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CONTENTS

MUNTEAN I. : Extensions of some mean value theorems.....	7
TÓTH L. : Some inverse Hölder inequalities	25
VORNICESCU N. : A note on Wirtinger's integral and discrete inequalities	31
MOCANU P. T. : On a class of first-order differential subordinations	37
SĂLĂGEAN G. Ș. : On univalent functions with negative coefficients	47
MUREȘAN M. : A note on partial stability for differential inclusions	55
TOADER G. : Invariant transformations of p, q -convex sequences	61
MITREA A. I. : On the convergence of numerical differentiation for Hermite nodes	65
MUSTĂȚA C. : Extension of Hölder functions and some related problems of best approximation	71
BALOGH Z. and MOISESCU M. : On vector topologies on finite -dimensional vector spaces	87
ANISIU M-C. : On fixed point theorems for mappings defined on spheres in metric spaces	95
KASSAY G. : On Brézis-Nirenberg-Stampacchia's minimax principle	101

MUSTĂŢA C. : On a problem of extremum	107
BALÁZS M. and GOLDNER G. : Fréchet derivative as a limit of divided differences	115
PĂVĂLOIU I. : On the convergency of a Steffensen-type method	121
PĂVĂLOIU I. : Remarks on the secant method for the solution of nonlinear operatorial equations.....	127
DIACONU A. : Sur l'approximation des solutions des problèmes polylocaux pour des équations différentielles non-linéaires ..	133
POTRA T. : Boundary element Galerkin method for a nonlinear boundary value problem	151
DUCA E. and DUCA D. : Vectorial fractional optimization in complex space	163

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EXTENSIONS OF SOME MEAN VALUE THEOREMS

Ioan Muntean

1. Introduction. The classical mean value theorems for real functions, usually attributed to M. Rolle (1691), J. L. Lagrange (1797) and A.-L. Cauchy (1823), together with the famous rule of G. F. l'Hôpital (1696), constitute today the central part of real analysis and its applications. In the last century and more, numerous researches and extensions of these theorems have been appearing. They concern functions: 1) taking complex values or values in a normed space (even in a topological vector space), 2) possessing a derivative in an extended sense (infinite, lateral, Dini, approximate), 3) deprived of any derivative (even in extended senses) at some points. Other investigations refer to mean value theorems for divided differences or to various converses of the classical mean value theorems.

In this paper we limit oneself to real functions of a real variable, and we propose to attenuate the continuity hypothesis, which is present in almost all mean value theorems, until the primitivability or even the Darboux condition for involved functions. So attenuated, these requirements are combined in

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ON VECTOR TOPOLOGIES ON FINITE-DIMENSIONAL VECTOR SPACES

Zoltan Balogh and Marius Moisesescu

1. Introduction.

This article grew out from some basic ideas presented by I. Muntean in his book [1]. We wrote it thinking that the question of the independence of the defining axioms of a vector topology is still an interesting one.

Let X be a vector space on K (K being $\mathbb{R}$ or $\mathbb{C}$, endowed with the Euclidian topology). Let O be the origin of X .

A topology τ in X is named vector topology if it satisfies:

- TV1) The addition "+" : $X \times X \rightarrow X$, $+(x,y) = x + y$ is continuous.
TV2) The multiplication "·" : $K \times X \rightarrow X$, $\cdot(\alpha,x) = \alpha x$ is continuous.

Generally, the axioms TV1) and TV2) are independent.

2. Independence of the axioms.

If X is a vector space, $X \neq \{0\}$ with the discrete topology, TV1) is verified without TV2) being verified.

In 1988, V. Anisiu gave an exemple of a vector topology wich verifies TV2) without TV1) being satisfied. The vector space he considered was infinite dimensional. We give now an exemple of

vector topology which satisfies TV2) without TV1) being satisfied and the vector space is finite dimensional.

2.1. Example.

Let $X = \mathbb{R}$ a vector space over $\mathbb{R}$. We define the map $\mathcal{V}: X \rightarrow \mathcal{P}(\mathcal{P}(X))$ by:

$$\mathcal{V}(x) = \begin{cases} \{V \subset X : \exists \varepsilon > 0 \text{ such that }]x-\varepsilon, x+\varepsilon[\subset V\} & \text{for } x \neq 0, \\ \{\mathbb{R}\} & \text{for } x=0. \end{cases}$$

The axioms of neighbourhood map are easily checked. We can also check that TV2) holds but not TV1).

The example above has the trouble that the topology is not Hausdorff. We give now another example of topology which satisfies TV2) without TV1) being satisfied and the topology is Hausdorff.

2.2. Example.

Let $X = \mathbb{R}^2$ the vector space on $\mathbb{R}$ and the topology τ generated by the following neighbourhood function: $\mathcal{V}: X \rightarrow \mathcal{P}(\mathcal{P}(X))$

$$\mathcal{V}(x) = \begin{cases} \{V \subset X : \exists \varepsilon > 0 \text{ such that } x+]-\varepsilon, \varepsilon[\subset V\} & \text{for } x \neq 0, \\ \{V \subset X : \exists \varepsilon > 0 \text{ such that } B(0, \varepsilon) \subset V\} & \text{for } x=0, \end{cases}$$

where $0 = (0,0)$ and $B(0, \varepsilon)$ is the Euclidean ball from $\mathbb{R}^2$ of radius ε .

The neighbourhood function axioms are immediately verified for and the topology generated by $\mathcal{V}$ is stronger than the Euclidean one. It follows that the space is separated Hausdorff.

We show that TV2) holds. There are three cases:

a) $x = 0$.

The axiom is verified for $\alpha \cdot 0 = 0$, since the neighbourhoods of 0 are coinciding with those from the Euclidean topology.

b) $x \neq 0, \alpha \neq 0$.

We define an orthogonal coordinate system on $\mathbb{R}^2$ as to $0x_1$ be the line which passes through 0 and x . In this coordinate system we have $x = (x_1, 0)$, $\alpha x = (\alpha x_1, 0)$; $V \in \mathcal{V}(\alpha \cdot x)$ implies that there exists a line segment on $0x_1$ centered in αx_1 contained in V if and only if there exists an $\varepsilon_1 > 0$ such that $]\alpha x_1 - \varepsilon_1, \alpha x_1 + \varepsilon_1[\times \{0\} \subset V$.

The usual topology on $\mathbb{R}$ satisfies TV2) so there exists $\delta > 0$ and $\varepsilon_2 > 0$ such that:

$$] \alpha - \delta, \alpha + \delta[\cdot]x_1 - \varepsilon_2, x_1 + \varepsilon_2[\subset] \alpha x_1 - \varepsilon_1, \alpha x_1 + \varepsilon_1[.$$

It is easy to see that $U =]x_1 - \varepsilon_2, x_1 + \varepsilon_2[\times \{0\} \in \mathcal{V}(x)$ and $] \alpha - \delta, \alpha + \delta[\cdot U \subset V$.

c) $x \neq 0, \alpha = 0$.

We reason in the same way as we did at b), taking a line segment with the center in 0 on $0x$ which is contained in the neighbourhood of 0 we have already considered.

3. Main result.

In this section we give sufficient conditions to be satisfied TV1) in the presence of TV2), in the case of finite dimensional spaces. We note by τ_E the Euclidean topology in X .

Definition. We name open line segment that pass through 0 $\in X$ any set of the form $D(0, \varepsilon) \cdot x$, where $x \in X \setminus \{0\}$ and $D(0, \varepsilon) = \{\alpha \in K : |\alpha| < \varepsilon\}$.

3.1. THEOREM. Let X be a vector space over K of finite dimension $m \geq 1$, and τ be a topology in X satisfying TV2)

and the following properties:

1° τ has a countable basis of the origin.

2° The addition "+" : $X \times X \rightarrow X$ is continuous in every point of the form $(0, x)$.

Then TV1) holds.

Proof: A) Let $x \in X \setminus \{0\}$. We show that any V in $\mathcal{V}(0)$ contains an open line segment that passes through the origin, with direction x .

From $0 \cdot x = 0$ and TV2) it results that there exists an $\varepsilon > 0$ such that $D(0, \varepsilon) \cdot x \subset V$.

E) We show that for any $V \in \mathcal{V}(0)$ there exists a ball (in the sense of τ_E) centered at 0, with a sufficiently small radius, which is included in V .

It is sufficient to show that V contains a hyperparallelepiped centered in 0. Using the equality $0 + 0 = 0$ and condition 2°, it follows that there exists $V_1, V_2 \in \mathcal{V}(0)$ such that $V_1 + V_2 \subset V$. Denoting by $\{e_1, \dots, e_m\}$ the canonical base of X and using A) it follows that there is an $\varepsilon_1 > 0$ such that

$$(1) \quad D(0, \varepsilon_1) \cdot e_1 + V_2 \subset V$$

Following the same reasoning, after a finite number of steps we have: $D(0, \varepsilon_1) \cdot e_1 + \dots + D(0, \varepsilon_m) \cdot e_m + V_{m+1} \subset V$. This implies that $H = D(0, \varepsilon_1) \cdot e_1 + \dots + D(0, \varepsilon_m) \cdot e_m \subset V$, H being a hyperparallelepiped centred in $0 \in X$.

C) For any $x \in X$ and any $V \in \mathcal{V}(x)$ there is a ball centred at x and included in V .

The proof use B) and 2°

D) There is a bounded neighbourhood of 0 (in the sense of τ_E).

X has a countable basis of neighbourhoods of the origin, so X has a descending countable basis of neighbourhoods of the origin $\{U_k\}_{k \in \mathbb{N}}$; we have $U_m \subset U_n$, for $m > n$. From $0 \cdot 0 = 0$ it results that there exists $\varepsilon_k > 0$ and $V_k \in \mathcal{V}(0)$ such that $D(0, \varepsilon_k) \cdot V_k \subset U_k$.

We suppose that 0 has no bounded neighbourhood; it results that V_k is unbounded in τ_E . It follows that there is a sequence $\{x_i^k\}_{i \in \mathbb{N}}$, $x_i^k \in V_k$ such that $\|x_i^k\| \rightarrow \infty$ when $i \rightarrow \infty$ ($\|\cdot\|$ is the Euclidean norm on X). There is i_k such that $\|x_{i_k}^k\| > \frac{1}{\varepsilon_k}$ for any $i \geq i_k$, whence we conclude that $\frac{1}{\|x_{i_k}^k\|} \in D(0, \varepsilon_k)$ and $\frac{1}{\|x_{i_k}^k\|} \cdot x_{i_k}^k \in U_k$. Denoting $y_k = \frac{1}{\|x_{i_k}^k\|} \cdot x_{i_k}^k$ we obtain a sequence $(y_k)_{k \in \mathbb{N}}$ such

that $y_k \in U_k$ and $\|y_k\| = 1$ for any $k \in \mathbb{N}$. From $\{y_k\}_{k \in \mathbb{N}}$ bounded in τ_E it follows we can find a subsequence with $y_k \rightarrow y$ when $k \rightarrow \infty$ and $\|y\| = 1$. So $y \neq 0$ and from the fact that τ is a Hausdorff topology it follows that y and 0 are separated Hausdorff.

We will show that y and 0 are not separated Hausdorff and this will be a contradiction.

Let $V \in \mathcal{V}(y)$, $W \in \mathcal{V}(0)$. From C) it results that there exists $\varepsilon > 0$ such that $B(y, \varepsilon) \subset V$; but it results that there exists $p_1 \in \mathbb{N}$ such, as $y_{p_1} \in B(y, \varepsilon)$ for every $p \geq p_1$. So, we have $y_p \in V$ for all $p \geq p_1$. On the other hand, there is p_2 such that $U_p \subset W$ for all $p \geq p_2$, ($\{U_p\}_{p \in \mathbb{N}}$ descending basis of neighbourhoods) and $y_k \in U_k$. We obtain $y_p \in W$ for $p \geq p_2$. If we take $p_3 = \max\{p_1, p_2\}$ it results $y_{p_3} \in V$ and $y_{p_3} \in W$ so $V \cap W \neq \emptyset$. Because V and W were taken arbitrarily it results that y and 0 can't be separated Hausdorff, contradiction.

E) Let $x \in X$. There exists a bounded neighbourhood of x (in the sense of τ_E). Supposing contrary, every neighbourhood of x is unbounded. Let U be a bounded neighbourhood of 0 (the existence of U is ensured by D)). Then from $0 \cdot x = 0$ and TV2) it results the existence of $\varepsilon > 0$ and $V \in \mathcal{V}(x)$ such $D(0, \varepsilon) \cdot V \subset U$. But V is unbounded so it results that U is unbounded, too; contradiction.

We pass now at the proper demonstration. We shall suppose that TV1) does not hold. For $x, y \in X$ we suppose that there exists $U \in \mathcal{V}(x+y)$ such that for any $V \in \mathcal{V}(x)$, $W \in \mathcal{V}(y)$, $V+W \not\subset U$ holds. Let $\{V_k\}_{k \in \mathbb{N}}$ and $\{W_k\}_{k \in \mathbb{N}}$ be descendent bases of bounded neighbourhoods of x and y , respectively (their existence is ensured by E)). There exists $B(x+y, 2\varepsilon) \subset U$. From $V_k + W_k \not\subset U$ it follows $V_k + W_k \not\subset B(x+y, 2\varepsilon)$ and at least one of $V_k \not\subset B(x, \varepsilon)$, $W_k \not\subset B(y, \varepsilon)$ holds.

Let's admit that the first relation is verified by an infinite number of indices (on the contrary, the second relation is verified by an infinite number of indices).

So, there exists an infinite number of indices k such that $x_k \in V_k \setminus B(x, \varepsilon)$ exists; after an eventually renumbering it results that there is a sequence $\{x_k\}_{k \in \mathbb{N}}$ such that $x_k \in V_k \setminus B(x, \varepsilon)$. $x_k \in V_k \subset V_1$ so $\{x_k\}_{k \in \mathbb{N}}$ is bounded. There is a subsequence such that $x_k \rightarrow x'$ when $k \rightarrow \infty$. We show that x and x' are not separated Hausdorff. This will be a contradiction because $x' \notin B(x, \varepsilon)$.

Let $V \in \mathcal{V}(x)$, $V' \in \mathcal{V}(x')$. There is $B(x', \varepsilon') \subset V'$ (from C)) and for any $k \geq k_1$ we have $x_k \in B(x', \varepsilon')$ (from $x_k \rightarrow x'$ in the Euclidean topology). So $x_k \in V'$ for $k \geq k_1$; for $k \geq k_2$, $x_k \in V_k$ and

$V_k \subset V$ ($\{V_k\}_{k \in \mathbb{N}}$ basis of neighbourhoods). It follows that $x_k \in V$ for $k \geq k_2$. Taking $k_3 = \max\{k_1, k_2\}$ we obtain that $x_{k_3} \in V \cap V'$.

Because V and V' were taken arbitrarily it follows that x and x' are not separated Hausdorff.

3.2. COROLLARY. Let X be a vector space over K with dimension 1 and τ a topology on X so that τ is Hausdorff and with countable basis of the origin. If TV2) is satisfied, then TV1) is verified.

Proof. We do not need A) and B). We prove C) using $1 \cdot x = x$ and TV2). The rest of the proof is analogous.