_\Delta ; [a,b]) = \{ F : F \in C^1[a,b] ,\ F|_\Delta = f|_\Delta ;\ \|F\|_L \le \|f\|_L \} .$$

The set $\mathcal{E}^1(f|_\Delta ; [a,b])$ is contained in $\mathcal{E}^1(f|_\Delta ; [a,b])$ and is convex, bounded and equicontinuous (therefore relatively compact in $C[a,b]$) and its closure in $C[a,b]$ is contained in $\mathcal{E}(f|_\Delta ; [a,b])$, that is

$$\overline{\mathcal{E}^1(f|_\Delta ; [a,b])} \subseteq \mathcal{E}(f|_\Delta ; [a,b]) \quad (13)$$

As $\mathcal{E}^1(f|_\Delta ; [a,b])$ is not closed it is not sure that there exist $\overline{f}$ and $\underline{f}$ in $\mathcal{E}^1(f|_\Delta ; [a,b])$ for which the evaluations (7) are attained. The following theorem holds :

THEOREM 2. <u>For every</u> $\varepsilon > 0$ <u>there exist the functions</u> $G_1 , G_2 \in \mathcal{E}^1(f|_\Delta ; [a,b])$ <u>such that</u> $\|F_1\|_L = \|G_1\|_L = \|F_2\|_L = \|G_2\|_L$ <u>and</u>

$$\|F_1 - G_1\| < \varepsilon \ , \ \|F_2 - G_2\| < \varepsilon \quad (14)$$

<u>where</u> F_1 , F_2 <u>are given by</u> (8) .

<u>Proof</u>. Let $i \in \{o,1,\dots,n-1\}$. Consider the functions F_1, F_2 given by (8) on the interval $[x_i , x_{i+1}]$ and construct the functions G_1 , G_2 verifying the conditions (14) on this interval (see Fig. 1)

The function G_1 is constructed in the following way : The graph of F_2 on $[x_i , x_{i+1}]$ is formed of the segments A_iB_i , B_iA_{i+1} and the graph of F_1 is formed of the segments A_iC_i , C_iA_{i+1} .

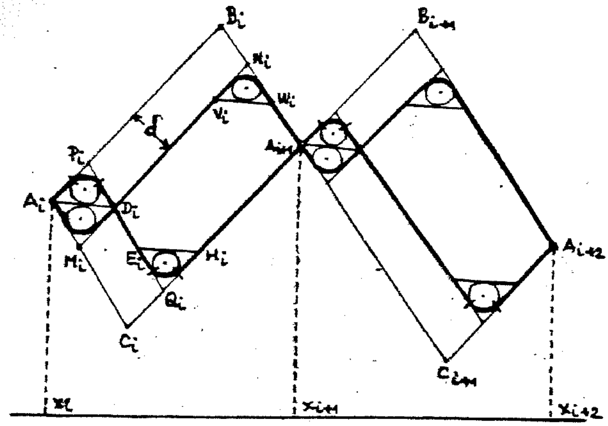

Figure 1 .

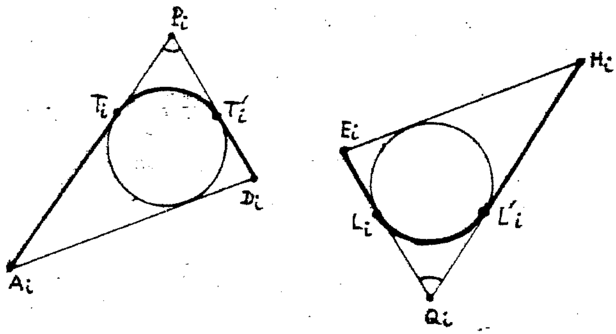

Figure 2 .

Construct M_iN_i A_iB_i and P_iQ_i A_iC_i such that the distance d between A_iB_i and M_iN_i and between P_iQ_i and A_iC_i verifies the inequality

$$d < \frac{\varepsilon}{2\sqrt{1 + \|f\|_L^2}} .$$

The circle inscribed in the triangle $A_iP_iD_i$ is tangent to A_iP_i and P_iD_i in the points T_i and T_i', respectively (Fig. 2).

Let the triangle $E_iH_iQ_i$ be congruent to the triangle $A_iP_iD_i$. The circle inscribed in the triangle $E_iH_iQ_i$ is tangent to E_iQ_i and to Q_iH_i in the points L_i and L_i', respectively.

On the interval $[x_i , x_{i+1}]$, the graph of G_1 is formed of the line segment A_iT_i, the circle arc $\overset{\frown}{T_iT_i'}$, the line segment T_iL_i, the circle arc $\overset{\frown}{L_iL_i'}$ and the line segment $L_i'A_{i+1}$.

The graph of G_2 is obtained similarly on $[x_i , x_{i+1}]$ using the circles inscribed in the congruent triangles $A_iM_iD_i$ and $V_iN_iW_i$.

Repeating the construction on the interval $[x_{i+1}, x_{i+2}]$ as long as $i+2<n$, one obtains the function G_1 and G_2 on $[a,b]$. Theorem 2 is proved.

From this Theorem one obtains the following corollary :

COROLLARY. <u>For</u> $f \in \mathcal{E}^1(f|_\Delta;[a,b])$ <u>and</u> $\varepsilon>0$ <u>there exist</u> $G_1, G_2 \in \mathcal{E}^1(f|_\Delta;[a,b])$ <u>such that</u>

$$\min\{\|s-G_1\|, \|s-G_2\|\}-\varepsilon \le \|s-f\| \le \max\{\|s-G_1\|, \|s-G_2\|\}+\varepsilon$$

<u>where</u> s <u>is given by</u> (3).

<u>Proof</u>. Since $f \in \mathcal{E}^1(f|_\Delta;[a,b])$ implies $f \in \mathcal{E}(f|_\Delta;[a,b])$, the inequalities (6) give

$$\min\{\|s-F_1\|, \|s-F_2\|\} \le \|s-f\| \le \max\{\|s-F_1\|; \|s-F_2\|\}.$$

By Theorem 2, for every $\varepsilon>0$, there exist G_1, G_2 such that $\|F_1-G_1\|<\varepsilon$ and $\|F_2-G_2\|<\varepsilon$. Therefore

$$\varepsilon > \|F_1-G_1\| = \|F_1-s+s-G_1\| \ge \big|\|F_1-s\|-\|s-G_1\|\big| ,$$

which implies

$$-\varepsilon \le \|F_1 - s\| - \|s - G_1\| \le \varepsilon$$

$$\|s - G_1\| - \varepsilon \le \|F_1 - s\| \le \|s - G_1\| + \varepsilon .$$

Similarly

$$\|s - G_2\| - \varepsilon \le \|F_2 - s\| \le \|s - G_2\| + \varepsilon$$

These inequalities give

$$\max\{\|F_1 - s\| , \|F_2 - s\|\} \le \max\{\|s - G_1\| , \|s - G_2\|\} + \varepsilon$$

and

$$\min\{\|s - G_1\| , \|s - G_2\|\} - \varepsilon \le \min\{\|F_1 - s\| , \|F_2 - s\|\} ,$$

and the Corollary is proved .

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