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NORMALITY, REGULARITY, LATTICIALITY AND ORDER COMPLETENESS OF ORDERED FRÉCHET SPACES

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Abstract. The Fréchet spaces in which every closed normal cone is regular, and those in which every latticially ordered closed subspace with continuous lattice operations is order complete with order continuous quasinorm, are the same.

All vector spaces dealt with in this note will be over the reals.

A cone in the vector space E is a set K such that $K + K \subset K$, $tK \subset K$ for every non-negative real number t, and $K \cap (-K) = \{0\}$. The cone K induces a partial ordering in E by considering $x \leq y$ whenever $y-x$ is in K. Then E is called an ordered vector space with the positive cone K.

If E is an ordered vector space which is endowed with a locally convex topology, then E (and its positive cone) is called normal if there is a family $P = \{p\}$ of seminorms which generates the topology of E and which has the property that if p is in P, and $x, y \in K$, with $x \leq y$, then $p(x) \leq p(y)$.

The ordered locally convex space E (and its positive cone K) is regular if each decreasing sequence in K is convergent.

E (and its positive cone) is called *fully regular* if each decreasing topologically bounded sequence in it is convergent.

An ordered vector space E is said *lattice ordered* or a *vector lattice* if every pair x, y of its elements possesses a supremum (or equivalently, an infimum) denoted by $x \vee y$ (and respectively, by $x \wedge y$). The positive cone of a vector lattice is called *latticial*. We denote as usual $x^+ = x \vee 0$, $x^- = (-x) \vee 0$, $|x| = x^+ + x^-$. The operations $x \mapsto x^+$, $x \mapsto x^-$, $x \mapsto |x|$, $(x,y) \mapsto x \vee y$ and $(x,y) \mapsto x \wedge y$ are called *lattice operations*.

A vector lattice E is called *order complete* if every upper bounded (lower bounded) subset in it possesses a supremum (an infimum) in E.

Let E be a vector lattice. A seminorm (norm) p on E is called *lattice seminorm* (*lattice norm*) if $|x| \leqslant |y|$ implies $p(x) \leqslant p(y)$ for all x, y in E. The vector lattice E is a *locally convex (vector) lattice* if there exists a family of lattice seminorms which generates the topology in E. If p is a lattice norm on E, then (E,p) is called *normed (vector) lattice*; if in addition (E,p) is norm complete, it is called a *Banach lattice*. Locally convex lattices (and normed and Banach lattices as well) are in particular normal ordered locally convex spaces.

A filter F in the ordered vector space E is called *order convergent* to x, if F contains a family of order intervals (sets of form $\{z : a \leqslant z \leqslant b\}$) with intersection $\{x\}$. A functional f on E is called *order continuous* at x if any filter $f(F)$ converges to $f(x)$, whenever F is a filter which is order convergent to x.

The normal and regular locally convex spaces play a central role in ordered vector space theory. Strongly related to vectorial optimization, they have occured in some fields of the functional analysis as operator theory, basis theory, numerical analysis, etc. Their appearance in the basis theory furnishes some interesting characterizations of various types of bases (see e. g. [8], Theorem II. 16.3 and Proposition II. 16.2). Conversely, basis theory yields useful characterizations of spaces in which every closed normal cone is regular. A first result of this kind for Banach spaces is in implicite form contained in the paper [1] of Bessaga and Pelczyński. It was stated in ordered vector space setting and was extended to Fréchet spaces by McArthur in [4].

The characterization of the Banach spaces in which every closed subspace organized as a Banach lattice with continuous lattice operations is also order complete with order continuous norm, is another problem of similar kind. Results concerning it have their origin in the monographs of Kantorovič, Vulikh and Pinsker [2] and of Nakano [5]. Deep investigations about the problem were made by Mayer-Nieberg [3].

The above described studies were done independently. An important tool in both of them is the possibility of the imbedding in the respective Fréchet or Banach spaces of the space c_o of real sequences converging to 0. This circumstance is not accidental. In fact it expresses the same structural feature and suggests that the two different directions can be joined. This fact is illustrated by the following result :

THEOREM . <u>Let</u> E <u>be a Fréchet space. Then the following conditions are equivalent</u> :

(i) Every closed normal cone in E is fully regular.

(ii) Every closed normal cone in E is regular.

(iii) Each closed subspace of E which is lattice ordered with continuous lattice operations is also order complete and the quasi norm induced from that of E is order continuous.

(iv) E has no subspace isomorphic (as a topological vector space) to c_0.

All the steps of the proof of this theorem can be built up from known facts in the literature. The equivalences (i)$\Leftrightarrow$(ii) $\Leftrightarrow$(iv) is in fact the content of Theorem 1 in [4]. The proof of the equivalence (iii)$\Rightarrow$(iv) for Banach lattices can be carried out putting together Theorem II. 5.10, Proposition II. 5.15, Propositions II. 10.5 and II. 10.6, and finally Exercise II. 17. (b) in [7]. We shall outline a proof in which all the steps are rather simple, except the last one : the proof of (iv)$\Rightarrow$(i) for which we refer to [4]. The cited proof uses many facts from basis theory. It seems that a certain amount of basis theory is inavoidable also in a shortened proof following this line. Another way would be eventually the transposition to Fréchet lattices of the results cited above from [7].

Proof. The implication (i)$\Rightarrow$(ii) follows from the topologycal boundedness of order bounded sets in ordered normal locally convex spaces. For (ii)$\Rightarrow$(iii) we observe first that from the fact that the lattice operations in a lattice ordered Fréchet space are continuous, it follows that the positive cone is closed and from Theorems 8.1 and 8.2 in [6] it is also normal. Hence by (ii) it is also regular. Using a transfinite induction method in the way it was done in the proof of (d)$\Rightarrow$(a) in

Theorem II. 5.10 in [7] , we conclude that the respective Fréchet lattice is order complete and has order continuous quasinorm.

(iii) $\Rightarrow$ (iv) . If (iv) does not hold, then E contains a subspace E_o isomorphic to c , the space of all convergent sequences with real terms, since c and c_o are isomorphic. Let us consider then c with its canonical order structure induced by the cone of sequences with non-negative terms. This cone is closed, normal and latticial. Hence the lattice operations in c are continuous. Let us consider the sequence (x_n) in c defined as follows :

$$x_1 = (0,1,\dots,1,\dots)$$
$$x_2 = (0,0,1,\dots,1,\dots)$$
$$\dots\dots\dots\dots\dots\dots$$
$$x_n = (\underbrace{0,\dots,0}_{n \text{ times}},1,\dots,1,\dots)$$
$$\dots\dots\dots\dots\dots\dots$$

Then $\inf \{x_n\} = (0,\dots,0,\dots)$ and hence (x_n) order converges to the zero element of the space. But (x_n) is not norm convergent to this element. (We can show also by a different way that c is not order complete.) The obtained contradiction with (iii) proves the required implication.

Finally we refer to [4] for the proof of (iv) $\Rightarrow$ (i) .

Remark. The equivalent conditions in the theorem can be completed with other ones by using vector lattice theoretic notions as disjoint sequence, ideal, band, lattice homomorphism etc. in the spirit it was done for Banach lattices in Chapter II. of [7] .

R E F E R E N C E S

[1] Bessaga, C., Pelczyński, A., On bases and unconditional convergence of series in Banach spaces, Studia Math. 17 (1958), 151-164.

[2] Kantorovic, L.V., Vulkh, B.C., Pinsker, A.G., Functional Analysis in Partially Ordered Spaces (Russian), Moskow - Leningrad, 1950 .

[3] Mayer-Nieberg, P., Characterisierung einiger topologischer und ordnungstheoretischer Eigenschaften von Banachverbänden mit Hilfe disjunkter Folgen, Arch. Math. 24 (1973), 640-647.

[4] McArthur, C.W., In what spaces is every closed normal cone regular ? , Proc. Edinburgh Math. Soc. 17 (series II) (1970), 121-125.

[5] Nakano, H., Modulared Semi-Ordered Linear Spaces, Tokyo, 1950.

[6] Namioka, I., Partially Ordered Linear Topological Spaces, Memoirs of the Amer.Math. Soc., Nr. 24, 1957.

[7] Schaefer, H.H., Banach Lattices and Positive Operators, Springer-Verlag, Berlin - Heidelberg - New York, 1974.

[8] Singer, I., Bases in Banach Spaces I., Springer-Verlag, Berlin - Heidelberg - New York, 1980.

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This note is in final form and no version of it is or will be submitted for publication elsewhere.