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REGULARITY OF SETS IN ORDERED LOCALLY CONVEX SPACES AND THE EXISTENCE OF PARETO EFFICIENT POINTS

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Introduction. The existence of Pareto efficient points (or efficient points, or minimum points) for a set in an ordered topological vector space was investigated recently by different authors. The sufficiency conditions for the existence of such points were obtained either for a rather large class of sets, the sets having closed and order bounded lower sections in an ordered locally convex space with a "good" positive cone (see e.g. Theorem 1 (a) and (b) in (B) and the similar results in (P) or Theorem 2 in (I)), or for a restricted class of sets having certain compacticity like property in an ordered locally convex space with a positive cone which is closed (se e.g. Theorem 1 (c) in (B) or Theorem 1 in (C)).

The common machinery which ensures the existence of efficient points is in fact the convergence of some decreasing lower bounded nets in the respective sets. By introducing a sort of relative regularity of sets the above two different aspects can be handled in a unified way when the underlying ordered topological vector space is normal. It turns out that some regularity type condition is necessary for existence of efficient points. Even not explicitely stated this fact derives from our results in (N1) and (N2) and from some auxiliary results used in (N3).

The aim of the present note is to emphasize on the common part of the above considered two approaches on existence problems of efficient points introducing the notion of regularity of sets in ordered topological vector spaces. We shall see that every closed lower bounded set in a regular ordered locally convex space is regular and so does every compact set in a locally convex space ordered by a closed cone.

The basic tool in our proofs is the employing of an approximative efficiency, called here H-efficiency which was introduced and named H near to minimality in (N1) and (N2). In the proofs of our principal results comprised in Theorems 5 and 6 the condition of the normality of the space is essential. Hence they cannot be considered generalizations of the results in (N2) where normality is not used. It remans to see that by introducing spaces with two cones in the spirit of (N2) and (N3) this condition can be weakened or not.

I. Definitions and preparatory results

Let Y be a vector space over the reals. If K is a cone in Y (i.e., if (i) $K+K \subset K$; (ii) $tK \subset K$ whenever t is a non-negative real, and (iii) $K \cap (-K) = \{0\}$), then there is an antisymmetrical partial order on Y defined by $u \leq v$ exactly when $v-u \in K$. This order is translation invariant and invariant with respect to the multiplication with non-negative reals. The resulting object (Y,K) is called an ordered vector space (o.v.s.) with the positive cone K .

A set M in (Y,K) is said to be lower (upper) bounded if it exists some y in Y such that $y \leq x$ (respectively, $x \leq y$) for every x in M . If M is both lower and upper bounded, it is said to be order bounded. The set $[u,v] := \{z \in Y : u \leq z \leq v$ is called the order interval determined by u and v . Accordingly a set is order bounded if it is contained in some order interval. A set M is called full if $u, v \in M$ imply that $[u,v] \subset M$.

A section of the set M will mean a nonempty set of form $(y-K) \cap M$ with some y in Y.

Suppose that (Y,K) is an o.v.s. and Y is endowed with a vector space topology which is Hausdorff. If K is closed and the topology is locally convex we shall say following (PE) that (Y,K) is an ordered locally convex space (o.l.c.s.). We consider that the context of o.l.c.s.-s is reach enough to allow of a comprehensive treatement of the most important problems in our attention.

An o.l.c.s. (Y,K) or its positive cone K is called normal if Y has a base of neighbourhoods of 0 consisting from full sets.

A decreasing net in (Y,K) is by definition a totally ordered set which is considered directed downward by the ordering of the space. The o.l.c.s. (Y,K) or its positive cone K is called [quasi] regular if every decreasing lower bounded net in K is convergent [is Cauchy].

It is known and easy to check that (Y,K) is [quasi] regular if and only if every decreasing sequence in K is convergent [is Cauchy] (N2). This circumstance allows to use summation criteria for [quasi] regularity of the space. By particularization we can deduce from the results in (N2) and (N3) the following

CRITERION. The o.l.c.s. (Y,K) is quasi regular if and only if for every sequence (y_i) in K the condition $y_i \notin U$ for some neighbourhood U of 0 and for all i implies that the set $\{\sum_{i=1}^{n} y_i : n \in N\}$ cannot be upper bounded.

If K is complete "quasi regular" can be changed in "regular". The questions discussed in the introduction justifies the introduction of a notion of relative regularity of a set as follows:

1. DEFINITION. Let (Y,K) be an o.l.c.s. The set M in Y will be called [quasi] regular if every decreasing net in M converges to an element of M [is Cauchy].

Remarks. It is immediate that a complete set M is regular if and only if it is quasi regular. This is the case also if M is closed and K is complete.

It can be easily shown by a way similar to that in Lemma 1 in (N2) (or (3.1) in (N3)) that "net" in the above definition can be changed in "sequence". That is to say a set is [quasi] regular if and only if it is "sequentially" [quasi] regular. Hence "complete" and "closed" in the above paragraph can be weakened to sequentially complete and sequentially closed.

Using the last remark we have in analogy with the above cited Criterion the following

2. LEMMA. The [complete] set M is quasi regular [regular] if and only if we cannot have a sequence (y_i) with $y_i \in K \setminus U$ for each $i \geq i_o$, where i_o is a given index and U is some neighbourhood of the origin, such that $s_n := -\sum_{i=1}^{n} y_i \in M$ for any n.

Proof. If it exists a sequence (y_i) with $y_i \in K \setminus U$ for $i \geq i_o$ and $s_n \in M$ for any n, then (s_n) is a decreasing sequence in M which is not Cauchy since $s_{n-1}-s_n = y_n \notin U$. Thus M cannot be quasi regular [regular].

If M does not be quasi regular [regular], then there exists a decreasing sequence (z_n) in M which is not Cauchy, i.e., there can be get some neighbourhood U of 0 and a decreasing subsequence of its, which we shall denote also by (z_n) such that $z_{n-1}-z_n \notin U$ for any n. But $z_{n-1}-z_n \in K$, hence $y_n := z_{n-1}-z_n \in K \setminus U$, if

$n \geq 2$, and $z_n = -\sum_{i=1}^{n} y_i \in M$ for each n. Accordingly we have got a contradiction with the condition of the lemma.

Q. E. D.

3. Examples.

(a) Using Definition 1 we see that the o.l.c.s. (Y,K) is quasi regular if and only if every lower bounded set in it is quasi regular. If K is complete then (Y,K) is regular if and only if every closed lowerbounded set in it is regular.

(b) The reasoning in the proof of (c) in Theorem 1 of (B) shows that every compact set in an o.l.c.s. is regular.

II. H-efficient points and regularity of sets

The notion of the H-efficient point of a set we shall introduce generalizes that of ε-efficient point used in the literature of vector minimization. Introduced in (N1) and (N2) it permits the developing of a technique which exploiting the summation criterion yields necessary and sufficient conditions of existence of H-efficient points for sets with lower bounded sections in the above cited papers. Let we remember this definition.

4. DEFINITION. Let (Y,K) be an o.v.s. and let H be a nonempty subset of $K \setminus \{0\}$. Given a set M in Y a point x in M is called an H-efficient point of M if there is no $y \in M$ such that $y \leq x-h$ for each h in H.

The condition in the definition means that there is no y in M which is also in $x-h-K$ for some h in H, that is, it holds

$$(x-H-K) \cap M = \emptyset ,$$

relation which will be taken as definition for H-efficient points

x in M . The set of H-efficient points of M will be denoted by H-eff M .

On the analogy of the results in section 5 of (N2) and section 4 of (N3) we have for regular sets the following

5. THEOREM. Let (Y,K) be a normal o.l.c.s. and let M be a [complete] set in Y . Then the following assertions are equivalent:

(a) M is quasi regular [regular] ;

(b) Each subset of M has H-efficient points for every $H \subset K$ such that $Y \setminus H$ is a neighbourhood of 0 ;

(c) Each closed denumerable subset of M has H-efficient points for every denumerable set $H \subset K$ such that $Y \setminus H$ is a neighbourhood of 0 .

Proof. (a) $\Rightarrow$ (b). Suppose that $M_o \subset M$ has no H-efficient point with $H \subset K \setminus U$ where U is a neighbourhood of 0 . That is,

$$(x-H-K) \cap M_o \neq \emptyset$$

for each x in M_o . Take $x_1 \in M_o$ arbitrarily and let we construct inductively the sequence (x_n) in M_o taking

$$x_{i+1} \in (x_i-H-K) \cap M_o' \quad , \quad i \in N .$$

then $x_{i+1} \leq x_i - h_i$, $i \in N$ for some h_i in H , wherefrom

$$h_i \leq x_i - x_{i+1} \quad , \quad i \in N .$$

By the normality of (Y,K) we can suppose that U is a full neighbourhood of 0 . Then since $h_i \notin U$ the last relation will imply that

$$y_i := x_i - x_{i+1} \notin U \quad , \quad i \in N .$$

But then

$$x_{n+1} = - \sum_{i=1}^{n} y_i + x_1 \in M \quad , \quad n \in N$$

with $y_i \in K \setminus U$ and by Lemma 2 we see that M cannot be quasi regular [regular] .

The implication (b) $\Rightarrow$(c) is trivial.

Let we prove (c)$\Rightarrow$(a) by assuming that M is not quasi regular [regular]. Then by Lemma 2 there exists a sequence (y_i) such that $y_i \in K \setminus U$ for some full neighbourhood U of 0 and each $i \geq i_o$ with i_o a given index, and $s_n := - \sum_{i=1}^{n} y_i \in M$ for each n . Consider $M_o := \{s_n : n \in N\}$ and $H := \{y_i : i \in N\}$. The set M_o is discrete since $s_n - s_{n+p} = \sum_{i=1}^{p} y_{n+i} \notin U$ for any $n, p \in N$. This follows using $y_{n+i} \notin U$ from the condition that U is full. Let x be an arbitrary element of M_o . Then $x = s_m = - \sum_{i=1}^{m} y_i$ for some $m \in N$. We have further

$$x - y_{m+1} = - \sum_{i=1}^{m+1} y_i \in M_o \quad \text{and} \quad x - y_{m+1} \in x - H \subset x - H - K$$

wherefrom

$$(x - H - K) \cap M_o \neq \emptyset \ .$$

Since x was arbitrarily chosen in M_o the above relation shows that M_o cannot have H-efficient points. Now, M_o is denumerable and discrete, hence closed subset of M and H is denumerable in $K \setminus U$. Thus M cannot have property (c) and the required implication follows.

Q. E. D.

III. Existence of efficient points

Let (Y,K) be an o.v.s. and let M be a set in Y . The point

$x \in M$ is called an efficient point (or Pareto efficientpoint, or minimum point) of M if the relations $y \leqslant x$ and $y \in M$ imply $y = x$. The condition of efficiency for a point $x \in M$ can be expressed by the relation

$$(x - K) \cap M = \{x\} .$$

The results in section II can be used to deduce necessary and sufficient conditions for existence of efficient points of a set. We have by analogy with Theorem 5 the following result :

6. THEOREM. Let (Y,K) be a normal o.l.c.s. and let M be a complete set in Y or let K be complete and M be a closed set in Y. Then the following assertions are equivalent:

(a) M is regular ;

(b) Each closed subset of M has efficient points ;

(c) Each closed denumerable subset of M has efficient points.

Proof. (a) $\Rightarrow$(b). Let M_o be a closed subset of M. By Zorn's lemma there exists a totally ordered subset Z in M_o which is maximal with respect to the set theoretic inclusion. Then Z can be considered to be a decreasing net in M_o. Since M is regular this net converges to $z \in M$. Because M_o is closed we have $z \in M_o$. By Corollary II. 3. 2 in (PE) we have also $z \leqslant x$ for each x in Z and since Z is a maximal totally ordered set in M_o it follows that $z \in Z$. The point z must be efficient point of M_o since if $(z-K) \cap M_o$ would contain any point different from z this would contradict the maximality of the totally ordered subset Z in M_o.

The implication (b)$\Rightarrow$(c) is trivial.

(c) $\Rightarrow$(a). Since each denumerable closed set in M has efficient points, it follows that each set of this kind has H-efficient points for every denumerable $H \subset K$ such that $Y \setminus H$ is a neighbourhood of 0. When M is complete it follows that it is regular by (c) of Theorem 5. If M is closed and K is complete then for every x in M the set $(x-K) \cap M$ will be a closed subset of $-K$ translated with its vertex in x. Hence the above set is complete being a closed subset of a complete set. Repeating verbatim the above reasonings for $(x-K) \cap M$ in place of M we deduce that this set is regular. Since x was arbitrarily chosen in M, we deduce that every decreasing net in M is convergent because it has sections in sets of form $(x-K) \cap M$. But these sections are obviously convergent to some element of this last set since this set is regular. This shows that M is itself regular.

IV. Comments

The normality of (Y,K) was used in the proofs of (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (a) of Theorem 5. It is perhaps an essential condition. For Fréchet spaces the normality follows from the regularity of the space (M). There exist non-complete normed regular spaces which are not normal.

Zorn's lemma was used in the proof of (a) $\Rightarrow$ (b) in Theorem 6. Its use is inavoidable. This results in the way followed in (B) section 3 where was investigated the relation of the existence of efficient points in a set with the axiom of choice.

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