University of Cluj-Napoca
Faculty of Mathematics and Physics
Research Seminar on Mechanics
Preprint Nr.1, 1988, pp.35-5o

FINDING VARIATIONAL FUNCTIONALS BY THE GALERKIN PROCEDURE. APPLICATIONS TO THE HEAT TRANSFER

Doina Brădeanu

Several methods are employed in order to determine functionals and variational principles for some boundary value problems (in mathematics, mechanics, physics). These methods apply: the energy functional, the Galerkin method, the Lax-Milgram lemma, the Gateaux derivatives or certain physical principles (the principle of the local potential).

1. Self-Adjoint Problems.

a) The Mixed Boundary Value Problem. Let us consider in the domain $\Omega\subset R^n$ a second order partial differential equation subject to conditions on the boundary $S\ (=\bar{\Omega}-\Omega)$

$$(1')\quad Au=\sum_{i,j=1}^{n} b_{ij}(x)\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i}+b_o(x)u(x)=f(x)$$

$$x\in\Omega\subset R^n$$

$$B_m u=h_m,\quad x\in S_m,\quad m=\overline{1,s}$$

where B_m are the formal operators of the boundary conditions, that may also be reduced to the identity operator I. If we put

$$b_{ij}=-a_{ij},\ a_i=b_i-\sum_{j=1}^{n}\frac{\partial}{\partial x_j}b_{ij}\ ,\ b_o=a_o$$

the equation (1') turns into the equation (1) (see below).

Let us consider the mixed boundary value problem on the domain bonded by S, for the function u ($\in C^2(\Omega) \cap C^1(\bar{\Omega})$)

$$(1) \quad Au = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j}) + \sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i} + a_o u = f(x), \quad x \in \Omega \subset R^n$$

$$(2) \quad B_m u = g_m = \begin{cases} h_m(x), & x \in S_m, \ m = \overline{1,p}, \ p < s \\ \alpha_m(u - \beta_m), & x \in S_m, \ m = \overline{p+1,s} \end{cases} \qquad (m = \overline{1,s})$$

where A is a formal operator of the 2^{nd} order of the equation, B_m are differential operators of the boundary conditions on S:

$$S = S_1 \cup \ldots \cup S_s, \quad S_i \cap S_j = \emptyset, \ i \neq j.$$

B_m cannot contain more than 1^{st} order derivatives or they can be reduced to the identity operator I on certain pieces in S, while the coefficients $a_{ij} \in C^2(\bar{\Omega})$, $a_i \in C^1(\bar{\Omega})$, $a_o \in C(\bar{\Omega})$, $f \in C(\bar{\Omega})$, $h_m \in C(S)$; α_m, β_m, $m = \overline{1,s}$ can be considered constant.

b) <u>The Finding of the Variational Functional for the Problem</u> (1)-(2) <u>by the Galerkin Method</u>. If the equality $\delta J(u)=o$ (where δ is the operator of the variation while J(u) is an integral functional) can be inferred from the problem (1)-(2), then J(u) is called a variational functional for the differential problem (1)-(2). The equality $\delta J(u)=o$ expresses a variational principle (stationarity) for the problem (1)-(2). This principle can be an extremal principle (minimum, maximum). Hence, if a functional J(u) exists (can be found) so that both the equation (1) and the conditions (2) [with $B_m \neq I$] should result from the condition $\delta J=o$ [i.e. (1) should be Euler-Lagrange equation for J], then J(u) is a variational functional for the problem (1)-(2). The boundary conditions

(2), that results from the condition $\delta J=0$ if $B_m \neq I$ are called natural boundary conditions. The conditions (2) with B=I (which does not result from $\delta J=0$) are called essential boundary conditions (they are verified by the set of the admissible functions in the problem of the stationarity of the functional J). From the definition of J(u) there results that the solution of the problem (1)-(2) is a stationary point for the variational functional associated to the problem.

In order to find the variational functional, provided it exists, u is assumed to verify (1)-(2) and we write the equality (Galerkin formulation)

$$\int_\Omega (Au-f)\delta u \, dx = \sum_{m=1}^{s} \int_{S_m} (B_m u - g_m)\delta u \, dS \tag{3}$$

where δu (the variation of u on $\bar{\Omega}$) is an arbitrary function of the same class with u on Ω with the property $\delta u=0$ on S_1, if u on S_1 is given (where $B_1=I$; essential boundary condition) and $\delta u \neq 0$ on S_m if, on S_m, $B_m u$ is a differential expression (natural boundary condition).

By using the Gauss-Ostrogradski formula, we obtain the identity (Green)

$$\int_\Omega (Au)\delta u \, dx = a(u,\delta u) - \sum_{m=1}^{s} \int_{S_m} \frac{\partial u}{\partial n_c}\delta u \, dS$$

if the bilinear form $a(u,\delta u)$ and the derivative along the conormal vector $\partial u/\partial n_c$ on S are introduced with the equalities

$$a(u,\delta u) = \int_\Omega \Big[\sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_j}\frac{\partial \delta u}{\partial x_i} + \sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i}\delta u + a_0 u \delta u\Big] dx$$

(4)

$$\frac{\partial u}{\partial n_c} = \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_j} n_i \, , \quad n_i = \cos(\vec{n}, Ox_i)$$

($\vec{n}$ is the unit vector of the exterior normal at S)

The formulation (3) turns into the following variational equation

(5) $$a(u,\delta u)-\delta\iint_{\Omega} fu\,dx-\int_{S_m}\sum_{n=1}^{s}(B_m u-g_m+\frac{\partial u}{\partial n_c})\delta u\,dS=0$$

since, from a variational point of view, f is a fixed function.

Let us assume, now, that the bilinear form $a(u,\delta u)$ is symmetrical:

(6) $a_{ij}=a_{ji}$; $a_i=0$ and, besides, that $a_{ij}=0$ for $i\neq j$

In this case the formal operator A is symmetrical (self-adjoint). Then, we have

$$a(u,\delta u)=\int_{\Omega}\Big[\sum_{i=1}^{n}a_{ii}\frac{\partial u}{\partial x_i}\frac{\partial(\delta u)}{\partial x_i}+a_o u\delta u\Big]dx=$$

$$=\delta\frac{1}{2}\int_{\Omega}\Big[\sum_{i=1}^{n}a_{ii}\Big(\frac{\partial u}{\partial x_i}\Big)^2+a_o u^2\Big]dx$$

Let us take the differential expressions

$$B_1 u=Iu=u,\ x\in S_1$$

$$B_m u=-\frac{\partial u(x)}{\partial n_c}=-\sum_{i=1}^{n}a_{ii}\frac{\partial u}{\partial x_i}n_i,\ x\in S_m,\ m=\overline{2,s}$$

Under these conditions, the equation (5) is transcribed in the form

$$\delta J(u)=0$$

where the functional $J:U\longrightarrow R^1$ is defined by means of the equality

$$J(u)=\frac{1}{2}\int_{\Omega}\Big[\sum_{i=1}^{n}a_i\Big(\frac{\partial u}{\partial x_i}\Big)^2+a_o u^2-2fu\Big]dx+$$

(7) $$+\sum_{r=2}^{p}\int_{S_r}h_r u\,dS+\frac{1}{2}\sum_{\gamma=1}^{s-p}\int_{S_{p+\gamma}}\alpha_{p+\gamma}(u-\beta_{p+\gamma})^2dS;$$

$$U=\{u\in C^2(\Omega)\cap C^1(\bar{\Omega})\mid B_1u\equiv u(x)=h_1(x)\text{ on }S_1\}$$

The functional J(u), (7) is the variational functional for the following mixed boundary value problem in the domain Ω with the boundary $S(= \bigcup_{m=1}^{s} S_m)$:

(8) $$Au = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (a_i(x) \frac{\partial u}{\partial x_i}) + a_o(x)u(x) = f(x) , \quad x \in \Omega$$

$$B_1 u \equiv u(x) = h_1(x), \quad x \in S_1$$

(9) $$B_m u \equiv - \sum_{i=1}^{n} a_i^{(m)} \frac{\partial u}{\partial x_i} n_i = \begin{cases} h_m(x), \ x \in S_m, \ m = \overline{2,p}, \ p < s \\ \alpha_m(u - \beta_m), \ x \in S_m, m = \overline{p+1,s} \end{cases}$$

$$(m = \overline{1,s}, \ a_i^{(m)}(x) = a_i(x), \quad x \in S_m \ ; \ S_i \cap S_j = \emptyset , \ i \neq j)$$

Reciprocally. Let us consider the functional $J:U \longrightarrow R^1$ given in (7). The linear space of the test functions (perturbation) U_o is associated to the set U of the admissible functions in the stationarity problem of the functional J:

$$U_o = \{ v \in C^2(\Omega) \cap C^1(\bar{\Omega}) \mid v(x) = o, \ x \in S_1 \}$$

The function f, of real variable $\varepsilon \xrightarrow{f} J(u+\varepsilon h)$ with $\varepsilon \in R^1$, $u \in U$ and $h \in U_o$ is introduced, supposing that $u \in U$ is the stationary point of J and $h \in U_o$ is an arbitrarly fixed element.

Some simple calculations shows that

(1o) $$f(\varepsilon) = J(u+\varepsilon h) = J(u) + \varepsilon J'(u,h) + \frac{1}{2} \varepsilon^2 J''(u,h,h)$$

that proves $J(u)$ to be a quadratic functional on U with Gateaux derivatives

(11) $$J'(u,h) = \int_{\Omega} (\sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i} \frac{\partial h}{\partial x_i} + a_o uh - fh)dx + \sum_{r=2}^{p} \int_{S_r} h_r h \, dS + \sum_{\gamma=1}^{s-r} \int_{S_{p+\gamma}} \alpha_{p+\gamma} (u - \beta_{p+\gamma})h \, dS \ ;$$

(12) $$J''(u,h,h) = \int_{\Omega} [\sum_{i=1}^{n} a_i (\frac{\partial h}{\partial x_i})^2 + a_o h^2] dx + \sum_{\gamma=1}^{s-p} \int_{S_{p+\gamma}} \alpha_{p+\gamma} h^2 \, dS$$

$$J^{(m)}(u,h,\dots,h) = 0, \ m = 3,4,\dots$$

It results that for $a_i > 0$, $i = \overline{0,n}$ and $\alpha_{p+\gamma} > 0$, $\gamma = \overline{1,s-p}$ we have, $\forall h \in U_0$, $J''(u,h,h) > 0$; then the stationary point u is a unique strong global (or local) minimizer of the quadratic functional J (according to (10)). Therefore, under these conditions, the functional $J(u)$, $u \in U$ is a minimizing functional.

The equations (8)-(9) are obtained from the stationarity condition $J'(u,h) = 0$, $\forall h \in U_0$, if we take into account that

$$\int_{\Omega} a_i \frac{\partial u}{\partial x_i}\frac{\partial h}{\partial x_i}\,dx = -\int_{\Omega} h \frac{\partial}{\partial x_i}\left(a_i \frac{\partial u}{\partial x_i}\right)dx + \int_S a_i \frac{\partial u}{\partial x_i} n_i h \, dS$$

Therefore, (8)-(9) are the Euler-Lagrange equations for the quadratic functional $J(u)$, (7). The function $u \in U$ that minimizes the functional $J(u)$, (7), is a solution for the differential problem (8)-(9). The solution of the problem (8)-(9) can be found by searching the function $u \in U$ [i.e. that must verify only the essential condition $u = h_1$ on S_1] that makes the functional $J(u)$, $u \in U$ stationary on the set U.

Exemple 1. **The Problem of Heat Conduction in Solid Bodies.** The conductive stationary heat transfer in the solid body from the domain Ω bounded by the sufficiently smooth surface S [$= S_1 \cup S_2 \cup S_3$; $S_i \cap S_j = \emptyset$, $i \neq j$] is characterized by divergence type equation and the mixed boundary conditions

(a) $\quad -\nabla\cdot(k(x)\nabla T(x)) = f(x), \quad x \in \Omega \subset R^n \ (n = 1,2,3)$

(b) $\quad T = T_1$ on S_1

(c) $\quad -k\,\vec{n}\cdot\nabla T = q_2$ on S_2; $-k\,\vec{n}\cdot\nabla T = \alpha(T - T_3)$ on S_3

$(T(x) > 0,\ k(x) > 0,\ x \in \bar{\Omega};\ \alpha(x) > 0,\ q_2(x) \geq 0;\ T_1, T_3 > 0;\ f \in C(\Omega))$

$$(T \in C^2(\Omega) \cap C^1(\bar{\Omega}))$$

where $T(x)$ is the body temperature at the point x; k, f, T_1, q_2, α, T_3 with known physical significance are given functions (some are practically constant). The boundary conditions on S_2 express the Fourier's law while the boundary condition on S_3 express Newton's law (convective heat exchange with the exterior on the S_3 piece of S); the coefficient α is considered constant in the case of many practical problems. T_3 is a constant temperature (the temperature of a fluid that moves in the exterior over S_3).

In this case $a_i = a_i^{(m)} = k$, $a_e = e$. The variational functional of the mixed boundary value problem (a)-(c), according with (7) is

$$\text{(d)} \qquad J(T) = \int_{\Omega} \left[\frac{1}{2} k |\nabla T|^2 - fT \right] dx + \int_{S_2} q_2 T \, dS + \frac{1}{2} \int_{S_3} \alpha (T - T_3)^2 dS$$

$$T \in U = \{ T \in C^2(\Omega) \cap C^1(\bar{\Omega}) \mid \quad T = T_1 \quad \text{on } S_1 \}$$

Obviously, the formula (d) can be obtained if the Galerkin method is used directly in (a)-(c).

<u>Remarks</u>. 1°. The conditions (c) are natural conditions for the problem (a)-(c).

2°. The solution to the problem (a)-(c) minimizes the functional $J(T)$ on U, and can be determined by searching the function $T \in U$ that minimizes the functional $J(T)$ on the set U; therefore the minimizer (stationarity point) T, for $J(T)$, has to verify only the essential conditions.

3°. If $k = k(T)$, Kirchoff transformation is performed by which instead of T, the unknown ϕ is introduced putting

$$\frac{d\phi}{dT} = k(T), \qquad \phi(T) = \int_{T_e}^{T} k(T') d\,T' \Rightarrow \nabla \phi = k \nabla T$$

The problem (a)-(c) appears in the form

$$-\Delta \phi = f(x)$$

$$\phi = \phi_1 \text{ on } S_1, \; -\vec{n} \cdot \nabla \phi = \phi_2 \text{ on } S_2, \; -\vec{n} \cdot \nabla \phi = \alpha (T(\phi) - T_3)$$

If the condition on S_3 is eliminated the problem is a linear one.

2. Non-Self-Adjoint Problems.

a) The Adjoint Problem. The function transformation $u=v+\varphi$ is performed, where v is the unknown function that verifies homogeneous boundary conditions, while φ is a given function that verifies the inhomogeneous boundary conditions. Then, according to the problem (1)-(2), we consider the mixed boundary problem with homogeneous conditions

$$(13)\quad A_1 v \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial v}{\partial x_j}\right) + \sum_{i=1}^{n} a_i \frac{\partial v}{\partial x_i} + a_o v = f_1(x)$$

$$(13a)\quad B_m v = \sum_{i=1}^{n} a_{mi}(x)\frac{\partial}{\partial x_i} v(x) + a_{mo} v(x) = o,\ m=\overline{1,s}$$

where

$$f_1(x) = f(x) - A_1\varphi,\quad \sum_{i=1}^{n} a_{mi}(x)\frac{\partial}{\partial x_i}\varphi(x) + a_{mo}\varphi(x) = h_m(x),$$

By means of direct calculation (integration by parts), Green's formula on $\Omega\subset R^n$ for the formal operator A_1 is obtained

$$(14)\quad \int_\Omega w A_1 v\, dx = \int_\Omega v A_1^* w\, dx + \int_S \left(v\frac{\partial w}{\partial n_c} - w\frac{\partial v}{\partial n_c}\right) dS + \int_S vw \sum_{i=1}^{n} a_i n_i\, dS$$

where $A_1^* w$ is the formal adjoint operator (or in the sense of Lagrange)

$$(15)\quad A_1^* w = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial w}{\partial x_j}\right) - \sum_{i=1}^{n}\frac{\partial}{\partial x_i}(a_i w) + a_o w$$

while $\partial h/\partial n_c$ is the derivative in the direction of the conormal vector at S.

Let us return to the boundary conditions. The boundary condition $B_m^* w = o$, $m = \overline{1,s^*}$ is called adjoint with $B_m v=o$, $m=\overline{1,s}$ if

(16) $B_m v=o$, $m=\overline{1,s}$ and $B_m^* w=o$, $m=\overline{1,s^*}$ $\quad v\dfrac{\partial w}{\partial n_c} - w\dfrac{\partial v}{\partial n_c} +$

$$+ vw \sum_{i=1}^{n} a_i n_i = o$$

The boundary value problem

(17) $A_1^* w = g_1(x)$, $x \in \Omega$, $B_m^* w(x) = o$, $x \in S$, $m = \overline{1,s^*}$

is called an adjoint problem with the problem (1)-(2).

Remarks. If $a_i = o$, $i=\overline{1,n}$, then the last integral on S disappears in (14) and A_1^* coincides with A_1 ; in this case A_1 is called a formal self-adjoint operator (or in the sense of Lagrange). A boundary problem that coincides with the adjoint problem is called a self-adjoint problem (in this case it is necessary that $A_1 = A_1^*$ and $B_m = B_m^*$).

b) Operatorial Formulation. The second order problem (13)-(13a) is reduced to the operatorial equation

(18) $$Av = f$$

where v is the unknown function, f is the given function, while the operator $A: D(A) \subset L_2(\Omega) \longrightarrow L_2(\Omega)$ is defined by

$$D(A) = \left\{ v \in C^2(\Omega) \cap C^1(\bar{\Omega}) \;\middle|\; Av \in L_2(\Omega),\ B_m v(x) = o,\ x \in S \right\}$$

$$Av = A_1 v$$

Thus, the functions v and f are considered as elements of the Hilbert space $H = L_2(\Omega)$; f is considered continuous and $f \in L_2(\Omega)$ - a condition, in fact non-obligatory in (13), however the latter provides the equivalence between the problems (13)-(13a) and (18).

With Ω- bounded, $D(A) \subset L_2(\Omega)$. It is known that the linear space $C_o^\infty(\Omega)$ is dense in $L_2(\Omega)$ and because $C_o^\infty(\Omega) \subset D(A)$ it results that $D(A)$ is a dense subspace of $L_2(\Omega)$. The operator A is not closed (however it admits closeness). But, since $\overline{D(A)} = L_2(\Omega)$, the operator A has an adjoint that we note with the symbol A^*. In order to define A^*, the scalar product $(Av,w)_{L_2(\Omega)}$, $v\in D(A)$, $\forall w\in L_2(\Omega)$ is used and Green's formula (integration by parts) is applied. We obtain

$$(Av,w) = (v,A^*w),\ v\in D(A),\ w\in D(A^*)$$

If $A = A^*$, $\forall v,w\in D(A)$, A is symmetrical and if, besides, $D(A) = D(A^*)$ the operator A is self-adjoint. If $A \neq A^*$, the operator A is non-self-adjoint.

Analogously, the problem (17) can be reduced to the operatorial equation

$$A^*_{op}w = g_1\ ,\ g_1\in L_2(\Omega)$$

where the operator $A^*_{op}: D(A^*_{op})\subset L_2(\Omega) \longrightarrow L_2(\Omega)$ is given by the formulas

$$D(A^*_{op}) = \left\{ w\in C^2(\Omega)\cap C^1(\bar{\Omega})\ \middle|\ A_1^*w\in L_2(\Omega),\ B_m^*w(x)=o,\ x\in S\right\},$$

$$A^*_{op}w = A_1^*w$$

By using the Green Formula (14) and the notation $(.,.)$ for the scalar product, we obtain

$$v\in D(A),\ \forall w\in D(A^*_{op})$$

$$\int_\Omega (wA_1v - vA_1^*w)dx = (Av,w)_{L_2(\Omega)} - (v,A^*_{op}w)_{L_2(\Omega)} =$$

(19)

$$= \int_S \left(v\frac{\partial w}{\partial n_c} - w\frac{\partial v}{\partial n_c} + vw\sum_{i=1}^{n} a_in_i\right)dS = o \Longrightarrow (Av,w) = (v,A^*_{op}w)$$

With $D(A^*_{op})\subset D(A)$ and $A^*_{op} = A^*$ on $D(A^*_{op})$ it results that the operator A^*_{op} is the restriction of the operator A^* on $D(A^*_{op})$ (symmetrical operator). Therefore, the operator of the adjoint problem, A^*_{op}, coincides on $D(A^*_{op})$ with the adjoint operator A^* of the direct boundary value problem.

c) <u>The Variational Functional</u>. Let us consider the problem (13)-(13a) and its adjoint (17). The problem (13)-(13a) is not self-adjoint ($A_1\neq A_1^*$). In this case a variational functional can be determined and a variational principle (the adjoint variational principle) can be formulated for the given problem and its adjoint.

By using the Galerkin procedure, where v and w verify (13)-(17) we write the equality

(2o) $\quad (Av-f, \delta\lambda)_{L_2(\Omega)} + (A^* w-g, \delta\mu)_{L_2(\Omega)} = o,$

where only the functions v and w are subject to variations.

We have

$$\left[(\cdot,\cdot) = (\cdot,\cdot)_{L_2(\;)} \text{ - scalar product in } L_2(\Omega)\right]$$

$$(Av, \delta\lambda)-(f, \delta\lambda)+(w, A\delta\mu)-(g, \delta\mu) = o$$

If we take $\delta\lambda = \delta w$ and $\delta\mu = \delta v$, we obtain

$$(Av, \delta w)+(w, \delta Av) - (f, \delta w) - (g, \delta v) = o$$

or

$$\delta(Av,w) - \delta(f,w) - \delta(g,v) = o$$

The variational functional J(v,w) for the problem (13) and its adjoint (17) is

(21) $\quad J(v,w) = (Av-f,w)-(g,v)$

<u>Remark</u>. From (2o) it can also be inferred that

$$(v, A^*\delta\lambda) + (A^* w, \delta\mu) = (f, \delta\lambda) - (g, \delta\mu) = o$$

We take $\delta\lambda = \delta w$, $\delta\mu = \delta v$ and, as above, we get

$$\delta(A^* w,v) - \delta(g,v) - \delta(f,w) = o$$

Thus, the variational functional in the form

(21') $\quad J(v,w) = (A^* w-g,v) - (f,w)$

is obtained.

Exemple 2. <u>The Adjoint Variational Functional in the Problem of Heat Transfer in a Fluid Flow</u>. Let us consider the energy equation for an incompressible fluid, in non-stationary motion and in the absence of interior sources, with mixed boundary conditions:

(a) $\quad Au = \frac{\partial u}{\partial t} + \vec{v}.\nabla u - \nabla.(K\nabla u) = o, \; (x,t) \in \Omega \times (O,T)$

(b) $\quad u(x,t) = o$ on S_1, $\; t \in (O,T]$

(c) $K\vec{n}.\nabla u + \alpha u = o$ on S_2, $t \in (0,T]$

(d) $u(x,o) = o$, $x \in \bar{\Omega} (= \Omega \cup S)$

$(\Omega \subset R^n,\ n = 1,2,3;\ S = \bar{\Omega} - \Omega = S_1 \cup S_2,\ S_1 \cap S_2 = \emptyset, \nabla.\vec{v} = o)$

where $u(x,t)$ is the fluid temperature, K and α are functions of x and t (not necessarily continuous) that can also be reduced to constants; $\vec{n}$ is the unit vector of the exterior normal at S, and $\vec{v} = \vec{v}(x,t)$ is also a given function: the velocity of an incompressible fluid in which the heat transfer with the coefficient K takes place. Hence, u is not a component of the velocity $\vec{v}$ and therefore A is a linear operator.

Let D(A) be the definition domain for the linear operator A. We assume that $D(A) \subset L_2(\Omega)$ for each t and that D(A) contains functions $u \in C^{2,1}(\Omega \times (0,T))$ that verify (b)-(c).

The scalar product (Au,w) in $L_2(\Omega)$ is calculated and we get

$(u \in D(A);\ \forall t,\ w \in L_2(\Omega)$ and $dx = dx_1 dx_2 dx_3)$

$$(Au,w)_{L_2(\Omega)} = -\int_o^T \int_\Omega \left[\frac{\partial w}{\partial t} + \vec{v}.\nabla w + \nabla.(K\nabla w)\right] u\,dx\,dt +$$

$$+ \int_o^T \int_{S_2} \left[uK\vec{n}.\nabla w - wK\vec{n}.\nabla u + uw\,\vec{v}.\vec{n}\right] dS\,dt + \int_\Omega (uw)\Big|_o^T dx$$

Hence, it results that A is not a symmetrical (self-adjoint) operator. Consequently, we associate the adjoint problem to the (a)-(d).

(a^*) $A^* w = \frac{\partial w}{\partial t} + \vec{v}.\nabla w + \nabla.(K\nabla w) = o$

(b^*) $B_1^* w = w = o$ on S_1, $t \in [0,T)$

(c^*) $B_2^* w = K\vec{n}.\nabla w + \alpha w + w\,\vec{v}.\vec{n} = o$ on S_2, $t \in [0.T)$

(d^*) $B_3^* w = w(x,T) = o, x \in \bar{\Omega}$

We notice that A^* does not coincide with A and the condition (c^*) differs from (c). The conditions (c) and (c^*) coincide only if $\vec{v}.\vec{n} = o$ on S, i.e. the fluid does not cross the surface S_2 (it does not go off it).

By using (21) the adjoint variational functional has the form

$$J(u,w) = \int_o^T \int_\Omega [K \nabla u . \nabla w \, dx \, dt +$$

(e)

$$+ \frac{1}{2} \int_o^T \int_\Omega \left[w \frac{\partial u}{\partial t} - u \frac{\partial w}{\partial t}) + \vec{v}.(w \nabla u - u \nabla w) \right] dx \, dt +$$

$$+ \int_o^T \int_{S_2} (\alpha + \frac{1}{2} \vec{v}.\vec{n}) \, uw \, dS \, dt$$

$$u \in V = \{ u \in Q, \ u = o \ \text{on} \ S_1, \ t \in (0,T] , \ u(x,0)=0, \ x \in \bar{\Omega} \};$$
$$w \in V^* = \{ w \in Q, \ w = o \ \text{on} \ S_1, \ t \in [0,T), \ w(x,T)=o, \ x \in \bar{\Omega} \};$$
$$\left(Q = C^{2,1}(\Omega \times (0,T)) \cap C^{1,o}(\bar{\Omega} - S_1) \right)$$

The condition $\delta J(u,w) = o$ expresses an adjoint variational principle: the functional J(u,w) is stationary on the solution to the direct and adjoint problem.

Verification. The function f of two real variables is calculated

$$\varepsilon_1, \varepsilon_2 \xrightarrow{f} J(u + \varepsilon_1 \varphi, w + \varepsilon_2 \psi)$$

u and w are considered the stationary function (points) for J(u,w) while $\varphi \in V$ and $\psi \in V^*$ are arbitrary test function (arbitrarily fixed).

Consequently, $\varphi(x,t) = o$, on S_1 and for t=o, $\psi(x,t)=o$ on S_1 and for t=T.

The Gateaux derivative $J'_w(u,\psi)$ of the functional J(u,w) in terms of the function w is given by the equality

(f) $\quad J'_w(u,\psi) = J(u,\psi)$

if we use the above function.

By means of identity

$$K\nabla u.\nabla\psi = \nabla.(K\psi\nabla u) - \psi\nabla.(Ku)$$

we deduce

$$J'_w(u,\psi) = \int_0^T\int_\Omega\left[-\nabla.(K\nabla u) + \frac{\partial u}{\partial t} + \vec{v}.\nabla u\right]\psi\, dx\, dt -$$

$$-\frac{1}{2}\int_0^T\int_\Omega\left[\frac{\partial}{\partial t}(\psi u) + \vec{v}.\nabla(\psi u)\right]\, dx\, dt +$$

$$+\int_0^T\int_\Omega\left[K\,\vec{n}.\nabla u + (\alpha + \frac{1}{2}\,\vec{v}.\,\vec{n})u\right] dS\, dt$$

If we also use the formulas

$$\int_0^T\int_\Omega \vec{v}.\nabla(\psi u)\, dx\, dt = \int_0^T\int_{S_2}\psi u\,\vec{v}.\vec{n}\, dS\, dt;$$

$$\int_0^T\int_\Omega \frac{\partial}{\partial t}(\psi u)dx\, dt = \int_\Omega(\psi u)_0^T\, dx \quad (= 0)$$

the condition $J'_w(u,\psi) = 0,\ \forall\psi\in V^*,\ u\in V$ provides:

1) the Euler-Lagrange equation for $J(u,w)$, supposing that only w is subject to variation (this equation is exactly (a));

2) the natural bondary condition (c).

These two results together with the fact that $u\in V$ represent the direct boundary problem (a)-(d) itself.

Analogously the condition $J'_u(w,\varphi) = 0,\ \forall\varphi\in V$ and $w\in V^*$ can be shown to lead to the adjoint problem (a^*)-(d^*).