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ON MODIFIED VARIATIONAL METHOD FOR THE STUDY OF LIQUID MOTION IN MOVING TANKS

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§ 1 The Approximate Solution of the Galerkin Variational Equation

a) Introduction. The linearized motion equations. The Galerkin variational equation. Let us consider a tank, partially filled with heavy, incompressible, ideal fluid. Both the tank and the fluid are in a state of motion. We note : Ω - the finite volume of fluid, inside the tank, S^* - the tank surface that is in contact with the fluid, S_o the undisturbed free surface (for the fluid at rest, t=o) and S - the disturbed free surface (at the moment t). Let us take $O_1x_1y_1z_1$ a fixed reference system in space and Oxyz a invariable mobile reference system connected with the tank. We shall consider S_o as a horizontal fluid surface, O is in the centre of S_o and the axis Oz is directed on the vertical of the Earth. The fluid moves due to some initial conditions (pressure) under the action of the active force (weight) $\vec{F} = -\operatorname{grad} U(x,y,z)$ with $U = gz$. We assume that the fluid motion is slow and the it represents small deviations from the state at rest. If

$$F(x,y,z,t) = o \text{ or } F \equiv z-f(x,y,t) = o$$

is the equation of the surface S, v_{Sn} is the relative velocity of the fluid particle in projection on the exterior normal $\vec{n}$ at S ($|\vec{n}| = 1$) and $\Phi(x,y,z,t)$ is the velocity potential ($\vec{v}_a = \nabla\Phi$, $\vec{v}_a$ - absolute velocity), then

$$|\nabla F| \approx 1 \,, \quad v_{Sn} \approx \frac{\partial f}{\partial t} \quad \text{and} \quad \frac{\partial \phi}{\partial n} \approx \frac{\partial \phi}{\partial z} \quad \text{on } S.$$

(for fixed $Oxyz$; $v_{Sn} \approx \frac{\partial \phi}{\partial z}$ on S); $|\nabla\phi|^2 = 0$, $\nabla\phi \cdot \nabla f = 0$

We admit that the boundary conditions on S are applied on S_o ; for example

$$\frac{\partial \phi}{\partial z}(x,y,z,t)\Big|_{z=f} \approx \frac{\partial \phi}{\partial z}(x,y,z,t)\Big|_{z=o} = \frac{\partial \phi(x,y,o,t)}{\partial z}$$

The linearized equations of the motion of the potential incompressible liquid and the boundary conditions are [1]:

$$\Delta\phi(x,y,z,t) = o\ ,\ (x,y,z,t) \in \Omega \times (t_o,t_1],\ t_o \geq o,\ t_1 < \infty \quad (1)$$

$$\frac{\partial \phi}{\partial n} = \vec{v}_t \cdot \vec{n} \quad \text{on } S^* \quad (2)$$

$$\frac{\partial \phi}{\partial z} = \vec{v}_t \cdot \vec{n} + \frac{\partial f}{\partial t} \quad \text{on } S_o \quad (3)$$

$$\frac{\partial \phi}{\partial t} + gf - \vec{v}_t \cdot \nabla\phi = o \quad \text{on } S_o \quad (4)$$

($\phi(x,y,z,t_o) = \phi_o(x,y,z)$, $f(x,y,t_o) = f_o(x,y)$, ϕ_o and f_o -given)

where $\vec{v}_t = \vec{v}_o + \vec{\omega} \times \vec{r}$ is the transport velocity ($\vec{v}_o$ - translation velocity and $\vec{\omega}$ - the rotation velocity of the tank), ϕ is the velocity potential,while the (4) is given by the Cauchy-Lagrange integral (g - gravitational acceleration).The unknown functions in (1)-(4) are $\phi(x,y,z,t)$ and $f(x,yt)$.

The variational equation of Galerkin type,for the problem (1)-(4), may be written in the form [1],[6]:

$$\int_{t_o}^{t_1}\Big\{\int_\Omega(-\Delta\phi)\delta\phi\, d\Omega + \int_{S*}\Big(\frac{\partial \phi}{\partial n} - \vec{v}_t \cdot \vec{n}\Big)\delta\phi\, dS +$$

$$+\int_{S_o}\Big(\frac{\partial \phi}{\partial z} - \vec{v}_t \cdot \vec{n} - \frac{\partial f}{\partial t}\Big)\delta\phi\Big|_{z=o} dS_o +$$

$$+\int_{S_o}\Big[\Big(\frac{\partial \phi}{\partial t} - \vec{v}_t \cdot \nabla\phi\Big)_{z=o} - (\vec{g}\cdot\vec{r})_{z=o}\Big]\delta f\, dS_o\Big\}\, dt \quad (4')$$

b) <u>The tank has only a translation movement</u> ($\vec{\omega}(t) = o, \vec{v}_t = \vec{v}_o(t)$).
The function ϕ is transformed into the function φ by writing

$$\Phi = \varphi + \varphi_o(x,y,z,t) + \alpha(t)$$

where

$$\Delta\varphi_o = o\ ;\ \frac{\partial \varphi_o}{\partial n} = \vec{v}_o . \vec{n} \text{ on } S^*;\ \alpha(t) = \int_{t_o}^{t} v_o^2(s)ds$$

Since

$$\text{on } S^*,\ \vec{v}_o . \vec{n} = \vec{n} . \nabla(\vec{v}_o . \vec{r}) = \frac{\partial}{\partial n}(\vec{v}_o . \vec{r})$$

we choose

$$\varphi_o = \vec{v}_o . \vec{r}\ ;\ (\text{with } \Delta(\vec{v}_o . \vec{r}) = o)$$

We use therefore the transformation

$$\Phi(x,y,z,t) = \varphi(x,y,z,t) + \vec{v}_o . \vec{r} + \int_{t_o}^{t} v_o^2(s)ds \qquad (5')$$

The boundary value problem of small movements of the fluid in a tank that is in translation at the speed $\vec{v}_o(t)$ with respect to functions φ and f is

$$\Delta\varphi(x,y,z,t) = o,\ (x,y,z,t) \in \Omega \times (t_o, t_1], t_o \geqslant o, t_1 < \infty$$

$$\frac{\partial \varphi}{\partial n} = o \quad \text{on } S^* \qquad (5'')$$

$$\frac{\partial \varphi}{\partial n} = \frac{\partial f}{\partial t} \text{ on } S_o \quad \text{and} \quad \frac{\partial \varphi}{\partial t} + (\vec{v}_o - \vec{g}) . \vec{r} - \vec{v}_o . \nabla\varphi = o \text{ on } S; (S \to S_o)$$

Solutions of the form $(5')$ with

$$\varphi(x,y,z,t) = \bar{\varphi}(x,y,z) \cos\omega t \qquad (5)$$

$$f(x,y,t) = -\omega \bar{f}(x,y) \sin\omega t \qquad (6)$$

are searched for $(4')$ and after the functions $\bar{\varphi}$, $\bar{f}$ have been introduced, with the help of the relations (5)-(6), the equation $(4')$ is integrated on the interval $[t_o = o, t_1 = 2\pi/\omega]$.

The variational equation of the problem, $(4')$, now gets the form

$$\int_{\Omega}(-\Delta\bar{\varphi})\delta\bar{\varphi}\,d\Omega+\int_{S^*}\frac{\partial\bar{\varphi}}{\partial n}\delta\bar{\varphi}\,dS^*+\int_{S_0}\left(\frac{\partial\bar{\varphi}}{\partial z}+\omega^2\bar{f}\right)\delta\bar{\varphi}\Big|_{z=0}dS_0+$$

$$+\omega^2\int_{S_0}(\bar{\varphi})_{z=0}\,\delta\bar{f}\,dS_0-\frac{\omega^2}{\pi}\int_0^{2\pi/\omega}(\dot{\vec{v}}_0-\vec{g}).\Big[\int_{S_0}(x\vec{i}+$$

$$+y\vec{j}-\omega\bar{f}\sin\omega t.\vec{k})\,\delta\bar{f}\,dS_0\Big]\sin\omega t\,dt+ \qquad (7)$$

$$+\frac{\omega^2}{2\pi}\int_0^{2\pi/\omega}\vec{v}_0\sin 2\omega t\,dt.\int_{S_0}(\nabla\bar{\varphi})_{z=0}\,\delta\bar{f}\,dS_0=0$$

c) Approximation. We assume the tank to be in a translation movement on the vertical Oz at the speed

$$\vec{v}_0(t)=v_{oz}(t)\,\vec{k}\,,\quad v_{oz}(t)=a_0\sin\sigma t$$

and we admit the approximation

$$v_{oz}(t)=a_0\sigma t=0\,,\quad a\equiv a_0\sigma\ \ (\text{with small }\sigma)$$

Then, the variational equation (7) has the form

$$\int_{\Omega}(-\Delta\bar{\varphi})\,\delta\bar{\varphi}\,d\Omega+\int_{S^*}\frac{\partial\bar{\varphi}}{\partial n}\delta\bar{\varphi}\,dS^*$$

$$+\int_{S_0}\left(\frac{\partial\bar{\varphi}}{\partial z}\delta\bar{\varphi}\right)_{z=0}dS_0+\omega^2\int_{S_0}\left\{\bar{f}\,\delta\bar{\varphi}+[\bar{\varphi}+(a+g)\bar{f}]\,\delta\bar{f}\right\}_{z=0}dS_0-$$

$$-\frac{a}{2}\int_{S_0}\left(\frac{\partial\bar{\varphi}}{\partial z}\right)_{z=0}\delta\bar{f}\,dS_0=0 \qquad (9)$$

By using a direct approximation method, the solutions of this equation are chosen in the following form [6]

$$\bar{\varphi}(x,y,z)=\sum_{k=1}^{N}a_k w_k(x,y,z)+\sum_{n=1}^{q}c_n\varphi_n(x,y,z) \qquad (10)$$

$$\bar{f}(x,y,t)=\sum_{i=1}^{q}b_i f_i(x,y) \qquad (11)$$

where c_n, a_k and b_i are real, unknown coefficients, while φ_n, w_k

are trial functions which are characterized by special properties, imposed by the procedures employed in solving the problem. The functions w_k are the common trial functions.

The choice of trial functions. The trial functions and the surfaces S^* and S_o will chosen in such a way that the following conditions should be fulfilled :

$$(L = S^* \cap S_o \ , \ \Delta = \nabla^2 \ , \ \vec{k} \text{ - unitary vector on } Oz, \ |\vec{k}|=1)$$

a) $$\vec{n}.\vec{k} = o \quad \text{on} \quad L$$

b) $$\Delta\varphi_i(x,y,z) = o \text{ in } \Omega \ ; \quad \frac{\partial\varphi_i}{\partial n} = o \quad \text{on } L \ , i=\overline{1,q} \tag{12}$$

c) $$\Delta w_k(x,y,z) = o \text{ in } \Omega \ , k=\overline{1,N}$$

d) The systems $\{w_k\}_{1,\Omega}^{\infty}, \{\varphi_i\}_{1,\Omega}^{\infty}, \{f_i\}_{1,S_o}^{\infty}$ are complete (in $L_2(\Omega), L_2(S_o)$)

e) $$\varphi_i(x,y,z) = e^{kz} f_i(x,y) \ , \ i=\overline{1,q} \ , k \in R^1$$

Here : k is an unknown parameter (it should be determined in the context of the problem), (e) reflects a property of boundary layer type [6] (in the neighbourhood of the free surface S, $z-f(x.y,t)=o$, φ_i is similar to f_i in behaviour) while $\{w_k\}_1^N$ and $\{\varphi_i\}_1^q$ are functions chosen from complete systems of harmonic functions. Therefore, the derivative $\partial\varphi_j/\partial n$ is written :

$$\text{on } S^*, \ \frac{\partial\phi_j}{\partial n} = \vec{n}.\nabla\varphi_j = e^{kz}\ \vec{n}.(\nabla f_i + k f_i \ \vec{k}\ ) \tag{13}$$

From (b) and (e) we infer the Helmholtz equation for f_i

$$\Delta f_i(x,y) + k^2 f_i(x,y) = o \ , \ i=\overline{1,q} \quad \text{on } S_o \tag{14}$$

and the condition

$$\frac{\partial f_i}{\partial n} = \begin{cases} o \ , & \text{in the case } \vec{n}.\vec{k} = o \text{ on } L \quad (15) \\ -k(\vec{n}.\vec{k}\,)f_i \ , & \text{in the case } \vec{n}.\vec{k} \neq o \text{ on } L \quad (15') \end{cases}$$

The algebraic equations system for the coefficients a_i, b_i, c_i and the circular frequency ω. By introducing the solutions (10)-(11) in (9) we obtain the identity (with respect to $\delta f_j, \delta c_m, \delta a_s$):

$$\int_{S_0 \cup S^*} \Big[\sum_{n=1}^{q} \frac{\partial \varphi_n}{\partial n} c_n + \sum_{k=1}^{N} \frac{\partial w_k}{\partial n} a_k\Big]\Big[\sum_{m=1}^{q} \varphi_m \delta c_m + \sum_{s=1}^{M} w_s \delta a_s\Big] dS_0 +$$

$$+\omega^2 \int_{S_0} \Big\{\Big(\sum_{i=1}^{q} f_i b_i\Big)\Big(\sum_{m=1}^{q} \varphi_m \delta c_m + \sum_{s=1}^{N} w_s \delta a_s\Big) + \tag{16'}$$

$$+\Big[\sum_{n=1}^{q} \varphi_n c_n + \sum_{k=1}^{N} w_k a_k + (a+g)\sum_{i=1}^{q} f_i b_i\Big]\Big(\sum_{j=1}^{q} f_j \delta b_j\Big)\Big\} dS_0 -$$

$$-\frac{a}{2}\int_{S_0} \Big(\sum_{n=1}^{q} \frac{\partial \varphi_n}{\partial z} c_n + \sum_{k=1}^{N} \frac{\partial w_k}{\partial z} a_k\Big)\Big(\sum_{j=1}^{q} f_j \delta b_j\Big) dS_0 \equiv 0$$

Now, the real (finite) numbers are introduced (in (16'))

$$\int_{S_0} f_j \varphi_n \, dS_0 = \alpha^{(1)}_{jn} , \qquad \int_{S_0} f_j w_k \, dS_0 = \alpha^{(2)}_{jk} ;$$

$$\int_{S_0} f_j f_i \, dS_0 = \alpha^{(3)}_{ji} \; (= \alpha^{(1)}_{ji}) ;$$

$$\int_{S_0} f_j \frac{\partial \varphi_n}{\partial z} dS_0 = \alpha^{(4)}_{jn} ; \qquad \int_{S_0} f_j \frac{\partial w_k}{\partial z} dS_0 = \alpha^{(5)}_{jk}$$

$$\int_{S_0 \cup S^*} \varphi_m \frac{\partial \varphi_n}{\partial n} dS = \beta^{(1)}_{mn} ; \qquad \int_{S_0 \cup S^*} \varphi_m \frac{\partial w_k}{\partial n} dS = \beta^{(2)}_{mk} ; \tag{16}$$

$$\int_{S_0} \varphi_m f_i \, dS_0 = \beta^{(3)}_{mi} ;$$

$$\int_{S_0 \cup S^*} w_s \frac{\partial \varphi_n}{\partial n} dS = \gamma^{(1)}_{sn} ; \qquad \int_{S_0 \cup S^*} w_s \frac{\partial w_k}{\partial n} dS = \gamma^{(2)}_{sk}$$

$$\int_{S_0} w_s f_i \, dS_0 = \gamma^{(3)}_{si}$$

We notice that we have (Green's formula is used)

$$\beta^{(3)}_{mi} = \alpha^{(1)}_{im}$$

$$\gamma^{(1)}_{sn} = \int_{\Omega} w_s \,\Delta\varphi_n \, d\Omega + \int_{\Omega} \nabla w_s \cdot \nabla\varphi_n \, d\Omega = \int_{S_0 \cup S^*} \varphi_n \frac{\partial w_s}{\partial n} \, dS = \beta^{(2)}_{ns}$$

$$\gamma^{(3)}_{si} = \alpha^{(2)}_{is} \quad ; \; (n,i,j,m = \overline{1,q} \; ; \; k,s = \overline{1,N}) \tag{17'}$$

The algebraic system of linear and homogeneous equations is deduced from identity (16′)

$$\sum_{n=1}^{q} \left(\alpha^{(1)}_{jn} - \frac{a}{2\omega^2}\alpha^{(4)}_{jk}\right) c_n + \sum_{k=1}^{N} \left(\alpha^{(2)}_{jn} - \frac{a}{2\omega^2}\alpha^{(5)}_{jk}\right) a_k + $$
$$+ (a+g)\sum_{i=1}^{q} \alpha^{(3)}_{ji} b_i = 0, \qquad j = \overline{1,q}$$
$$\sum_{n=1}^{q} (\beta^{(1)}_{mn} c_n + \sum_{k=1}^{N} \beta^{(2)}_{mk} a_k + \omega^2 \sum_{i=1}^{q} \beta^{(3)}_{mi} b_i = 0 \; , \; m=\overline{1,q} \tag{17}$$
$$\sum_{n=1}^{q} \gamma^{(1)}_{sn} c_n + \sum_{k=1}^{N} \gamma^{(2)}_{sk} a_k + \omega^2 \sum_{i=1}^{q} \gamma^{(3)}_{si} \, b_i = 0 \; , \; s=\overline{1,N}$$

in which the unknown are the coefficients a_k, c_n, b_i.

The matrix form of the system (17). We introduce the matrices

$$\underset{\sim}{a} = \begin{Bmatrix} a_1 \\ \vdots \\ a_N \end{Bmatrix} \qquad \underset{\sim}{b} = \begin{Bmatrix} b_1 \\ \vdots \\ b_q \end{Bmatrix} \qquad \underset{\sim}{c} = \begin{Bmatrix} c_1 \\ \vdots \\ c_q \end{Bmatrix}$$

$$(i,j,m,n = \overline{1,q} \; ; \; k,s = \overline{1,N})$$

$$A_1 = [\alpha^{(1)}_{jn}] \; , \; A_1^{(4)} = [\alpha^{(4)}_{jn}] \; , \; A_2 = [\alpha^{(2)}_{jk}] ; \tag{18}$$

$$A_2^{(5)} = [\alpha^{(5)}_{jk}], \; A_3 = [\alpha^{(3)}_{ji}] ;$$

$$B_1 = [\beta^{(1)}_{mn}] \; , \; B_2 = [\beta^{(2)}_{mk}], \; B_3 = [\beta^{(3)}_{mi}] ;$$

$$C_1 = [\gamma^{(1)}_{sn}] \; , \; C_2 = [\gamma^{(2)}_{sk}], \; C_3 = [\gamma^{(3)}_{si}] .$$

By using (17$'$) we obtain the equalities (T - indicates the transposed matrix)

$$A_3 = A_1 \; , B_3 = A_1^T \; , \; C_1 = B_2^T \; , \; C_3 = A_2^T \qquad (18')$$

The system (17) is written in the homogeneous matrix form

$$(\omega^2 A_1 - \frac{a}{2} A_1^{(4)}) \underset{\sim}{c} + (\omega^2 A_2 - \frac{a}{2} A_2^{(5)}) \underset{\sim}{a} + (a+g)\omega^2 A_3 \underset{\sim}{b} = 0$$

$$B_1 \underset{\sim}{c} + B_2 \underset{\sim}{a} + \omega^2 B_3 \underset{\sim}{b} = 0 \qquad (19)$$

$$C_1 \underset{\sim}{c} + C_2 \underset{\sim}{a} + \omega^2 C_3 \underset{\sim}{b} = 0$$

Or

$$KX = 0$$

where K is a block matrix and X is a column block matrix (they are read in (19)).

The first equation from (19) is solved, with respect to $\underset{\sim}{b}$, and the result is introduced in the last two equations in (19). We obtain the relation

$$\underset{\sim}{b} = - \frac{1}{(a+g)\omega^2} A_3^{-1} \Big[(\omega^2 A_2 - \frac{a}{2} A_2^{(5)}) \underset{\sim}{a} + $$
$$+ (\omega^2 A_1 - \frac{a}{2} A_1^{(4)}) \underset{\sim}{c} \Big] \qquad (20)$$

and the linear and homogeneous system

$$(B_2 - B_3 A_3^{-1} A_2^{(\omega)}) \underset{\sim}{a} + (B_1 - B_3 A_3^{-1} A_1^{(\omega)}) \underset{\sim}{c} = 0$$
$$(C_2 - C_3 A_3^{-1} A_2^{(\omega)}) \underset{\sim}{a} + (C_1 - C_3 A_3^{-1} A_1^{(\omega)}) \underset{\sim}{c} = 0 \qquad (21)$$

where the matrix have been introduced

$$A_1^{(\omega)} = \frac{\omega^2}{a+g} A_1 - \frac{1}{2} \frac{a}{a+g} A_1^{(4)}$$

$$A_2^{(\omega)} = \frac{\omega^2}{a+g} A_2 - \frac{1}{2} \frac{a}{a+g} A_2^{(5)}$$

If $M^{(\omega)}$ is the matrix of the system (21) then the circular frequency (pulsation parameter) $\omega^2/(a+g)$ will be given by the equation

$$\det M^{(\omega)} = 0 \tag{22}$$

which represents the necessary condition for the system to admit non-trivial solutions a and c too. Subsequently, the vector $\underset{\sim}{b}$ is determined by means of (2o). The acceleration ratio a/g also appears in the system (21).

Remark. The parameter a/(a+g) is arbitrary one. If a = o, i.e. the tank is fixed in space, all the above formulas, including the variational equation (4′) and the system (21), stand good in a simplified form.

§2 An Application. The Fluid Motion in a Spherical Tank which is in a Vertical Uniformly Accelerated Motion

The fluid fills the semi-sphere of radius $R_o = 1$. Let us assume (we maintain the notations used in §1) that the domain that the fluid fills is the semi-sphere :

$$\Omega = \{(x,y,z) \mid x^2+y^2+z^2 \leq R_o^2 \ , \ -R_o \leq z \leq o \text{ with } R_o=1\}$$

in the system Oxyz with the origin in the center of the equatorial cyrcle and with the plane Oxy over the undisturbed free surface S_o; The fluid depth is $R_o = 1$. Take into account the symmetry, the cylindrical coordonates (r,η,z) - with $r\in[o,1]$, $\eta\in[o,2\pi]$, $z\in[-1,o]$ and $r^2+z^2 = 1$ on the sphere - are used.

1. Determination of functions $f_i(r,\eta)$. In agreement with the relation (b) and (e) from (12), f verifies the Helmholtz equation

$$r\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+\frac{\partial^2 f}{\partial \eta^2}+k^2r^2f = o \text{ on } S_o \tag{23}$$

$$\text{with } \vec{n}.\nabla f = o \text{ on } L = S^*\cap S_o, \vec{n} = (1,o,o) \tag{23a}$$

By using the separation $f(r,\eta) = R(r)u(\eta)$ and considering

$$u''/u = -m^2$$

where m is an arbitrary real constant, we obtain (A,B - arbitrary constants)

$$r^2R'' + rR' + (k^2r^2 - m^2)R = 0$$

$$u(\eta) = A\cos m\eta + B\sin m\eta$$

We find the solution $R(r) = J_m(kr)$ where J_m is the Bessel function of m order, while the admissible constant k depends on m ($k = k_m$).

The particular solutions for (23) are

$$f_m(r,\eta) = J_m(k_m r)\begin{matrix}\cos m\eta\\ \sin m\eta\end{matrix} \qquad (24)$$

The values k_m are determined with (23a), which, with $\vec{n} = (1,0,0)$ on L gets reduced to the equation

$$J_m'(k_m) = 0 \qquad (25)$$

This equation has an infinite number of solutions : $k_{m1}, k_{m2}, \ldots, k_{mn}$, ... which represents the eigenvalues of the equation (23). The functions f_i are chosen in the form

$$f_i = f_{mi}(r,\eta) = J_m(k_{mi})\begin{matrix}\cos m\eta\\ \sin m\eta\end{matrix} \qquad \text{with } m \in R^1 \text{ given}$$

We take

$$m = 1,\ i = \overline{1,2}$$

The solutions of the equation (25), [5], [6], and the functions f_i will be

$$k_{11} = 1{,}84118\ ;\ k_{12} = 5{,}33144$$

$$f_i(r,\eta) = J_1(k_{1i}r)\cos\eta \quad \text{or } f_i(r,\eta) = J_1(k_{1i})\sin\eta$$

$$(i = \overline{1,2})$$

2. The finding of the functions w_k. We choose harmonic polynomials for the functions $w_k(r,\eta,z)$. The polynomials are represented in spherical coordinates thus [6],[4]:

$$U_n^m = R^n P_n^m(\cos\theta) \begin{matrix}\cos m\eta \\ \sin m\eta\end{matrix} = w_n^m(r,z) \begin{matrix}\cos m\eta \\ \sin m\eta\end{matrix}$$

where $P_n^0 \equiv P_n$ and P_n^m represent Legendre polynomials. We consider the harmonic polynomial in cylindrical coordinates (in the meridian plane (r,z) with $R^2 = r^2 + z^2$ and $\cos\theta = z/R$)

$$w_n^m(r,z) = b_{nm} R^n P_n^m(\cos\theta)\ ,\ b_{nm} = \frac{2^n\, m!(n-m)!}{(n+m)!}$$

$$(n = 0,1,2,\dots\ ;\ m \leqslant n)$$

By using known recurrence relations ($P_0^0 = P_0(\mu) = 1, P_1(\mu) = \mu, \dots,$ $\mu = \cos\theta = z/R$) the following expressions are inferred :

The case $m = 0$:

$$w_0^0 = P_0(\mu) = 1,\ w_1^0 = b_{10} R P_1^0 = z\ ,\ w_2^0 = z^2 - \frac{1}{2} r^2, w_3^0 = z^3 - \frac{r^2}{3}\left(2z + \frac{5}{2}\right), \dots.$$

The case $m = 1$

$$w_1^1 = r\ ,\ w_2^1 = rz\ ,\ w_3^1 = rz^2 - \frac{1}{4} r^3,\ w_4^1 = rz^3 + \frac{3}{4} r^3 z, \dots, w_N^1, \dots$$

Harmonic polynomials of the form w_n^m with $m = 1$ and $n \geqslant m$ may be used for the problem under study.

3. The trial functions f_i and w_k . If we consider an approximation with $m = 1$, $N = 1$, $q = 2$ and we use the Bessel function $J_1(x)$, the trial functions are chosen as follows :

$$\begin{aligned} f_j(r,\eta) &= J_1(k_{1j} r) \cos\eta \\ \varphi_j(r,\eta,z) &= e^{k_{1j} z} J_1(k_{1j} r) \cos\eta \end{aligned} \quad ; \quad j = \overline{1,2}$$

4. The calculation of coefficients α, β, γ. We use the formulas, [7]:

(a) $$\int_0^1 J_1(k_{1m}r)J_1(k_{1n}r)r\,dr = \begin{cases} \frac{1}{2}\left(1-\frac{1}{k_{1n}^2}\right)J_1^2(k_{1n}), m=n, J_1'(k_{1n})=0 \\ 0, \quad m\neq n, J_1'(k_{1m})=J_1'(k_{1n})=0 \end{cases}$$

(b) $$\int_0^1 J_1(k_{1j}r)r^2\,dr = \frac{1}{k_{1j}}\left[\frac{2}{k_{1j}}J_1(k_{1j})-J_0(k_{1j})\right], j=\overline{1,2}$$

(c) $$J_1'(z)=J_0(z)-\frac{1}{z}J_1(z)\ ;\ (J_1'(z)\equiv dJ_1/dz)$$

The coefficients $\alpha_{ij}^{(n)}$ (with $i,j=\overline{1,2}$, $n=\overline{1,5}$) have the expressions

$$\alpha_{jj}^{(1)} = \frac{\pi}{2}\left(1-\frac{1}{k_{jj}^2}\right)J_1^2(k_{1j})$$

$$\alpha_{ij}^{(1)} = 0, \text{ with } i\neq j$$

$$\alpha_{jj}^{(4)} = k_{1j}\,\alpha_{jj}^{(1)}$$

$$\alpha_{ij}^{(4)} = 0, \quad i\neq j$$

$$\alpha_{j1}^{(2)} = \frac{\pi}{k_{1j}}\left[\frac{2}{k_{1j}}J_1(k_{1j})-J_0(k_{1j})\right], \quad j=\overline{1,2}$$

$$\alpha_{j1}^{(5)} = 0,\ j=\overline{1,2}\ ;\ \alpha_{jj}^{(3)}=\alpha_{jj}^{(1)}\ ;\ \alpha_{ij}^{(3)}=\alpha_{ji}^{(3)}=\alpha_{ij}^{(1)}=0,\ i\neq j$$

The coefficients β. We use the formulas (16). If we assume the surface integrals to be finite, we obtain the equalities

$$\beta_{ij}^{(1)} = \int_{S_0\cup S^*}\varphi_i\frac{\partial\varphi_1}{\partial n}\,dS = \int_{S_0}\varphi_i\frac{\partial\varphi_1}{\partial z}\,dS_0 + \int_{S^*}\varphi_i\frac{\partial\varphi_1}{\partial n}\,dS\ ;\ i,j=\overline{1,2}$$

Hence we deduce the calculation formulas ($r = \cos\varphi$)

$$\beta_{ij}^{(1)} = k_{1j}\alpha_{ij}^{(1)} +$$

$$+ k_{1j}\pi\int_0^{\pi/2} e^{-(k_{1i}+k_{1j})\sin\varphi} J_1(k_{1i}\cos\varphi)\times$$

$$\times\left[J_1'(k_{1j}\cos\varphi)\cos\varphi - J_1(k_{1j}\cos\varphi)\sin\varphi\right]\cos\varphi\, d\varphi$$

If we use Simpson's rule (1/3) for the approximate calculation of an integral,we obtain the approximation formula

$$\beta_{1j}^{(1)} \simeq k_{1j}\left\{\alpha_{ij}^{(1)} + \frac{\pi^2}{6} e^{-\frac{1}{2}(k_{1i}+k_{1j})} J_1\left(\frac{k_{1i}}{\sqrt{2}}\right)\times\right.$$

$$\left.\times\left[J_1'\left(\frac{k_{1j}}{\sqrt{2}}\right) - J_1\left(\frac{k_{1j}}{\sqrt{2}}\right)\right]\right\} \quad ; i,j = \overline{1,2}$$

$$\left(J_1'\left(\frac{k_{1j}}{\sqrt{2}}\right) = J_0\left(\frac{k_{1j}}{\sqrt{2}}\right) - \frac{\sqrt{2}}{k_{1j}} J_1\left(\frac{k_{1j}}{\sqrt{2}}\right)\right)$$

From (16) we find out the formula ($r = \cos\varphi$)

$$\beta_{i1}^{(2)} = \int_{S_0\cup S^*} \varphi_i \frac{\partial w_1}{\partial n}\, dS = \int_{S^*} \varphi_i \frac{\partial w_1}{\partial n}\, dS^* =$$

$$= \pi\int_0^{\pi/2} e^{-k_{1i}\sin\varphi} J_1(k_{1i}\cos\varphi)\cos^2\varphi\, d\varphi \quad , \quad i = \overline{1,2}$$

If Simpson s rule (1/3) is used,we obtain the approximate calculation formula

$$\beta_{i1}^{(2)} \simeq \frac{\pi^2}{12}\left[J_1(k_{1i}) + 2e^{-\frac{1}{\sqrt{2}}k_{1i}} J_1\left(\frac{k_{1i}}{\sqrt{2}}\right)\right] , i=\overline{1,2}$$

For $\beta_{ij}^{(3)}$ we also have the formulas

$$\beta_{ii}^{(3)} = \alpha_{ii}^{(3)} \quad , \; i=\overline{1,2}$$

$$\beta_{ij}^{(3)} = 0 \; ; \; i,j=\overline{1,2}; \quad i\neq j$$

The coefficients $\gamma_{ij}^{(n)}$ $(i,j = \overline{1,2}\ ;\ n = \overline{1,3})$ are estimated by means of the formulas

$$\gamma_{11}^{(1)} = \beta_{11}^{(2)}\ ,\quad \gamma_{12} = \beta_{21}^{(2)}$$

$$\gamma_{11}^{(2)} = \pi\int_0^1 \frac{r^3 dr}{(1-r^2)^{1/2}} = \frac{2\pi}{3}\ ,\quad \gamma_{11}^{(3)} = \alpha_{11}^{(2)}\ ,\quad \gamma_{12}^{(3)} = \alpha_{21}^{(2)}$$

5. The equations for the coefficients a_1, c_1, c_2 and for the frequency ω. By using the Bessel functions values $J_0(x)$ and $J_1(x)$ given in tables [3] and the above mentioned formulas for α, β, γ the following numerical expressions are inferred for the matrices in the system (21)

$$A_1 = \begin{bmatrix} \alpha_{11}^{(1)} & \alpha_{12}^{(1)} \\ \alpha_{21}^{(1)} & \alpha_{22}^{(1)} \end{bmatrix} = \begin{bmatrix} 0.374937 & 0 \\ 0 & 0.181565 \end{bmatrix};$$

$$A_2 = \begin{Bmatrix} \alpha_{11}^{(2)} \\ \alpha_{21}^{(2)} \end{Bmatrix} = \begin{Bmatrix} 0.538060 \\ -0.037961 \end{Bmatrix};\qquad A_3 = A_1$$

$$A_1^{(4)} = \begin{bmatrix} 0.690327 & 0 \\ 0 & 0.9680003 \end{bmatrix};\ A_2^{(5)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix};$$

$$A_3^{-1} = A_1^{-1} = \frac{1}{|A_1|}\begin{bmatrix} \alpha_{22}^{(1)} & -\alpha_{12}^{(1)} \\ -\alpha_{21}^{(1)} & \alpha_{11}^{(1)} \end{bmatrix} = \begin{bmatrix} 2{,}667132 & 0 \\ 0 & 5.507705 \end{bmatrix};$$

$$B_1 = \begin{bmatrix} \beta_{11}^{(1)} & \beta_{12}^{(1)} \\ \beta_{21}^{(1)} & \beta_{22}^{(1)} \end{bmatrix} = \begin{bmatrix} 0.653849 & -0.012453 \\ -0.000145 & 0.967969 \end{bmatrix}$$

$$B_2 = \begin{Bmatrix} \beta_{11}^{(2)} \\ \beta_{21}^{(2)} \end{Bmatrix} = \begin{Bmatrix} 0.712139 \\ -0.283727 \end{Bmatrix}$$

$$B_3 = A_3^T = A_1^T = A_1$$

$$C_1 = B_2^T \text{ , } C_2 = (2{,}094395) \text{ , } C_3 = A_2^T$$

The algebraic system (21) is written in the matrix form

$$M^{(\bar{\omega})} X = 0 \text{ , } \bar{\omega} = \frac{\omega^2}{a+g}$$

where the matrix $M^{(\bar{\omega})}$ and the column vector are

$$M^{(\bar{\omega})} = \begin{bmatrix} p_{11}+q_{11}\bar{\omega} & p_{12}+q_{12}\bar{\omega} & p_{13}+q_{13}\bar{\omega} \\ p_{21}+q_{21}\bar{\omega} & p_{22}+q_{22}\bar{\omega} & p_{23} \\ p_{31}+q_{31}\bar{\omega} & p_{32} & p_{33}+q_{33}\bar{\omega} \end{bmatrix} ; X = \begin{Bmatrix} a_1 \\ c_1 \\ c_2 \end{Bmatrix}$$

in which the coefficients p and q have the expressions

$$(g_1 = a + g)$$

$$(\mathrm{I}) \begin{cases} p_{11} = 2.094395 \quad ; \quad q_{11} = -0{,}780095 \\ p_{12} = 0.712139 + 0.495336 \dfrac{a}{g_1} \quad ; \quad q_{12} = -0.538060 \\ p_{13} = -(0.283727+0.101194 \dfrac{a}{g_1}) \quad ; \quad q_{13} = 0.037961 \end{cases}$$

$$(\mathrm{II}) \begin{cases} p_{21} = 0.712139 \; ; \; q_{21} = -0.538060 ; \; p_{22} = 0{,}653849+0.345164\dfrac{a}{g_1} \\ q_{22} = -0.374937 \quad ; \quad p_{23} = -0.012453 \end{cases}$$

$$(\mathrm{III}) \begin{cases} p_{31} = -0.283727 \quad ; \quad q_{31} = 0.037961 \; ; \; p_{32} = -0.000145 \\ p_{33} = 0.967168+0.484002 \dfrac{a}{g_1} \quad ; \quad q_{33} = -0.181565 \end{cases}$$

The equation of the circular frequency parameter

$$\det M^{(\bar{\omega})} = 0$$

is reduced, after calculations are performed, to the equation

$$b_2 \bar{\omega}^2 + b_1 \bar{\omega}^2 + b_0 = 0$$

(The $\bar{\omega}^3$ coefficient is $q_{11} q_{22} q_{33} - q_{12} q_{21} q_{33} - q_{13} q_{22} q_{31} = -4.10^{-7} (\approx 0)$)

where ($\bar{a} = a/(a+g)$):

$b_2 = 0.090166 + 10^{-6}\bar{a} \approx 0,090166$;

$b_1 = -(0.626642 + 0.305421\bar{a} + 4.10^{-7}\bar{a}^2 \approx$
$\approx -(0,626642 + 0,305421\ \bar{a})$;

$b_0 = 0,784626 + 0,730894\bar{a} + 0,169249\bar{a}^2$

$\bar{\omega} = \omega^2/\bar{a}$

Table 1

a	$\bar{\omega}_{11}$	$\bar{\omega}_{12}$
0	1,638319	5,311551
0,5	1,672066	5,442079
1	1,702715	5,560499
1,5	1,730642	5,668477
2	1,756215	5,767286

In the Table 1 the values of the frequency parameter $\bar{\omega}$ are given for m = 1, q = 2 (and N = 1). The values calculated in Table 1 are in accordance with [6] where is considered the case a = 0.

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SUMMARY

The variational Galerkin equation is applied to the study of small motions of an ideal fluid in mobile tanks. In the case of tank translation, solutions in the modified Ritz form are chosen for both the velocity potential and for the free surface. The equations for the coefficients and frequency are determined. The results are applied to the study of fluid oscillations in a spherical tank (the fluid fills the semi-sphere of radius $R_0=1$ in a uniformly accelerated motion).

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This paper is in final form and no version of it will be submitted for publication elsewhere.